1 Derivation of eq. (10.79) in Ljungqvist & Sargent - p. 262

First, L&S make the assumption that the returns and consumption growth follow the following process:

\[
\frac{c_{t+1}}{c_t} = \bar{c}_{\Delta} \exp \{ \varepsilon_{c,t+1} - \sigma_c^2/2 \} \quad (1)
\]

\[
1 + r_{i,t+1} = (1 + \bar{r}_i) \exp \{ \varepsilon_{i,t+1} - \sigma_i^2/2 \} \quad i = s, b \quad (2)
\]

It is assumed that the means of the innovations \((\varepsilon_{t+1})\) are 0. And also that they are normally distributed.

The basic asset pricing equation is:

\[
1 = \beta E \left[ \left( 1 + r_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right) \right] \quad (3)
\]

Using the assumed processes for the returns and consumption growth (eqs. (1) and (2)) yields:

\[
1 = \beta (1 + \bar{r}_i) \bar{c}_{\Delta}^{-\gamma} E \{ \exp [\varepsilon_{i,t+1} - \sigma_i^2/2 - \gamma (\varepsilon_{c,t+1} - \sigma_c^2/2)] \} \quad (4)
\]

Note that eq.(1) has simply been raised to the power of \(-\gamma\). The term in braces can be written as:

\[
\varepsilon_{i,t+1} - \gamma \varepsilon_{c,t+1} - \sigma_i^2/2 + \gamma \sigma_c^2/2 \quad (5)
\]

That last two terms are constants. Define the variable \(x\) as:

\[
x_{t+1} = \varepsilon_{i,t+1} - \gamma \varepsilon_{c,t+1} \quad (6)
\]

Since \(\varepsilon_i\) and \(\varepsilon_c\) are jointly normally distributed, the mean and variance of \(x\) is:

\[
E(x_{t+1}) = 0 \quad (7)
\]

\[
\sigma_x^2 = \sigma_i^2 + \gamma^2 \sigma_c^2 - 2\gamma \text{Cov}(\varepsilon_i, \varepsilon_c) \quad (8)
\]

This implies that the random variable \(\exp(x_{t+1})\) is log normally distributed so using the formulas given in L&S we have:

\[
E[\exp(x_{t+1})] = \exp \left[ \frac{\sigma_i^2 + \gamma^2 \sigma_c^2 - 2\gamma \text{Cov}(\varepsilon_i, \varepsilon_c)}{2} \right] \quad (9)
\]

First re-writing eq.(4) as:

\[
1 = \beta (1 + \bar{r}_i) \bar{c}_{\Delta}^{-\gamma} E \{ \exp[x_{t+1}] \} \exp (-\sigma_i^2/2 + \gamma \sigma_c^2/2) \quad (10)
\]
And using eq. (9) we have

\[ 1 = \beta \left( 1 + \bar{r} \right) \bar{c}^{-\gamma} \exp \left[ \frac{\sigma_i^2 + \gamma^2 \sigma_c^2 - 2\gamma \text{Cov} (\varepsilon_i, \varepsilon_c)}{2} - \sigma_i^2/2 + \gamma \sigma_c^2/2 \right] \]

or

\[ 1 = \beta \left( 1 + \bar{r} \right) \bar{c}^{-\gamma} \exp \left[ \gamma (1 + \gamma) \frac{\sigma_c^2}{2} - \gamma \text{Cov} (\varepsilon_i, \varepsilon_c) \right] \tag{11} \]

which is eq. (10.79) in L&S.

For \( i = b \) (i.e. riskless bonds), \( \text{Cov} (\varepsilon_b, \varepsilon_c) = 0 \). Then, taking logs and rearranging terms we have

\[ \bar{r}^b = -\ln \beta + \gamma \ln \bar{c} - \gamma (1 + \gamma) \frac{\sigma_c^2}{2} \]

Notice the implications that this expression has for how risk aversion (or intertemporal substitution) and the variance of consumption have on the risk-free rate.

For the risk premium, we have:

\[ \bar{r}^s - \bar{r}^b = \gamma \text{Cov} (\varepsilon_s, \varepsilon_c) \]

Since \( \bar{r}^s - \bar{r}^b \approx 0.06 \) and \( \text{Cov} (\varepsilon_s, \varepsilon_c) = 0.00219 \), this requires \( \gamma \approx 27 \) which is too high - this is the equity premium puzzle.