1 Homework #2 - Due April 30

1. Consider the discrete time optimal growth model with no uncertainty, no labor, and no technological growth. That is, the social planner chooses a path for consumption and capital in order to solve the following problem:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{subject to } c_t + k_{t+1} = k_t^\alpha$$

(a) Set up the problem as a dynamic programming problem.
(b) Find the value function by the conjecture method. This involves several steps:
   i. Conjecture that the value function is linear and takes the form $V(k) = A + B \ln k$.
   ii. Use the conjecture to solve for the policy functions $c(k)$ and $k(k)$.
   iii. Substitute these policy functions into the original function that was maximized. For the original conjecture to be correct, the constant must be equal to $A$ and the coefficient of $\ln k$ must be equal to $B$.

2. Consider an economy populated by a representative agent with preferences given by:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t \ln c_t \right]$$

In each period, the individual must allocate output, $y_t$, between consumption and saving. Saving takes the form of investment in two linear technologies both of which are subject to random technology shocks. The technologies are used to produce output in the next period, i.e.

$$y_t = z_1_t k_1_t + z_2_t k_2_t$$

where $z_1_t$ and $z_2_t$ represent the technology shocks to the technologies and $k_1_t$ and $k_2_t$ represent the beginning of period capital stock for each technology. The depreciation rate of both capital stocks is 100% and the shocks are $i.i.d.$

(a) Express this problem as a dynamic programming problem. Derive and interpret the first-order conditions.
(b) Characterize the behavior of consumption and investment. (Hint: Use the conjecture and verify method.) Since the technologies are identical, why aren’t the quantities $k_1_t$ and $k_2_t$ indeterminate?

3. Consider once again the stochastic growth problem with 100% depreciation:

$$\max E_0 \left[ \sum_{t=0}^{\infty} \beta^t c_t^{1-\gamma} \right]$$

subject to $c_t + k_{t+1} = z_t k_t^\alpha$. Assume that $\gamma \neq 1$ (recall that if $\gamma = 1$, then preferences are logarithmic). Note that it is assumed that depreciation is 100%; also $z_t \sim i.i.d.$ Suppose one used the conjecture that savings was a constant fraction of output - demonstrate that this conjecture is not verified. Is there a value of $\alpha$ that will make the conjecture work?

4. A firm chooses a path of sales, $q_t$, labor, $n_t$, and investment, $x_t$, in order to maximize discounted profits

$$E_0 \left[ \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t (p_t q_t - w n_t) \right]$$

where $w$ is the (constant) wage rate. Output, $y_t$, is produced through the following production function:

$$y_t = A k_t + B n_t - \frac{C}{2} \left( \frac{q_t}{x_t} - \frac{q_{t-1}}{x_{t-1}} \right)^2$$

where $A$, $B$, and $C$ denote positive constants. The first two terms imply that output is positively related to the two inputs: beginning-of-period capital, $k_t$, and labor. The third term represents the resource costs of changes in the firm’s sales to investment ratio. Output in each period is allocated to sales and investment, i.e.

$$y_t = q_t + x_t$$

It is assumed that capital depreciates at the rate $\delta$ implying the law of motion for the capital stock is $k_{t+1} = k_t (1 - \delta) + x_t$. Uncertainty is due to randomness in the price of output, $p_t$. It is assumed that $p_t$ is independently and identically distributed over time.
(a) Formulate the firm’s maximization problem as a dynamic programming problem. Be explicit in identifying the state and control variables.

(b) Derive and interpret the first-order conditions associated with the solution of this problem.

(c) Apply the envelope theorem to the conditions in (b) in order to establish that \( \frac{\partial x}{\partial x_t} \) is positively related to the price of output. What is the intuition behind this result? (Hint: The derivative of the value function with respect to \( k_t \) will involve an expression that can be recursively substituted; doing this yields a simple expression.)