1 Homework #1

1. Within the context of the Ramsey-Cass-Koopmans model of optimal growth, assume that agents have instantaneous utility given by:

\[ u(c) = \frac{e^{-\theta c}}{c} \]

where \( \theta > 0 \). In the economy, there is no technological progress, no labor growth, and assume that capital does not depreciate. Agents discount future utility at the subjective rate of \( \rho > 0 \). Technology is represented by \( Y(t) = F(K(t), L(t)) \) where \( F(.) \) exhibits constant returns to scale. Agents are endowed with one unit of labor.

(a) Relate \( \theta \) to the concavity of the utility function. Compute the elasticity of intertemporal substitution (between consumption at dates \( t_1 \) and \( t_2 \)) which is defined as:

\[ \sigma = \frac{d \ln (c(t_2)/c(t_1))}{d \ln (u'[c(t_1)] / u'[c(t_2)])} \]

(b) Express the social planner’s problem as a Hamiltonian and derive the associated necessary conditions.

(c) Construct the phase diagram associated with this economy.

ANSWER: This utility function exhibits constant absolute risk aversion:

\[ ARA = -\frac{u''}{u'} = \theta. \]

Hence, greater values implies greater curvature but note that this functional form exhibits increasing relative risk aversion \( (RRA) \) as a function of consumption. Since relative risk aversion is the inverse of the elasticity of intertemporal elasticity of substitution, we know that the IES is decreasing in consumption. To see this, assume that the ratio \( \frac{c_2}{c_1} \) is changing strictly through a change in \( c_2 \) (a convenience which we can do since utility is time separable). Also note that

\[ d \ln (u'[c(t_1)] / u'[c(t_2)]) = -d \ln (u'[c(t_2)] / u'[c(t_1)]). \]

So we have:

\[ \sigma = -\frac{d \left( \frac{c_2}{c_1} \right)}{d \left( \frac{u_2}{u_1} \right)} \frac{u_2}{u_1} \frac{d \left( \frac{c_2}{c_1} \right)}{d \left( \frac{u_2}{u_1} \right)} = \frac{u_2}{u_1} \frac{d u_2}{d c_2} c_2 = \frac{1}{RRA} = \frac{1}{\theta c_2} \]

The Hamiltonian associated with this problem is:

\[ H = e^{-\rho t} \left( -\frac{e^{-\theta c(t)}}{\theta} \right) + \lambda(t) \left[ f(k(t)) - c(t) \right] \]

The associated necessary conditions are:

\[ \frac{\partial H}{\partial c} = 0 \Rightarrow e^{-\rho t} e^{-\theta c(t)} - \lambda(t) \quad (1) \]

\[ \frac{\partial H}{\partial k} = -\dot{\lambda}(t) \Rightarrow \lambda(t) f'(k(t)) = -\dot{\lambda}(t) \quad (2) \]

\[ \frac{\partial H}{\partial \lambda} = \dot{k}(t) \Rightarrow \dot{k}(t) = f'(k(t)) - c(t) \quad (3) \]

Take the time derivative of eq.(1), simplifying and using in eq.(2) yields the Keynes-Ramsey condition:

\[ \dot{c}(t) = \frac{f'(k(t)) - \rho}{\theta} \quad (4) \]

Note that the phase diagram associated with this economy is very similar to that discussed in class with the only real difference that the \( \dot{k}(t) = 0 \) locus is simply the production function.
2. Within the context of the optimal growth model, suppose technology is given by the constant elasticity of substitution (CES) production function:

\[ Y = A \left( a \left( K^\alpha L^{1-\alpha} \right)^\psi + (1-a) \Lambda^\psi \right)^{1/\psi} \]

where \( \Lambda \) denotes land; the supply of land is fixed. It is assumed that \( A > 0, a > 0, 0 < \alpha < 1, \) and \( \psi < 1. \) There is no technological progress and \( L \) grows at the constant rate \( n > 0. \) Capital does not depreciate. It is assumed that households own the factors of production.

(a) In a competitive economy, firms purchase inputs to maximize profits. Demonstrate that, in equilibrium, factor payments exhaust total output.

(b) Under what conditions on \( \psi \) is the level of per capital output, \( y, \) constant in the steady-state? Under what conditions does \( y \) decline steadily in the long run? What do the results suggest about the role of a fixed factor like land in the growth process?

ANSWER: Since in a competitive equilibrium, factors are paid their marginal products, the answer to (a) is a straightforward application of Euler’s theorem for homogeneous functions so I do not present the details. For part (b) first rewrite the production in intensive (i.e. per-capita) form as:

\[ y(t) = A \left[ a \left( k(t)^\alpha \right)^\psi + (1-a) \lambda(t)^\psi \right]^{1/\psi} \]

where \( \lambda(t) = \frac{A}{L(t)} \). Since \( \lambda(t) \) is declining in a growing economy, a balanced growth path in which \( \dot{k}(t) = \dot{y}(t) = 0 \) does not exist for any value of \( \psi \in (0,1) \). Again restricting our attention to a balanced growth path in which \( \frac{\dot{y}(t)}{y(t)} = \frac{\dot{k}(t)}{k(t)} \), take the time derivative of the intensive production function (denoted \( f(\cdot) \)) we have:

\[ \dot{y}(t) = f_k \dot{k}(t) + f_\lambda \dot{\lambda}(t) \]

Expressing as growth rates, (where \( \alpha_i \) is share to factor \( i \)):

\[ \frac{\dot{y}(t)}{y(t)} = \alpha_k \frac{\dot{k}(t)}{k(t)} + \alpha_\lambda \frac{\dot{\lambda}(t)}{\lambda(t)} \]

But \( \frac{\dot{\lambda}(t)}{\lambda(t)} = -n \) so that in a balanced growth path we have:

\[ \dot{y}(t) = -\frac{\alpha_\lambda}{1-\alpha_k} n < 0 \]

Romer discusses the Cobb-Douglas case in his book - see his eq. 1.49.

3. Suppose that there is no population growth and that the per-capita production function is linear:

\[ y(t) = rk(t) \]

Also assume that agents instantaneous preferences are given by

\[ u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta} : \theta \neq 1 \\ \ln c : \theta = 1 \end{cases} \]

Capital depreciates at the rate of \( \delta \) and there is no technological progress. Do the following:

(a) Express the social planner’s problem in the form of a Hamiltonian and derive the associated necessary conditions.
(b) Define the steady-state in this economy.

(c) Suppose the economy is not at the steady-state. For this class of preferences and technology, we can solve for the policy function \( c(t) = c[k(t)] \). Conjecture that the policy function is linear: \( c(t) = \eta k(t) \) where \( \eta \) is a constant (to be determined). Show that this conjecture is verified and solve for \( \eta \).

(d) Interpret your answer - in particular, what is the role of \( \theta \)? Note that we require that \( \eta > 0 \).
What restrictions does this imply?

ANSWER: The Hamiltonian is:

\[
H = e^{-\rho t} \left( \frac{c(t)^{1-\theta}}{1-\theta} \right) + \lambda(t) [rk(t) - c(t) - \delta k(t)]
\]

Proceeding as in Q1 yields the Keynes-Ramsey condition:

\[
\frac{\dot{c}(t)}{c(t)} = \frac{r - \delta - \rho}{\theta}
\]

This condition along with the law of motion for the capital stock:

\[
\dot{k}(t) = rk(t) - c(t) - \delta k(t)
\]

Because the marginal product of capital is a constant in this economy, a steady-state requires \( r = \delta + \rho \). Even with this restriction, there is nothing to pin down the scale of the economy (i.e. \( \bar{k} \)). Setting \( \dot{k}(t) = 0 \) implies

\[
r - \delta = \rho = \frac{\bar{c}}{\bar{k}}
\]

So the steady-state capital consumption ratio is pinned down, but not the level of \((\bar{c}, \bar{k})\).

If we are not in steady-state then we make the conjecture that

\[
c(t) = \eta k(t)
\]

(This is in part inspired by the fact that consumption is proportional to the capital stock in steady-state.) If this is true, then the law of motion for capital implies:

\[
\frac{\dot{k}(t)}{k(t)} = r - \eta - \delta
\]

But, since \( c(t) = \eta k(t) \), then consumption growth will also be given by:

\[
\frac{\dot{c}(t)}{c(t)} = r - \eta - \delta
\]

For this to be consistent with the Keynes-Ramsey condition, we require:

\[
\eta = (r - \delta) - \frac{1}{\theta} (r - \delta - \rho)
\]

Note that if the net MPK \((r - \delta)\) is equal to agents’ discount factor \((\rho)\), then the consumption profile is flat. If the net MPK > \(\rho\), then the consumption profile is upward sloping and the IES determines the consumption fraction \((\eta)\). Note that if \(\theta\) is very small, it is possible that \(\eta < 0 \) which is clearly not permissible in equilibrium. Here we see that a linear technology can cause problems for the existence of equilibrium.
4. In empirical studies of growth (i.e., growth accounting), one often constructs the Solow residual, or total factor productivity, by the growth in output that is not explained by factor growth. Assuming Cobb-Douglas technology, technological growth is thus measured as:

\[ \tilde{g} = \frac{\dot{Y}}{Y} - \alpha \frac{\dot{K}}{K} - (1 - \alpha) \frac{\dot{L}}{L} \]

(a) Suppose an economy is on its balanced growth path. What fraction of growth in output per worker does this kind of growth accounting attribute to growth in capital per worker? What fraction does it attribute to technological progress?

(b) How can you reconcile your results in (a) with the fact that the Solow model implies that the growth of output per worker on the balanced growth path is determined solely by the rate of technological progress?

ANSWER: First rewrite the expression in per-capita terms:

\[ \tilde{g} = \frac{\dot{y}}{y} - \alpha \frac{\dot{k}}{k} \]

so that the growth of output per worker would be given by:

\[ \frac{\dot{y}}{y} = \tilde{g} + \alpha \frac{\dot{k}}{k} \]

But this seems inconsistent with the Solow growth model where \( \frac{\dot{k}}{k} \) is entirely due to technology. But there is no inconsistency since the change in \( \frac{\dot{k}}{k} \) is also due to technology. Consider a numerical example: Assume the economy is characterized by the Solow model with labor productivity growth of 1% and with a labor force that is growing at 2%. Suppose \( \alpha = 0.3 \). In this economy \( \frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = 3\% \). And growth accounting would imply

\[ \tilde{g} = 3\% - 0.3 (3\%) - (0.7) (2\%) = 3 - 0.9 - 1.4 = 0.7\% \]

But this undervalues the true increase in productivity = 1% - what is going on?

For this example, our Cobb-Douglas production function is:

\[ Y = K^{0.3} (AL)^{0.7} \]

So a 1% increase in \( A \) corresponds to a 0.7% increase in \( Y \). But the 1% increase in \( A \) also implies a 1% increase in \( K \) which causes an additional increase of 0.3% in \( Y \). So the total increase in \( Y \) is 1% --- the same as the increase in productivity. (Here I have done the example in levels of output rather than in per-capita terms but the argument goes through - convince yourself of this.)