Midterm Answer Key

1. The associated dynamic programming problem is:

\[ V(k_t, z_t) = \max_{(c_t, k_{t+1})} \{ \ln c_t + \beta E[V(k_{t+1}, z_{t+1})] + \lambda_t (z_t k_t^\alpha - c_t - k_{t+1}) \} \]

The intertemporal necessary condition is (using the envelope theorem so that \( \frac{\partial V(k_t, z_t)}{\partial k_t} = \lambda_t z_t k_t^\alpha \)):

\[
\frac{1}{c_t} = \alpha \beta E \left[ \frac{1}{c_{t+1}^{\alpha k_{t+1}^{-1}}} \right]
\]

The conjecture is that \( c_t = \theta y_t = \theta k_t^\alpha \). Using this in the necessary condition (and also, from the resource constraint we have \( k_t = (1 - \theta) y_t \)):

\[
\theta = 1 - \alpha \beta
\]

Since this relationship was seen repeatedly in class, almost everyone got this result.

The equilibrium interest rate in this economy would be determined by evaluating the necessary condition associated with bond purchases at the market clearing quantities:

\[
\frac{1}{c_t} = (1 + r_t) \beta E \left( \frac{1}{c_{t+1}} \right)
\]

Or

\[
(1 + r_t) = \frac{1}{\alpha \beta E \left( \frac{1}{c_{t+1}} \right)} = \frac{1}{\theta y_t \beta E \left( \frac{1}{y_{t+1}} \right)} = \frac{1}{\alpha \beta^{1-\alpha} z_t^{1-\alpha} \kappa_t^{\alpha(1-\alpha)} E \left( z_{t+1}^{-1} \right)}
\]

This implies that interest rates and the technology shock will be negatively correlated - this makes sense since consumption is increasing in the technology shock (as demonstrated in part (i)). With the relative
abundance of consumption today, the price (i.e. the interest rate falls). With regard to the MPK, this is:

\[ MPK_t = \alpha z_t k_t^{\alpha-1} \]

Note that in any given period, \( k_t \) is fixed - so the contemporaneous movements of \( r_t \) and \( MPK_t \) are entirely determined by \( z_t \). The above expressions imply that \( Corr(r_t, MPK_t) < 0 \).

iii. The risk premium on capital (which I will denote as \( rpk_t \)) is given by

\[ rpk_t = E(\alpha z_{t+1} k_{t+1}^{\alpha-1}) - (1 + r_t) \]

Many students did not get this expression correct so the rest of the answer was wrong. A risk premium is always defined as the expected return on the risky asset minus the riskless return. In this case, \( E(MPK_{t+1}) \) is the expected return on a unit of investment. Using the previous results, the risk premium on capital can be written as:

\[ rpk_t = \alpha^\alpha \beta^{\alpha-1} z_t^{\alpha-1} k_t^{\alpha(\alpha-1)} \left[ E(z_{t+1}) - \frac{1}{E(z_{t+1})} \right] \]

Convince yourself that the term in brackets is indeed positive. (There are many ways to prove this result - this is one.) The intuition, as many of you pointed out, is straightforward: since the return to capital is positively related to consumption, this implies that the sign of the covariance between the MPK and agents’ marginal utility of consumption will be negative. Based on the CCAPM, this implies a positive risk premium on capital.

2. Because of habit persistence in consumption, lagged consumption will be a state variable in addition to the typical beginning of period capital and the technology shock. That is, the dynamic programming problem is:

\[ V(k_t, z_t, c_{t-1}) = \max_{(c_t, k_{t+1})} \{ U(c_t^*) + \beta E_t V(k_{t+1}, z_{t+1}, c_t) \} + \lambda_t \left[ z_t k_t^\alpha + k_t (1 - \delta) - c_t - k_{t+1} - \frac{q}{2} (k_{t+1} - k_t)^2 \right] \]

where \( c_t^* = c_t - hc_{t-1} \) and note that I have used the law of motion for capital to substitute out for investment. This is convenient but not
necessary. I will present below the alternative of having investment as the control variable.

ii. The necessary conditions are (once again, I have used the envelope theorem):

\[ \lambda_t = U'(c_t^*) - h \beta E_t \left[ U'(c_{t+1}^*) \right] \]

\[ \lambda_t \left( 1 + q (k_{t+1} - k_t) \right) = \beta E_t \left\{ \lambda_{t+1} \left[ \alpha k_{t+1}^{\alpha - 1} + 1 - \delta + q (k_{t+2} - k_{t+1}) \right] \right\} \]

With habit persistence, agents are concerned about the change in the marginal utility (note the expression for \( \lambda_t \)). Also, choosing capital today affects adjustment costs both today and tomorrow.

iii. The presence of habit persistence will, ceteris paribus, cause consumption to become extremely smooth - that is agents want the change in MU to be smooth across time periods. With adjustment costs on capital, investment will, ceteris paribus, be smoother. Hence, if one wanted to use habit persistence to explain the equity premium puzzle in an economy with endogenous consumption, consumption would be too smooth relative to what is observed. By imposing adjustment costs in investment, consumption volatility (and its mirror, investment) can be brought into line with the properties of the data.

ii. Alternative - suppose you had not (as some students chose) substituted out for capital but had written the dynamic programming problem as:

\[ V(k_t, z_t, c_{t-1}) = \max_{(c_t, i_t)} \left\{ U(c_t^*) + \beta E_t V(k_{t+1}, z_{t+1}, c_t) \right\} + \lambda_t \left[ z_t k_t^\alpha - c_t - i_t - \frac{q}{2} (i_t - \delta k_t)^2 \right] \]

(Note that in the adjustment cost term I have eliminated \( k_{t+1} \).) The necessary condition for investment would be:

\[ \beta E_t \left[ \frac{\partial V(k_{t+1}, z_{t+1}, c_t)}{\partial k_{t+1}} \right] - \lambda_t [1 + q (i_t - \delta k_t)] = 0 \quad (1) \]

Now to use the envelope theorem, we take the following derivative:

\[ \frac{\partial V(k_t, z_t, c_{t-1})}{\partial k_t} = \beta E_t \left[ \frac{\partial V(k_{t+1}, z_{t+1}, c_t)}{\partial k_{t+1}} \right] \left( 1 - \delta \right) \frac{\partial k_{t+1}}{\partial z_t} + \lambda_t [\alpha z_t k_t^{\alpha - 1} + q (i_t - \delta k_t) \delta] \quad (2) \]
Using eq.(1) in eq.(2) yields

$$\frac{\partial V(k_t, z_t, c_{t-1})}{\partial k_t} = \lambda_t \left[ \alpha z_t k_t^{\alpha-1} + 1 - \delta + q (i - \delta k_t) \right]$$

Updating this and using it in eq.(1) generates the same expression as in (ii). So there are many paths to the correct answer.

3. Define the bonds (one and two period) in terms of pure discount bonds in which you receive one unit of consumption at maturity. Then the necessary conditions associated with the two bonds are:

$$p_1 \frac{1}{c_t} = \beta E_t \left( \frac{1}{c_{t+1}} \right) \quad (3)$$

$$p_2 \frac{1}{c_t} = \beta^2 E_t \left( \frac{1}{c_{t+2}} \right) \quad (4)$$

Of course, the associated interest rates are defined as $(1 + r_1) = \frac{1}{p_1}, (1 + r_2)^2 = \frac{1}{p_2}$. Using the fact that the endowment = consumption is growing at the stochastic rate $\lambda_t$, the equilibrium price of bonds is determined by:

$$p_1 = \beta E_t \left( \lambda_{t+1}^{-1} \right) \quad (5)$$

$$p_2 = \beta^2 E_t \left( \lambda_{t+1}^{-1} \lambda_{t+2}^{-1} \right) \quad (6)$$

The term premium is defined as

$$\frac{E_t (p_{t+1})}{p_{t+2}} - \frac{1}{p_t} \quad (7)$$

That is, next period a two-period bond purchased at time $t$ will sell at the same price as a one-period bond; consequently, the expected rate of return is just the expected change in the prices. At this point, there are a couple (at least!) ways to go. First, appeal directly to the consumption-based capital asset pricing theory which says that the sign of the risk premium on an asset is the negative of the sign of the covariance between the return on the asset and investor’s MU of consumption. It is clear that the return from selling the two period bond is completely determined by the price of a one-period bond. Given that it is assumed that $\pi > 1/2$, it is immediate that $p_{1} > p_{2}$. (Why? - Think of the relative scarcity of consumption
between \( t \) and \( t+1 \) in the two states in a growing economy.) Suppose \( \lambda_{t+1} = \lambda_1 \). This implies that \( MU_{t+1} \) is high while the return from selling the two period bond is high. Hence the critical covariance is positive. The same relationship holds if \( \lambda_{t+1} = \lambda_2 \) so that the risk premium - the same as the term premium here - is negative.

The other way to go is to use the definition of the covariance in eq. (6) to express this as:

\[
p2_t = \beta^2 \left[ \text{Cov}_t (\lambda_{t+1}^{-1}, \lambda_{t+2}^{-1}) + E_t (\lambda_{t+1}^{-1}) E_t (\lambda_{t+2}^{-1}) \right]
\]

and the law of iterated expectations so that

\[
E_t (p1_{t+1}) = E_t \left[ \beta E_{t+1} (\lambda_{t+2}^{-1}) \right] = \beta E_t (\lambda_{t+2}^{-1})
\]

Using these in the definition of the term premium and, by assumption, \( \text{Cov}_t (\lambda_{t+1}^{-1}, \lambda_{t+2}^{-1}) > 0 \), leads to the same result. For full credit, I wanted some discussion of the result in the context of the CCAPM.

4. There is no contradiction. The fact that savings is constant in the economy regardless of the persistence of the shock stems from three auxiliary assumptions. First, the shock is due to the rate of return on capital, not to income directly, so this implies conflicting income and substitution effects. Second, since capital completely depreciates each period, a persistent shock is irrelevant. These two combined with log utility imply the income and substitution effects with persistent shocks negate each other so that savings is constant.