A SIMPLE PERFECT FORESIGHT MONETARY MODEL

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1. Introduction

This paper develops a simple model which sharpens understanding of the forces that determine the equilibrium path of the price level and money income over time. The model is basically a formalization of part of Friedman's Optimum Quantity of Money discussion [Friedman (1969)]. We consider an economy in which individuals receive both a fixed endowment of real income and nominal transfer payments in the form of bills printed by the government during each period of life. These bills are desired for their use in easing transactions, saving labor involved in trips back and forth to the bank, for their own sake, or because there is a law in the land that states 'this note is legal tender for all debts, public and private'.¹ Individuals seek to maximize their lifetime utility which is equal to the discounted sum of the utility which they derive in each period from consumption of commodities and consumption of the services of money balances.

Since individuals live more than one period and since money balances carry over from one period to the next, individual's decisions in any particular period are influenced by what they think that their current money balances will be worth in future periods. Thus, a model of how individuals form their expectations regarding the future behavior of prices is an essential part of our analysis. Many different models of expectations formation have been discussed in the literature. The particular model which we will employ is a model of 'rational expectations'.² This particular concept has been selected out of the many

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¹For our analysis we need to be able to generate a direct or indirect utility-function that includes real balances as an argument. This real-balances-in-the-utility-function approach has been controversial ever since Samuelson's Foundations (1947, pp. 117–122). We defend studying it because of the very real possibility that one may be able to construct a function that includes real balances as an argument such that the individual acts *as if* he is maximizing it. [See Fischer (1972) for an argument in the similar case of real balances in the production function.]

²The concept of 'rational expectations' has been formalized in a variety of ways; see Black (1972), Brock (1972a), Grossman (1972), Lucas and Prescott (1972), Radner (1970) and Roll (1971).

possible specifications of the mechanism of expectation formation for the following reasons: First, it seems to be the notion that Friedman (1969, p. 45) has in mind in that part of his essay where he states:

This paper has had little or no overlap with the earlier literature, but it yields, as that literature does not, a specific and potentially objective criterion for an optimum behavior of the price level.

Why this difference? The main reason is that the earlier discussion was almost entirely about *unanticipated* inflations or deflations while this paper is mostly about *anticipated* inflations or deflations.

Second, models of 'rational expectations' provide the most appropriate analytical basis for studying the 'optimum quantity of money'. Models with rational expectations operate like models of competitive equilibrium over time in which welfare analysis may proceed along standard lines. In models with other expectation formation mechanisms, the welfare effects of errors of expectations tend to be confused with the welfare effects of a correctly anticipated inflation. Third, the model of rational expectations will enable us to study the effect on current price level of an anticipated, future change in monetary policy.

The following questions will be examined in this paper: (i) How is the equilibrium path of the price level determined in a simple, monetary model with rational expectations? (ii) How is the equilibrium path of the price level changed by a *current* change in the rate of growth of the money supply due to take place T_0 periods from today? (iii) What is the rate of growth or contraction of the money supply which will maximize welfare across the set of equilibria? In addition to analyzing these questions, we will also examine some possible pitfalls in the concept of the 'optimum quantity of money'.³

Since we will be focusing on questions whose analysis is not aided by the presence of bonds and capital, we will analyze a very simple model without bonds and capital where representative individuals receive initial endowments of real goods and transfer payments in the form of fiat money each period.

2. The equilibrium path of the price level

A simple, monetary model which captures the idea of rational expectations is the following: The representative individual seeks to maximize his lifetime utility,⁴

$$U \equiv \sum_{t=1}^{T} \beta^{t-1} u(c_t, m_t), \qquad (1)$$

³A more detailed, rigorous treatment of some of the issues discussed in this paper is given in Brock (1972b).

⁴We assume that all individuals in our society are identical, and speak of the behavior of the 'representative individual'.

subject to
$$p_t c_t + M_t - M_{t-1} = p_t y + H_t$$
, (2)
 M_0 given,

where c_t , M_t , p_t , H_t , y_t are real consumption, nominal balances, price level, cash transfers, and real income, all at time t; u(c, m) is the one-period utility function and β is the subjective time discount factor on future utility. The value of y and the time path of H are taken as given by the representative individual. The representative individual seeks to maximize U by choice of the time paths of c and M, subject to the constraint (2) conditional on the expectations which the individual holds concerning the behavior of the price level.

We say that expectations on $\{p_t\}$ are *rational* if planned demand for real consumption is equal to the real income and planned demand for nominal balances equals nominal supply of cash, at each moment of time. Mathematically this means that if the individual takes the sequences, $\{p_t\}_{t=1}^T$, as given,⁵ solves (2) to lay out planned demand for consumption and planned demand for nominal balances, it must turn out that c_t^d is planned demand for consumption and M_t^d is planned demand for nominal balances.⁶ This is just a dynamic generalization of the fact that, in equilibrium, the price level must adjust so that the existing stock of cash is willingly held.

Let p_t , H_t be given and let $M_0 + H_1 + \ldots + H_t = \sigma^t M$, where $\sigma > 0$. In other words, let the money supply expand or contract proportionately with factor σ . The necessary conditions generated by (2) assuming an interior solution are

$$u_{1}(c_{t}, m_{t})/p_{t} = u_{2}(c_{t}, m_{t})/p_{t} + \beta u_{1}(c_{t+1}, m_{t+1})/p_{t+1},$$

$$u_{1}(c_{T}, m_{T}) = u_{2}(c_{T}, m_{T}),$$
 (3)

where

 $u_1 = \hat{c}u/\hat{c}c, \qquad u_2 = \partial u/\partial m.$

Eq. (3) requires that the marginal utility which is derived from holding a dollar in the form of cash balances must equal the marginal utility which would be derived from spending the dollar on consumption. For the last period, T, money has no store of value function, and the level of m_T must be chosen so that the marginal utility of the services of real balances is equal to the marginal utility of consumption. For all periods but the last, an additional dollar held as money not only yields services at the marginal rate $u_2(c_t, m_t)/p_t$, but also acts as a store

⁵Note that the representative man takes the distribution of transfers $\{H_t\}$ as given exogenously. Roll (1971) studies a situation where transfers are effected by the representative consumer's money hold ags.

⁶The reader should be warned this equilibrium problem is *not* solved in the same way as a control theory problem where the objective function is maximized and the necessary conditions for optimality are written down in order to characterize the optimal path. What we are solving here is an *equilibrium* problem; i.e., we must find a sequence $\{p_t\}$ such that, when (1) is solved for $\{M_t\} \{c_t\}$, it turns out that $M_t = \sigma^t M$, $c_t = y$ for each t. Thus, the structure of our problem is similar to that of the standard existence of general equilibrium problem.

of value. The marginal utility of the consumption which the individual will be able to derive in the next period from consumption of the additional dollar saved in this period is $u_1(c_{t+1}, m_{t+1})$. Multiplying by the discount factor, β , converts this marginal utility into units of utils of period t. The sum of the marginal utility of the money services plus the discounted marginal utility of the additional consumption in the next period must equal the marginal utility of an additional dollar's worth of consumption in period $t, u_1(c_t, m_t)/p_t$.

To study the equilibrium path of the price level it is convenient to rewrite (3) in the real balance form,

$$u_1(c_t, m_t)m_t = u_2(c_t, m_t)m_t + \beta u_1(c_{t+1}, m_{t+1})m_{t+1}(M_t^d/M_{t+1}^d),$$

$$u_1(c_T, m_T) = u_2(c_T, m_T).$$
 (4)

(4) is obtained from (3) as follows: multiply each side of (3) by M_t^d for each t = 1, 2, ..., T. Put $m_t = M_t^d/p_t$. To see the form of the last term of the right-hand side of the *t*th equation of (4) note that $M_t^d/p_{t+1} = (M_{t+1}^d/p_{t+1})(M_t^d/M_{t+1}^d)$.

What must be satisfied if $\{p_i\}$ is an equilibrium price level sequence? First, planned consumption c_t must equal the endowment y in each period. Thus $c_t = y$ for all t. Second, people must willingly hold the existing stock of money. This implies $M_t^d/M_{t+1}^d = 1/\sigma$ for all t. To fix the ideas let us assume that u(c, m)is concave and separable; i.e., there are concave functions u(c) and v(m), and u(c, m) = u(c) + v(m). This allows us to write (4) as

$$[u'(y) - v'(m_t)]m_t = (\beta/\sigma)u'(y)m_{t+1},$$

$$u'(y) = v'(m_T).$$
(5)

This is a simple difference equation with m_T fixed by the terminal condition $u'(y) = v'(m_T)$. If we let A(m) = [u'(y) - v'(m)]m, $B(m) = \beta/\sigma u'(y)m$, then $(5)^7$ amounts to $A(m_t) = B(m_{t+1})$, for t < T and $A(m_T) = 0$. A(m) may be thought of as the net utility return from consumption of m units of real balances.

The equilibrium path of real money balances may be found by working backward from time T, using the difference equation (4). The procedure is illustrated in fig. 1. Use the condition $A(m_T) = 0$ to determine m_T . (We assume that v'' < 0 so that v' is decreasing in m.) Next, find m_{T-1} so that $A(m_{T-1}) = B(m_T)$. Then, use m_{T-1} and the condition $A(m_{T-2}) = B(m_{T-1})$ to find m_{T-2} ; and so forth.

An economic explanation of this procedure may be given as follows: In equilibrium, real balance holdings in the last period are determined by equating the marginal utility of consumption of the exogenously given real endowment,

⁷The reader will note that (5) is critically dependent on additivity of the utility function. The general case is much more complicated. Most of our results break down in the general case. We defend studying this special case because it is simple and a good number of useful insights may be harvested off of this simple case.

y, of each consumer with the marginal utility of services rendered from an extra unit of real balances. This determines real balances and implicitly the price level that makes the representative individual content to consume his endowment and hold the existing stock of money in the final period of his life. Look at matters from period T-1 on. Our representative man must equate marginal utility of consumption in period T-1 to marginal utility of money services plus present value of marginal utility of consumption in period T taking into account the difference in p_{T-1} , p_T . Now p_T is known and p_{T-1} must be set such that he is content at the margin with the existing stock of cash and with the consumption of his endowment. Thus p_{T-1} is determined. Continue on in this manner to determine the equilibrium path of the price level.



Fig. 1

In summary we have the following.

Theorem 1. Let u and v be concave functions with u' > 0, v'' < 0, $u'(0) = +\infty$, $v'(\infty) = 0$. Let $\sigma/\beta > 1$. Then for the finite horizon problem, the difference equation $A(m_t) = B(m_{t+1})$ with terminal condition $A(m_T) = 0$ generates the unique equilibrium path of real balances and hence of prices.

It is clear from this theorem and from the argument which has been used to prove it that as the time horizon T grows to infinity, the solutions $\{m_t^T\}_{t=1}^T$ to the finite horizon, T, equilibrium problems converge to the steady state value of real money balances, \overline{m} ; i.e., $m_t^T \to \overline{m}$ as $T \to \infty$ for each t. The proof of this corollary is easy and is left to the reader – just examine fig. 1.

The case $T = \infty$, the infinite horizon case, is a little more complicated. When

u(c, m) is separable, the necessary condition for a path to be equilibrium is

$$A(m_{t}) = B(m_{t+1}).$$
(6)

This is the same difference equation as (4), but there is no terminal condition. There are three possible cases: (a) m_t decreases to zero; (b) m_t is equal to the steady state value of m, \bar{m} for all t, where \bar{m} satisfies $A(\bar{m}) = B(\bar{m})$; (c) m_t increases to infinity. We will now establish the following results: (i) Under general conditions, paths of type (a) cannot be equilibrium. (ii) All paths which satisfy the difference equation $A(m_t) = B(m_{t+1})$ [and hence all candidates for equilibrium paths] also satisfy a certain critical inequality which is useful in determining whether they are equilibrium paths. (iii) The paths of type (b) are always equilibrium. (iv) Under certain conditions, paths of type (c) cannot be equilibrium. Later in section 5, we will discuss circumstances in which paths of type (c) may be equilibrium.

First, case (a) is easy to rule out because marginal utility of money services is increasing to infinity [assume that $v'(0) = +\infty$]. Hence at some point the representative man would rather consume a little less and harvest a large marginal utility from money services. Thus, a path of type (a) cannot be equilibrium.

Second, let us establish the following.

Lemma 1. If \tilde{m}_t is a solution of $A(m_t) = B(m_{t+1})$, and \tilde{p}_t is the price path associated with this solution, i.e., $\tilde{p}_t = \sigma^t M / \tilde{m}_t$, then for any other path m'_t and associated paths c'_t , M'_t which satisfies the budget constraint

$$\tilde{p}_{t}c_{t}' + M_{t}' - M_{t-1}' = \tilde{p}_{t}y + H_{t}, \tag{7}$$

the following inequality must hold:

$$\sum_{t=1}^{T} \beta^{t-1} [u(c_t') + v(M_t'/\tilde{p}_t) - u(y) - v(\tilde{m}_t)]$$

$$\leq \beta^T [u'(y)\tilde{m}_{T+1}](1/\sigma).$$
(8)

Proof. The following inequality is true for each T, for any solution $\{\tilde{m}_t\}$ of $A(m_t) = B(m_{t+1})$:

$$\sum_{t=1}^{T} \beta^{t-1} [u(c_t') + v(M_t'/\tilde{m}_t) - u(y) - v(\tilde{m}_t)]$$

$$\leq \beta^T [u'(y)\tilde{m}_{T+1}](1/\sigma).$$
(9)

To see this use the concavity of u and v to write

$$\sum_{t=1}^{T} \beta^{t-1} \{ [u(\tilde{p}_{t}y + M'_{t-1} - M'_{t})/\tilde{p}_{t}] + v(M'_{t}/\tilde{p}_{t}) - u(y) - v(\tilde{m}_{t}) \}$$

$$\leq \sum_{t=1}^{T} \beta^{t-1} [u'(y) \{ y + (M'_{t-1} - M'_{t})/\tilde{p}_{t} - y \} + v'(\tilde{m}_{t})(M'_{t} - \sigma^{t}M)/\tilde{p}_{t}].$$

Use $u'(y)\tilde{m}_t = v'(\tilde{m}_t)\tilde{m}_t + (\beta/\sigma)u'(y)\tilde{m}_{t+1}$ to simplify the R.H.S. of the latter expression to

$$(\sigma^T M - M'_T)[\{u'(y) - v'(\tilde{m}_T)\}/\tilde{p}_T]\beta^{T-1}.$$

Rewrite this as

$$\{(\sigma^T M - M_T') / \sigma^T M\} [u'(y) - v'(\tilde{m}_T) (\sigma^T M / \tilde{p}_T)] \beta^{T-1}.$$

Since $\sigma^T M/\tilde{p}_T = \tilde{m}_T$, $A(\tilde{m}_T) = B(\tilde{m}_{T+1})$, and $(\sigma^T M - M'_T)/M'_T \leq 1$, the last expression is bounded above by

 $\beta^T[(u'(y)\tilde{m}_{T+1})/\sigma].$

Thus, (8) is established.

Using Lemma 1 we may prove:

Theorem 2. A sufficient condition for a solution $\{\tilde{m}_t\}$ of $A(m_t) = B(m_{t+1})$ to be an equilibrium is that $\beta^t u'(y)\tilde{m}_t \to 0$ as $t \to \infty$.

Proof. To establish this theorem, we must show that given the price path \tilde{p}_t , there is no alternative path of consumption $\{c'_t\}$ and real money balances $\{m'_t\}$ which satisfies the budget constraint (7) and yields a higher lifetime utility than the path $\tilde{c}_t = y$ and $m_t = \tilde{m}_t$. Applying Lemma 1, we know that the gain in utility over the time interval from t = 1 to t = T from pursuing the alternative path is $\beta^T[u'(y)\tilde{m}_{T+1}/\sigma]$. By the hypothesis of this theorem, however, $\beta^T[u'(y)\tilde{m}_{T+1}/\sigma]$ converges to zero as $T \to \infty$. It follows that the alternative path cannot be any better than the proposed equilibrium path, and, hence, that the proposed equilibrium path is an equilibrium path.

Third, as a corollary of this theorem, we may prove that the steady state path is always an equilibrium:

Corollary 1. The path $p_t = \sigma^t M / \bar{m}$ is an equilibrium path.

Proof. Apply Theorem 2 noting that for the steady state path $m_t = m$ for all t.

Fourth, we are left with the paths of type (c). We will now prove a theorem which eliminates these paths as potential equilibria for an economically reasonable set of conditions. We will later discuss possibly unreasonable conditions under which these paths may be equilibria.

Theorem 3. Suppose that $\sigma > \beta$; i.e., the money supply grows faster than people discount the future. Assume that v' is always positive and that there exists a $\lambda < 0$ such that for sufficiently large m, v'(m) < m^{λ}. Then, paths of type (c) cannot be equilibrium.

Proof. Along a path of type (c), as $t \to \infty$, $m_t \to \infty$, $v'(m_t) \to 0$, and the difference equation $A(m_t) = B(m_{t+1})$ implies that the price level falls with factor β in the long run. In order to see that the price level falls with factor β in the long run, we first calculate the rate of growth of real balances from the difference equation $A(m_t) = B(m_{t+1})$. Doing this we get

$$[u'(y) - v'(m_t)]m_t = (\beta/\sigma)m_{t+1}u'(y)$$

Thus,

$$u'(y)m_{t+1}/m_t = (\sigma/\beta)[u'(y) - v'(m_t)]$$

$$\rightarrow (\sigma/\beta)u'(y), \quad t \rightarrow \infty.$$

Hence, the rate of growth of real balances approaches σ/β as $t \to \infty$. Now the price level is given by

$$p_t = \sigma' M/m_t.$$

It is now obvious that the growth factor of the price level must approach β as $t \to \infty$. Now will the consumer be willing to hold real balances growing with factor σ/β while the price level is falling at rate β ? He will not. Hence, paths of type (c) cannot be equilibrium. At some point in time, *T*, the act of taking one dollar out of cash balances will yield him $u'(y)/p_T$ utils at the margin. His cash balances are depleted by one dollar for all $t \ge T$. This loss of money services generates a utility loss

$$\sum_{t=T}^{\infty} \beta^{t-T} v'(m_t) (1/p_t),$$

which because p_t moves with factor β' ; therefore, since m_t grows with factor σ/β and $v'(m_t) \leq m_t^{\lambda}$, a constant k may be chosen so that the loss is bounded above by

$$\left\{k\sum_{\tau=\tau}^{\infty}\left[(\sigma/\beta)^{\lambda}\right]^{\tau}\right\} / p_{T},$$

where k is a constant independent of T. Since $(\beta/\sigma)^{\lambda} < 1$ upon cancelling p_T , obviously it follows that

$$u'(y) > k \sum_{t=T}^{\infty} [(\sigma/\beta)^{\lambda}]^{t},$$

for T large enough. This ends the proof.

3. Comparative dynamics

The diagrammatic apparatus of fig. 1 may be used for comparative dynamics analysis. For example, consider the effects of an increase in σ or a decrease in β . When σ increases, the opportunity cost of obtaining a given amount of money services is higher because the rate of inflation is higher. Thus, the stock of real balances should fall in each period. If β decreases, the future is worth less. Thus, the incentive to hold cash in order to buy goods in future periods is weakened. This leads to a lower equilibrium level of real balances. These two results appeal to common sense and thus serve as a check on the model.

Our apparatus is also useful for studying the following question. Assume money has been growing at rate σ_1 . Let the economy be in equilibrium in our sense; i.e., at each moment of time people are content to hold the existing stock of money and price expectations are being fulfilled. Now let a change take place. At date 1, 'today', each individual receives a notice that at date T_0 , transfer payments of money to him will grow at factor $\sigma_2 > \sigma_1$. It follows that the aggregate nominal money supply will grow with factor σ_2 starting at T_0 . Further, effects of this change are perfectly foreseen (and actually comes to pass). What will happen?

Consider, specifically, the infinite horizon case. Before the announcement, the economy is in a steady state with $m_t = \bar{m}_{\sigma_1}$ and $p_t/p_{t-1} = \sigma_1$. To fix the ideas suppose that $T_0 = 3$ is the date that each man's checks from the government will start growing at rate σ_2 . At the old expected price path, each man now feels richer. Each man attempts to buy more goods, putting upward pressure on the price level. The attempt to buy goods in the first period, however, is fairly small because the happy event is several periods away. Thus, the price level must rise in the first period above its previously expected level in order to induce people to hold the existing stock of money. Similarly the second period. In period 3, money is growing with factor σ_2 . Thus, as in Theorem 1, the economy must remain at the steady state \bar{m}_{σ_2} for all $t \ge 3$. Formally, we have

$$A(m_{1}) = (\beta/\sigma_{1})u'(y)m_{2},$$

$$A(m_{2}) = (\beta/\sigma_{1})u'(y)m_{3},$$

$$A(m_{3}) = (\beta/\sigma_{2})u'(y)m_{4}, \qquad m_{3} = \bar{m}_{\sigma_{2}}.$$
(10)

This solution is depicted in fig. 2.



Fig. 2

In general, if the change is to take place at T_0 the new equilibrium path of real balances is gotten by

$$A(m_1) = (\beta | \sigma_1) u'(y) m_2,$$

$$A(m_2) = (\beta | \sigma_1) u'(y) m_3, \dots$$

$$A(m_{T_0}) = (\beta | \sigma_2) u'(y) m_{T_0+1}, \qquad m_{T_0} = \bar{m}_{\sigma_2}.$$
(11)

In the face of an announcement that T_0 periods from now each man's transfer income, that is printed on checks which are sent to him by the government, will increase by factor $\sigma_2 > \sigma_1$, real balances fall in each period until at T_0 the new equilibrium steady state \bar{m}_{σ_2} is reached. Thus, the growth factor of price level rises from σ_1 at time 1 to σ_2 at time T_0 .

What lessons can we learn from this exercise? One lesson is that the anticipation of a rise in the rate of growth of the money supply will cause a rise in the price level and thus a rise in money GNP, even before the rate of monetary expansion increases, provided that the representative man perfectly foresees the consequence of the future increase in σ . To the extent that such foresight is present in the 'real world', empirical studies on the effect of money on GNP may produce misleading results. The observation of 'business leading money' may reflect the anticipation effect of future changes in monetary policy.⁸

⁸Needless to say such perfect foresight effects are only one of a multitude of possible causes of business leading money.

The following parable is an instance of how business may lead money. The world has been run by a very conservative group, call it R, which has not been expanding the money supply at all. Prices are stable, and the people are forecasting the future price level perfectly. A glamourous candidate, call him T.K., from the opposition party, call it D, appears. It looks certain that this guy will be president in three years. From the kind of noises T.K. is making it is clear that he will print much money to give to the elderly in the form of higher social security checks, more public projects and the like. The consuming public may not try to forecast the forthcoming bonanza but entrepreneurs will start planning. They start expanding their operations preparing for the expected rash of consumer spending during the reign of T.K. This puts pressure on factor prices, wages, etc., today. The increased wage income finds its way into the goods market pushing up goods prices. Expectations are revised upwards generating even more pressure on the price level. Thus much of the adjustment in real balance holdings and the price level may possibly take place before T.K. actually sets the printing presses in motion!

4. The optimum quantity of money

We may use the model of section 2 to sharpen our understanding of the optimum quantity of money. This idea is exposited in Friedman (1969). What does it mean in our model? Friedman concentrated his attention on monetary policies of the form $M_t = \sigma^t M$. Let us do likewise. Since we have only one type of individual, an unambiguous measure of welfare is available, viz., the man's utility. To find the optimum quantity of m, choose σ so that

$$\sum_{i=1}^{\infty} \beta^{i-1} [u(y) + v(\bar{m}_{\sigma})] = [u(y) + v(\bar{m}_{\sigma})]/(1-\beta),$$

subject to $[u'(y) - v'(\bar{m}_{\sigma})] = \beta/\sigma u'(y),$

is maximum. Here \bar{m}_{σ} is equilibrium real balances associated with σ .

If v'(m) > 0 for all $m \ge 0$, we are in trouble. In this case the optimum quantity of money does not exist. You name a rate of shrinkage of the money supply and I will ask that Friedman's (1969, p. 16) furnace operate even faster – making society better off. This result is reasonable and is well-known. If there is something that is costless to create and society prefers more of it to less of it, then society is best off by manufacturing an infinite amount of the stuff.

To avoid this problem let us assume, as does Friedman, that there is m^* such that $v'(m^*) = 0$. With this assumption the optimum quantity of money is achieved when $\beta = \sigma$. Friedman's (1969, p. 16) furnace should burn up the existing money supply at a factor that is just equal to the discount factor on

future utility. This is the policy which maximizes individual and, hence, total utility when the inflationary (deflationary) effects of money creation (destruction) are perfectly anticipated.

In our model certain technical problems arise with the notion of the optimum quantity of money. Troubles crop up from two quarters: (a) For the separable utility function, u(c, m) = u(c) + v(m) for certain values of the three quantities β , σ , $\lim_{m\to\infty} v'(m)$, the steady state m^* may not be the only equilibrium for the economy; there may be a whole range of perfect foresight paths which do not converge to the steady state. (b) If we drop the assumption of separability of the utility function, there may be a discrete set of multiple steady state equilibria with society better off on equilibria with a higher level of real balances. Further, in both cases (a) and (b), there does not seem to be an obvious way to guide the economy to the 'best' equilibrium.

5. Indeterminacy of the perfect foresight path

Let us take up (a) first. Record (7) for convenience,

$$\sum_{t=1}^{T} \beta^{t-1} [u(c_t) + v(M_t/\tilde{p}_t) - u(y) - v(\tilde{m}_t)] \leq \beta^T [u'(y)\tilde{m}_{T+1}](1/\sigma), \quad (12)$$

for any solution $\{\tilde{m}_t\}_{t=1}^{\infty}$ of $A(m_t) = B(m_{t+1})$. Recall Theorem 1 which says any solution $\{\tilde{m}_t\}$ of $A(m_t) = B(m_{t+1})$, where \tilde{m}_t remains non-negative and grows less rapidly than β^T , is an equilibrium. This is so because the R.H.S. of (12) goes to zero as $T \to \infty$, which means that people are content to hold $\{\tilde{m}_t\}$ real balances and consume y each period. Recall from the discussion in section 2 that the solutions of $A(m_t) = B(m_{t+1})$ that decrease to zero become infeasible, i.e., $\tilde{m}_t < 0$ for large t. This is because the marginal utility of money services, $v'(\tilde{m}_t)$, becomes larger than the marginal utility of consumption, u'(y), as $\tilde{m}_t \to 0$. Hence, the only possible troublemakers are solutions of $A(m_t) = B(m_{t+1})$ such that $\tilde{m}_t \to \infty$. Let us estimate from $A(m_t) = B(m_{t+1})$ how fast m_t can grow,

$$A(\tilde{m}_{t}) \equiv [u'(y) - v'(\tilde{m}_{t})]\tilde{m}_{t} = (\beta/\sigma)u'(y)\tilde{m}_{t+1} \equiv B(\tilde{m}_{t+1}).$$
(13)

Thus,

$$[1 - v'(\tilde{m}_t)/u'(y)](\sigma/\beta) = \tilde{m}_{t+1}/\tilde{m}_t.$$
(14)

To get a feeling for this rate of growth of real balances suppose that $v'(\tilde{m}_t) \rightarrow -a < 0, t \rightarrow \infty$; i.e., marginal utility of money services falls to a negative constant as the quantity of real balances goes to infinity. Then the long-run growth factor of real balances implied by (14) is finite and equals [1 + a/u'(y)] (σ/β) . Call this g. Now look at (12). If $g\beta < 1$ then $\{\tilde{m}_t\}$ is an equilibrium. In words, if $\{\tilde{m}_t\}$ grows slowly enough relative to the fall in worth of future utilities

then people will be content to hold increasing real cash balances provided that the intertemporal efficiency conditions are satisfied.⁹ When is it likely that $g\beta < 1$? This is most likely to obtain precisely at the optimum quantity of money, $\sigma = \beta$. In this case $g = \beta [1 + a/u'(y)]$. So if a is small then there is a whole continuum of equilibria. Namely every solution of $A(m_t) = B(m_{t+1})$ such that $\tilde{m}_t \to \infty$ is equilibrium. Furthermore society is worse off on the higher real balance equilibria.

It may be helpful to look at the following example to understand why these multiple equilibria may crop up. Put $v(m) = \log m - am$, a > 0. Choose $u(\cdot)$ so that u'(y) = 1. This will save us from carrying the u'(y) term in (13). Eqs. (13) and (14) become

$$A(m_t) \equiv [1 - 1/m_t + a]m_t = \beta/\sigma m_{t+1} \equiv B(m_{t+1}), \qquad (15)$$

$$[1 - 1/m_t + a](\sigma/\beta) = m_{t+1}/m_t.$$
(16)

The steady state equilibrium level of real balances is defined by

$$[1-1/\bar{m}+a] = \beta/\sigma, \tag{17}$$

or

$$\bar{m} = 1/(1+a-\beta/\sigma). \tag{18}$$

Also notice that $v'(\hat{m}) = 0$ when $\hat{m} = 1/a$. Thus society is satiated with real balances when $\hat{m} = 1/a$. Take any $\tilde{m}_1 > \tilde{m}$ and solve the difference equation (15). Let $\{\tilde{m}_t\}_{t=1}^{\infty}$ be the solution. This solution may be given in closed form but

⁹This result is odd enough to merit further explanation. The reader may ask: How can a path where real balances are going to infinity be an equilibrium? Because at some time T_0 I can take a dollar out of cash balances, use it to buy $1/p_{T_0}$ widgets which yields $u'(y)(1/p_{T_0})$ utils at the margin. Furthermore if I don't replace the dollar from time T_0 on I reduce cash balances. This act yields positive utility for all $t > T_0$ provided that $m_t > m^*$, where $v'(m) < 0, m > m^*$. Thus a path with $m_t \to \infty$ cannot be equilibrium.

The error in this argument is that a dollar taken out at T_0 must be replaced at some future time in order to keep cash balances non-negative. Cash balances are shrinking over time with factor $\sigma < 1$. Recall that $g\beta < 1$ implies that $\sigma < 1$. The economic content of (12) is that there is no sequence of increments to cash balances and consumption that yield a net utility gain over the path defined by $A(m_t) = B(m_{t+1})$ for the case $g\beta < 1$.

Another question the reader might have about paths of the form $A(m_t) = B(m_{t+1}), m_t \to \infty$ being equilibrium is: if the economy is oversatiated with cash balances, $v'(m_t) < 0$, why can't people throw the stuff away and make themselves better off? I.e., in mathematical terms replace equality budget constraints with inequalities.

There are two errors in this argument. First, whatever the price level may be consumers will spend all of their income because consumption yields positive utility even if real balances don't. Second, there is a difference between private cost and social cost. Suppose all consumers expect the price level path gotten from $A(m_t) = B(m_{t+1}), m_1 > \bar{m}$. I.e., $p_t \equiv \sigma^t M/m_t$. Eq. (12) tells us that the consumers are in equilibrium for this value of the price level. However, they could make themselves better off if they all agreed to expect $p_t = \bar{p}_t \equiv \sigma^t M/\bar{m}$ instead of $p_t = \sigma^t M/m_t$ and lay out their demands accordingly. This is a standard divergence of private cost from social cost problem.

we will not bore the reader with that algebra. Look at (16). Since $\tilde{m}_t < \infty$ (recall that we are assuming $\beta \leq \sigma$, a > 0), therefore from (16) $\tilde{m}_{t+1}/\tilde{m}_t \rightarrow (1+a)\sigma/\beta$, $t \rightarrow \infty$. This says that $\{\tilde{m}_t\}$ grows like $(1+a)\sigma/\beta$ for large t. Let us work out the price level $\{\bar{p}_t\}$ compatible with $\{\tilde{m}_t\}$. It is given by $(\sigma^t M)/\bar{p}_t = \tilde{m}_t$. Thus $\bar{p}_{t+1}/\bar{p}_t \rightarrow \beta/(1+a)$.

Now what does it mean to say that $\{\tilde{m}_t\}$ is an equilibrium? It means that if we insert \tilde{p}_t for p_t in (2) with $T = \infty$, $H_1 = \sigma^t M - \sigma^{t-1} M$, $M_0 = M$, then the representative consumer upon solving (2) will find it optimal to choose $c_t = y$, $M_t = \sigma^t M$ for all t. It turns out for certain values of a, β, σ he does just that. To show this let us use the fact that $A(\tilde{m}_t) = B(\tilde{m}_{t+1})$ to estimate the net utility gained by choosing an alternative path. An upper bound is given by (12) with u'(y) set equal to one and p_t set equal to \bar{p}_t . For what values does $\beta^T \tilde{m}_{T+1} \to 0$, as $T \to \infty$? Since \tilde{m}_t grows like $[(1+a)\sigma/\beta]^t$, $\beta^t \tilde{m}_{t+1}$ grows like $[\beta(1+a)(\sigma/\beta)]^t$ which approaches 0 if $(1+a)\sigma < 1$. Thus, if $(1+a)\sigma < 1$, the path $\{\tilde{m}_t\}$ is an equilibrium. And when is it likely that $(1+a)\sigma < 1$? This is likely when $\sigma = \beta$ and a is small. But $\sigma = \beta$ is precisely the optimum quantity rule.

Five comments are now in order concerning equilibrium paths other than the steady state path. First, it is clear that if such paths exist, the optimum quantity rule of setting σ equal to β does not necessarily insure that the economy will reach its highest possible level of utility. Given that $\sigma = \beta$, there may still be a whole continuum of possible, equilibrium paths of the price level. Among this continuum of paths, the steady state path is necessarily the one which yields the highest level of utility. But, there is no mechanism which insures that this is the equilibrium path which the economy will follow.

Second, along any equilibrium path other than the steady state path, the rate of change of the price level will differ from the rate of monetary expansion. Consider an equilibrium where real balances m_t are growing in the long run with factor $g = (\sigma/\beta)[1 + a/u'(y)]$. The price level is given by $p_t = \sigma^t M/m_t$. Thus the price level grows with factor $\sigma/g = \beta/[1 + a/u'(y)]$. We can say something about the size of this factor. Since $m_t > i\bar{m}$ for all t [here \bar{m} is defined by $A(\bar{m}) = B(\bar{m})$] we have $\lim_{s\to\infty} -v'(m_s) = a > -v'(m_t) > -v'(\bar{m})$ for all t. Thus $\beta/[1 + a/u'(y)] < \beta/[1 - v'(m_t)/u'(y)] < \beta/[1 - v'(\bar{m})/u'(y)] = \sigma$ for t large. The latter equality follows from definition of \bar{m} as the intersection point: $A(\bar{m}) = B(\bar{m})$. Hence the price level along such an equilibrium grows more slowly than the money supply. This shows that in a world that lives forever it is not necessarily true that the price level grows at the same rate as the money supply in long-run equilibrium.

Third, it can be shown that introducing the possibility of borrowing and lending does not eliminate the possibility of multiple equilibrium price paths. Suppose that a loan market were opened up in which people can borrow and lend at a real market rate of interest r. With borrowing and lending permitted, individual optimization requires that the marginal rate of substitution between c_t and c_{t+1} , $u'(c_t)/\beta \cdot u'(c_{t+1})$ equal 1+r. Since along any equilibrium path $c_t = c_{t+1} = y$, it follows the marginal rate of substitution equals β . Hence, along any

equilibrium path at a market rate of interest $v = 1/\beta - 1$; the loan market will clear without any transaction actually taking place. It follows that any price path which is an equilibrium path when borrowing and lending are prohibited remains an equilibrium path when borrowing and lending are permitted at a market clearing rate of $r = 1/\beta - 1$.¹⁰

Fourth, using this extension to the case of borrowing and lending there is an alternative way of stating the condition which a path must satisfy in order to be a perfect foresight path. Calculate the present value of real monetary transfers using $r = 1/\beta - 1$: The discount rate is β and

$$PV = \sum_{t=1}^{\infty} \beta^{t-1} [(\sigma^t M - \sigma^{t-1} M)/\bar{p}_t].$$

PV is finite exactly on those paths $\{\tilde{m}_t\}$ that solve (13) that give $\bar{p}_t = \sigma^t M/\tilde{m}_t$ growing with factor $\beta/[1+a/u'(y)]$ such that $[1+a/u'(y)]\sigma < 1$. To see this we just calculate the growth factor of a summand of PV. This is given by $(\sigma\beta)$ $\{\beta[1+a/u'(y)]\} = [1+a/u'(y)]\sigma$. Thus, equilibrium paths are precisely those solutions of the difference equation (13) that generate finite present values of real monetary transfers. To be honest we should point out that we have not proved that paths with $[1+a/u'(y)]\sigma \ge 1$ are not equilibria. We are confident that this can be done for $[1+a/u'(y)]\sigma > 1$.¹¹

Finally, the anomaly of indeterminacy of the perfect foresight path can be eliminated if one makes the reasonable assumption that $[1 + a/u'(y)]\sigma > 1$. When this condition is satisfied, the R.H.S. of (17) cannot be made to go to zero for any eligible path which satisfies $A(m_t) = B(m_{t+1})$. Clearly, for $\sigma > 1$ (a positive rate of monetary expansion), the condition for eliminating the indeterminacy is satisfied. For the optimum quantity rule, $\sigma = \beta < 1$, the condition for eliminating indeterminacy will be satisfied provided that *a* is large enough. This is a reasonable assumption since *a* measures the marginal disutility of money at an infinite level of real balances. One would think that if a large amount of *m* is noxious at the margin, then an infinite amount would be extremely noxious at the margin.

6. Non-separable utility and multiple steady states

Let us now turn to the second problem in our model with the optimum quantity of money: multiple steady state equilibria.

¹¹The borderline case $[1+a/u'(y)]\sigma = 1$ is usually difficult to resolve. We leave out this analysis in order to save space.

¹⁰This argument does not depend upon the absence of alternative assets. In Brock (1972b) we analyse a more complicated model with an alternative asset, call it 'capital'. It is shown there that the capital stock, k, converges to a steady state k. At k the model behaves exactly like the simple model discussed in this paper.

Suppose u(c, m) is not separable. Then the difference equation (4) becomes

$$u_{t}(c_{t}, m_{t})m_{t} = u_{2}(c_{t}, m_{t})m_{t} = (\beta/\sigma)u_{1}(c_{t+1})m_{t+1}, \qquad (19)$$

which in steady state reduces to

$$u_1(y, m)(1 - \beta/\sigma) = u_2(y, m).$$
⁽²⁰⁾



Fig. 3

The reader will recognize this as the necessary condition for the problem in ordinary demand theory,

Maximize
$$u(c, m)$$
,
subject to $c + (1 - \beta/\sigma)m = I$, (21)

where I is an undetermined level of income. Draw the income consumption curve for (21) with real balances inferior over some range. It is obvious from fig. 3 that utility functions exist that yield income consumption curves that cut the c = y line more than once. Each cut point is a solution to (20), over the range for which $u_2(y, m) > 0$, welfare increases with real balances along the set of equilibria. As in case (a) there seems to be no presumption that the economy will automatically converge to the 'best' of these equilibria.¹²

¹²One might try to argue that equilibria where real balances are inferior would be upstable in any reasonable adjustment scheme. There is no salvation in this argument, however. For each 'unstable' equilibrium there may be a 'stable' one with a higher level of real balances.

If $\beta = \sigma$ then $u_2(y, m) = 0$ so that there is only one equilibrium. Hence, the problem of multiple steady states does not arise. There is still difficulty for the optimum quantity concept if one interprets it to mean that the economy will automatically be better off by setting the growth factor of the money supply closer to β because with multiple equilibria the economy may go to a lower level of welfare. It should be emphasized, however, that the whole problem of multiple steady states can only arise in the unlikely circumstance where real money balances are an inferior good.

7. Summary

Let us sum up the value added of this exercise as we see it. First, we have laid out precisely, in a manageable way, the concept of anticipated inflation or deflation. Second, we have used our model to show how changes in money GNP may lead to changes in the money supply when the change in the money supply and the concomitant path of the price level is perfectly foreseen. Third, we have uncovered conditions that are needed to make sense out of the optimum quantity of money notion. We need equilibrium to be unique. We need to assume that marginal disutility of real balances becomes large as $m \to \infty$. Now, borrowing and lending, investment in physical capital, uncertainty, etc. may be introduced. The extra model building needed to do this seems straightforward. The extra analysis may be hard.

References

- Black, F., 1972, The uniqueness of the price level in monetary growth models with rational expectations, Money and Banking Workshop Paper (University of Chicago, Chicago, Ill.), forthcoming in Journal of Economic Theory.
- Brock, W.A., 1972b, On models of expectations that arise from maximizing behavior of economic agents over time, Journal of Economic Theory 5, 348-376.
- Brock, W.A., 1972b, Money and growth: The case of long run perfect foresight, University of Chicago, Money and Banking Workshop Paper (University of Chicago, Chicago, Ill.), forthcoming in International Economic Review.
- Brock, W.A. and L.J. Mirman, 1972, Optimal growth under uncertainty: The discounted case, Journal of Economic Theory, 479-513.
- Diamond, P., 1965, National debt in a neoclassical growth model, American Economic Review 55, 1126-1150.
- Fischer, S., 1972, Money and the production function, Workshop in Money and Banking Paper (University of Chicago, Chicago, Ill.).

Friedman, M., 1969, The optimum quantity of money and other essays (Aldine, Chicago, Ill.).

Grossman, S., 1972, A Bayesian approach to static, stochastic equilibria, Department of Economics mimeo (University of Chicago, Chicago, Ill.).

Lucas, R.E., Jr. and E. Prescott, 1972, Investment under uncertainty, Econometrica 39, 659-682.

- Muth, J., 1961, Rational expectations and the theory of price movements, Econometrica 29, 315-335.
- Radner, R., 1970, Equilibrium of plans, prices and price expectations in a sequence of markets, Center for Research in Management Science Technical Report no. 5 (University of California, Berkeley); and: 1972, Econometrica, 289–384.

- Roll, R., 1971, Rational response to the money supply, GSIA Working Paper no. 88-71-2 (Carnegie Mellon University, Pittsburgh, Penn.), forthcoming in Journal of Political Economy.
- Samuelson, P.A., 1958, An exact consumption-lean model of interest with or without the social contrivance of money, Journal of Political Economy 66, 467-482.
- Samuelson, P.A., 1947, Foundations of economic analysis (Harvard University Press, Cambridge, Mass.).
- Telser, L.G., 1973, Storage with flexible prices under uncertainty, mimeo. (University of Chicago, Chicago, Ill.).