An introduction to log-linearizations

Fall 2000

One method to solve and analyze nonlinear dynamic stochastic models is to approximate the nonlinear equations characterizing the equilibrium with log-linear ones. The strategy is to use a first order Taylor approximation around the steady state to replace the equations with approximations, which are linear in the log-deviations of the variables.

Let \( X_t \) be a strictly positive variable, \( X \) its steady state and

\[ x_t \equiv \log X_t - \log X \]  

the logarithmic deviation.

First notice that, for \( X \) small, \( \log(1 + X) \simeq X \), thus:

\[ x_t \equiv \log(X_t) - \log(X) = \log \left( \frac{X_t}{X} \right) = \log(1 + \%\text{change}) \simeq \%\text{change}. \]

1 The standard method

Suppose that we have an equation of the following form:

\[ f(X_t, Y_t) = g(Z_t). \]  

(2)

where \( X_t, Y_t \) and \( Z_t \) are strictly positive variables.

This equation is clearly also valid at the steady state:

\[ f(X, Y) = g(Z). \]  

(3)

To find the log-linearized version of (2), rewrite the variables using the identity \( X_t = \exp(\log(X_t))^1 \) and then take logs on both sides:

\[ \log(f(e^{\log(X_t)}, e^{\log(Y_t)})) = \log(g(e^{\log(Z_t)})). \]  

(4)

Now take a first order Taylor approximation around the steady state (\( \log(X) \), \( \log(Y) \), \( \log(Z) \)). After some calculations, we can write the left hand side as

\[ \log(f(X, Y)) + \frac{1}{f(X, Y)} [f_X(X, Y)X(\log(X_t) - \log(X)) + f_Y(X, Y)Y(\log(Y_t) - \log(Y))]. \]  

(5)

\(^1\)This procedure allows us to obtain an equation in the log-deviations.
Similarly, the right hand side can be written as
\[
\log(g(Z)) + \frac{1}{g(Z)} \left[ g'(Z)Z(\log(Z_t) - \log(Z)) \right]. \tag{6}
\]

Equating (5) and (6), and using (3) and (1), yields the following log-linearized equation:
\[
[f_1(X, Y)X_x + f_2(X, Y)Y_y] \approx [g'(Z)Zz_t], \tag{7}
\]
Notice that this is a linear equation in the deviations!

Generalizing, the log-linearization of an equation of the form
\[
f(x_1^t, \ldots, x_n^t) = g(y_1^t, \ldots, y_m^t)
\]
is:
\[
\sum_{i=1}^{n} f_i(x_1^t, \ldots, x_n^t)x_i^t_x^t \approx \sum_{j=1}^{m} g_j(y_1^t, \ldots, y_m^t)y_j^ty_j^t.
\]

2 A simpler method

However, in the large majority of cases, there is no need for explicit differentiation of the function \(f\) and \(g\). Instead, the log-linearized equation can usually be obtained with a simpler method. Let’s see.

Notice first that you can write
\[
X_t = X \left( \frac{X_t}{X} \right) = X e^{\log(X_t/X)} = X e^{x_t}
\]
Taking a first order Taylor approximation around the steady state yields
\[
X e^{x_t} \approx X e^0 + X e^0(x_t - 0) \\
\approx X(1 + x_t)
\]
By the same logic, you can write
\[
X_tY_t \approx X(1 + x_t)Y(1 + y_t) \\
\approx XY(1 + x_t + y_t + x_t y_t)
\]
where \(x_t y_t \approx 0\), since \(x_t\) and \(y_t\) are numbers close to zero.

Second, notice that
\[
f(X_t) \approx f(X) + f'(X)(X_t - X) \\
\approx f(X) + f'(X)X(\frac{X_t}{X} - 1) \\
\approx f(X) + f(X)(1 + \eta x_t)
\]

where \( \eta \equiv \frac{\partial f(X)}{\partial X} \).  

Now, the log-linearized equation can be obtained as follows. After having multiplied out everything in the original equation, simply use the following approximations:

\[
X_t \simeq X(1 + x_t) \quad (8)
\]

\[
X_t Y_t \simeq XY(1 + x_t + y_t) \quad (9)
\]

\[
f(X_t) \simeq f(X)(1 + \eta x_t) \quad (10)
\]

2.1 Some examples

2.1.1 The economy resource constraint

Consider the economy resource constraint

\[
Y_t = C_t + I_t,
\]

and rewrite it as

\[
1 = \frac{C_t}{Y_t} + \frac{I_t}{Y_t}.
\]

Using (9) we obtain

\[
1 \simeq \frac{C}{Y}(1 + c_t - y_t) + \frac{I}{Y}(1 + i_t - y_t)
\]

where \( i_t \) is the log-deviation of investment.

Since at the steady state

\[
Y = C + I,
\]

we can cancel out (some) constants and rearrange to obtain

\[
y_t \simeq \frac{C}{Y} c_t + \frac{I}{Y} i_t.
\]

2.1.2 The marginal propensity to consume out of wealth

Assume that the marginal propensity to consume out of wealth is governed by the following first order difference equation:

\[
R^{\sigma - 1}_{t+1} \beta^{\sigma} \frac{\Pi_t}{\Pi_{t+1}} = 1 - \Pi_t.
\]
Notice that at the steady state
\[ R^{\sigma-1}\beta^\sigma = 1 - \Pi. \]

and
\[ \Pi = 1 - R^{\sigma-1}\beta^\sigma. \]

Using (8) and (9) we can write the nonlinear difference equation as
\[ R^{\sigma-1}\beta^\sigma (1 + (\sigma - 1)r_{t+1} + \pi_t - \pi_{t+1}) \simeq 1 - (1 - R^{\sigma-1}\beta^\sigma)(1 + \pi_t). \]

Canceling out constants yields
\[ R^{\sigma-1}\beta^\sigma [(\sigma - 1)r_{t+1} + \pi_t - \pi_{t+1}] \simeq -(1 - R^{\sigma-1}\beta^\sigma)\pi_t. \]

Rearranging, we obtain
\[ \frac{R^{\sigma-1}\beta^\sigma - 1}{R^{\sigma-1}\beta^\sigma} \pi_t \simeq (\sigma - 1)r_{t+1} + \pi_t - \pi_{t+1} \]

and, finally,
\[ \pi_t \simeq R^{\sigma-1}\beta^\sigma [(1 - \sigma)r_{t+1} + \pi_{t+1}]. \]

2.1.3 The Euler equation

The consumption Euler equation is
\[ 1 = R_{t+1}\beta (C_{t+1}/C_t)^{-\gamma}. \]

Using (9) and (10) we can write it as
\[ 1 \simeq R\beta(1 + r_{t+1} - \gamma(c_{t+1} - c_t)). \]

Canceling out constants yields
\[ 0 \simeq r_{t+1} - \gamma(c_{t+1} - c_t) \]

and, rearranging,
\[ c_t \simeq -\sigma r_{t+1} + c_{t+1} \]

where \( \sigma = 1/\gamma \) is the intertemporal elasticity of substitution.
2.1.4 Multiplicative equations

If the equation to log-linearize contains only multiplicative terms, there is a faster procedure. Suppose we have the following equation:

\[ \frac{X_t Y_t}{Z_t} = \alpha \]

where \( \alpha \) is a constant. To log-linearize divide first by the steady state variables:

\[ \left( \frac{X_t}{X} \right) \left( \frac{Y_t}{Y} \right) \left( \frac{1}{Z} \right) = \frac{\alpha}{\alpha} = 1. \]

Now take logs:

\[ \log \left( \frac{X_t}{X} \right) + \log \left( \frac{Y_t}{Y} \right) - \log \left( \frac{Z_t}{Z} \right) = \log(1) = 0. \]

Using (1) we arrive then easily to the log-linearized equation:

\[ x_t + y_t - z_t = 0. \]

Notice that in this case the log-linearized equation is not an approximation!