



# Interpreting a stochastic monetary growth model as a modified social planner's problem

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## Abstract

Using a stochastic cash-in-advance growth model similar to that studied by Cooley and Hansen (1989), it is demonstrated that the competitive, monetary equilibrium can be expressed as the solution to a modified social planner's problem. Hence, by employing standard dynamic optimization techniques and the second welfare theorem to the modified social planner's problem, the equilibrium of the monetary economy can be studied.

*Key words:* Cash-in-advance; Monetary equilibrium

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## 1. Introduction

In the decade since Kydland and Prescott's (1982) seminal work on business cycles, the use of stochastic, dynamic economies to study a variety of economic phenomena has become commonplace. This has naturally led to settings in which, because of tax or monetary distortions, the implied competitive equilibrium is not equivalent to the solution of a social planner's problem. Since the second welfare theorem cannot be applied in this case, a competitive equilibrium must be solved for directly. Recently, several techniques have been developed to address this problem, e.g., Bizer and Judd (1988), Baxter (1988), and Coleman (1988).

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However, in some situations it is possible to interpret the competitive equilibrium in an economy with distortions as the solution to a modified social planner's problem, thus permitting the indirect application of standard dynamic optimization methods. For instance, Becker (1985) demonstrated this in studying the effects of distortionary taxes in the context of a growth model. In that economy, the presence of taxes could be incorporated into a modified social planner's problem in which agents' discount rate included the tax rate. In a similar vein, I demonstrate below that it is possible to solve a particular cash-in-advance growth model using this indirect approach.

Specifically, I solve for the equilibrium in a cash-in-advance stochastic growth economy analogous to one previously studied by Cooley and Hansen (1989). Because of the distortions introduced by inflation, they directly solved for a competitive equilibrium employing a solution algorithm due to Kydland (1989). However, I show that an indirect approach is possible – the restrictions on preferences and technology employed by Cooley and Hansen permit the competitive equilibrium to be interpreted as the solution to a modified social planner's problem in which agents' utility of consumption experiences a stochastic taste shock.

Clearly, the indirect approach which I use is not as general as direct solution methods; however, the loss in generality is compensated by the increased intuition garnered by seeking an indirect solution. That is, in attempting to cast an economy with distortions as a social planner's problem, the critical influence of the distortions on agents' decision making is underscored. In contrast, direct methods permit the application of a computer algorithm to solve for equilibrium that does not impose, as a prerequisite, much reflection on the nature of the problem by the researcher. As a consequence, the interpretation of equilibrium behavior can be made more difficult.

The model is described in the following section, and the equivalence between equilibrium and a social planner's problem is established in Section 3. In Section 4, some comments are offered.

## 2. A monetary growth economy

The economy is assumed to be populated by identical agents that make consumption, labor, and saving decisions in order to maximize:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \right\}, \quad (1)$$

where  $E$  denotes the expectations operator,  $c_t$  denotes consumption,  $h_t$  denotes labor, and  $0 < \beta < 1$  is agents' subjective discount rate. In each period, agents

combine current labor with beginning-of-period capital,  $k_t$ , in order to produce output,  $y_t$ , through a constant returns to scale production function denoted  $f(\cdot)$ .<sup>1</sup> That is, technology is described by

$$y_t = z_t f(k_t, h_t). \quad (2)$$

In Eq. (2),  $z_t$  denotes a stochastic shock to technology. It is assumed that capital depreciates at the constant rate of  $\delta$ .

In addition to random technology shocks, the aggregate money stock,  $\bar{M}_t$ , is assumed to grow stochastically at the rate  $\mu_t$ . Specifically, the behavior of the money stock is given by

$$\bar{M}_{t+1} = (1 + \mu_{t+1}) \bar{M}_t. \quad (3)$$

New money is distributed as a lump-sum transfer at the beginning of each period.

The motion of the vector,  $(z_t, \mu_t) = \theta_t$ , is described by a stationary Markov process with transition function  $F(\theta', \theta) = \Pr(z_{t+1} \leq z', \mu_{t+1} \leq \mu' \mid z_t = z \text{ and } \mu_t = \mu)$ . It is assumed that agents know  $F(\theta', \theta)$ .

In order to introduce a demand for money, a cash-in-advance constraint is imposed on the purchase of consumption goods.<sup>2</sup> Specifically, the following trading pattern occurs: At the beginning of each period, households, consisting of a shopper and a worker, observe the current realization of the technology shock and the monetary growth rate; this information is used to formulate consumption and investment plans. The shopper then uses the nominal balances acquired in the previous period together with the monetary transfer in order to purchase current consumption from another household. At the same time, the worker combines current labor and beginning-of-period capital to produce output. Part of this is purchased as consumption by a shopper from another household. Agents then use the cash raised by sales of current output, any money balances not spent on consumption, and any unsold output to acquire new money and increase the household's stock of capital. Note that this scenario implies that investment is not subject to the CIA constraint. Letting  $P_t$  denote

<sup>1</sup>In contrast to the economy studied by Cooley and Hansen, I do not specify a separate firm sector. This makes for expositional ease but in no way influences the equilibrium characteristics of the economy. To expand, suppose that households owned both inputs which are sold every period to the firm. The household's necessary conditions in this case would still be Eqs. (7) and (10) with Eq. (9) replaced by  $\lambda_t = \beta E_t \{ \lambda_{t+1} [r_{t+1} + (1 - \delta)] \}$  and Eq. (8) replaced by  $u_{2,t} = w_t \lambda_t$ , where  $r_t$  and  $w_t$  are the real prices of capital and labor, respectively. Since the firms' necessary conditions for profit maximization require the marginal products of inputs be equal to their respective prices, in equilibrium Eqs. (8) and (9) will apply.

<sup>2</sup>Note that since leisure is not subject to the CIA constraint, it is a 'credit good' as in the cash-credit good CIA model of Lucas and Stokey (1987).

The presence of inflation is seen in Eq. (10) which states that the marginal utility of real wealth at time  $t$  is equal to the expected increase in utility that a unit of real balances will provide next period. The role of inflation in distorting choices of labor and capital becomes more apparent by using Eq. (10) in Eqs. (8) and (9):

$$u_{2,t} = z_t f_{2,t} \beta E_t \left[ u_{1,t+1} \frac{P_t}{P_{t+1}} \right], \quad (11)$$

$$E_t \left[ u_{1,t+1} \frac{P_t}{P_{t+1}} \right] = \beta E_t \left( [z_{t+1} f_{1,t+1} + (1 - \delta)] E_{t+1} \left\{ u_{1,t+2} \frac{P_{t+1}}{P_{t+2}} \right\} \right). \quad (12)$$

Eq. (11) demonstrates that the choice of labor is influenced by the expected inflation rate since the nominal receipts from supplying labor in the current period cannot be used for consumption until the following period. With regard to capital, the left-hand side of Eq. (12) shows that the addition of current capital decreases expected utility next period because of the associated reduction of real balances. The right-hand side of Eq. (12) illustrates that the additional output generated next period by current investment cannot be used for consumption and, hence, augment utility, until period  $t + 2$  due again to the CIA constraint. An implication of Eq. (12) is that, even though investment is not subject to the CIA constraint and therefore a 'credit' good, investment choices will be influenced by changes in the conditional expectation of inflation.

A *stationary, competitive equilibrium* in this economy is one in which markets clear and agents' choices of consumption, capital and real money balances can be expressed as function of the state variables  $s_t = (k_t, z_t, \mu_t)$ . That is, equilibrium is defined by the functions

$$c_t = c(s_t), \quad (13)$$

$$k_{t+1} = k(s_t), \quad (14)$$

$$\bar{M}_t / P_t = m(s_t), \quad (15)$$

$$h_t = h(s_t). \quad (16)$$

To be a competitive equilibrium, these functions must satisfy Eqs. (11) and (12) (i.e., agents' are at an optimum) and the constraints Eqs. (4) and (5) (i.e., markets clear).

### 3. Equilibrium – Constructing a modified social planner's problem

As pointed out by Cooley and Hansen (1989), the fact that households' labor and capital decisions are influenced by the path of inflation implies that, in general, it is not possible to describe the competitive equilibrium as the solution to a social planner problem. However, I now demonstrate that under suitable restrictions on preferences and technology, there exists a modified social planner's problem which can be used to solve for the equilibrium in this monetary economy. The assumptions are identical to those used in Cooley and Hansen's previous analysis and common to many real business cycle models. These are:

$$(A.1) \quad u(c_t, 1 - h_t) = \ln c_t + \ln(1 - h_t),$$

$$(A.2) \quad f(k_t, h_t) = k_t^\alpha h_t^{1-\alpha},$$

(A.3)  $F(\theta', \theta)$  is such that the CIA constraint [Eq. (5)] always binds.

Note that this latter restriction implies that, in equilibrium, the inverse of the inflation rate can be expressed as

$$\frac{P_t}{P_{t+1}} = \frac{c(s_{t+1})}{c(s_t)} \left( \frac{1}{1 + \mu(s_{t+1})} \right). \quad (17)$$

Using Eq. (17) and the functional forms in (A.1) and (A.2) permits the agents' necessary conditions [Eqs. (11) and (12)] to be written as

$$\left( \frac{1}{1 - h(s_t)} \right) = (1 - \alpha) z_t k(s_{t-1})^\alpha h(s_t)^{-\alpha} \left( \frac{1}{c(s_t)} \right) \beta E_t \left[ \frac{1}{1 + \mu_{t+1}} \right], \quad (18)$$

$$E_t \left[ \frac{1}{1 + \mu_{t+1}} \right] \left( \frac{1}{c(s_t)} \right) = \beta E_t \left\{ [z_{t+1} \alpha k(s_t)^{\alpha-1} h(s_{t+1})^{1-\alpha} + (1 - \delta)] \right. \\ \left. \times \left( \frac{1}{c(s_{t+1})} \right) E_{t+1} \left[ \frac{1}{1 + \mu_{t+2}} \right] \right\}. \quad (19)$$

Since the monetary growth rate is exogenous, define the new random variable:

$$\psi_t \equiv \beta E_t \left[ \frac{1}{1 + \mu_{t+1}} \right]. \quad (20)$$

The presence of inflation is seen in Eq. (10) which states that the marginal utility of real wealth at time  $t$  is equal to the expected increase in utility that a unit of real balances will provide next period. The role of inflation in distorting choices of labor and capital becomes more apparent by using Eq. (10) in Eqs. (8) and (9):

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A *stationary, competitive equilibrium* in this economy is one in which markets clear and agents' choices of consumption, capital and real money balances can be expressed as function of the state variables  $s_t = (k_t, z_t, \mu_t)$ . That is, equilibrium is defined by the functions

$$c_t = c(s_t), \quad (13)$$

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$$\bar{M}_t/P_t = m(s_t), \quad (15)$$

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To be a competitive equilibrium, these functions must satisfy Eqs. (11) and (12) (i.e., agents' are at an optimum) and the constraints Eqs. (4) and (5) (i.e., markets clear).

Using this in the above expressions yields:

$$\left(\frac{1}{1-h(s_t)}\right) = (1-\alpha)z_t k(s_{t-1})^\alpha h(s_t)^{-\alpha} \psi_t \left(\frac{1}{c(s_t)}\right), \quad (21)$$

$$\psi_t \left(\frac{1}{c(s_t)}\right) = \beta E_t \left\{ [z_{t+1} \alpha k(s_t)^{\alpha-1} h(s_{t+1})^{1-\alpha} + (1-\delta)] \psi_{t+1} \left(\frac{1}{c(s_{t+1})}\right) \right\}. \quad (22)$$

In both expressions, the conditional expectation of the monetary growth rate enters as a multiplicative factor on agents' marginal utility of consumption. The critical implication is that the time path for consumption and capital implied by Eqs. (21) and (22) can be described as the solution to a social planner's problem with a technological disturbance,  $z_t$ , and a taste shock,  $\psi_t$ . Specifically, define the following social planner's problem:

$$(P1) \quad \max_{c_t, h_t, k_{t+1}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [\psi_t \ln c_t + \ln(1-h_t)] \right\}$$

$$\text{s.t. } c_t + k_{t+1} \leq z_t k_t^\alpha h_t^{1-\alpha} + (1-\delta)k_t.$$

Note that the transition function  $F(\theta', \theta)$  is still relevant for this problem. It is straightforward to establish that the necessary conditions associated with this problem are identical to Eqs. (21) and (22). Hence, since the restrictions on preferences and technology imply the solutions to both problems are unique, these solutions must be the same. Therefore, one could apply standard solution methods to the problem (P.1) (for instance, a linear-quadratic approximation method as described in Hansen and Sargent, 1988) in order to determine the path of equilibrium consumption and capital. Using the path of consumption in conjunction with the monetary growth rate in Eq. (17) would determine the behavior of inflation.<sup>4</sup>

## 5. Comments

A few comments on this equivalence result and the model are in order. First, note that the critical aspects of the restriction on preferences are separability and that the utility function for the cash good, i.e., that good which is subject to the

<sup>4</sup>As pointed out in Footnote 1, the equilibrium behavior of real wages and the capital rental rate would be given by the marginal product of these inputs evaluated at equilibrium quantities.

CIA constraint, be logarithmic. That is, any concave function representing the utility of leisure (including linear as implied by a Hansen, 1985, indivisible labor model), or in general, the utility from credit goods, could be used. The necessity of log utility of consumption of the cash good is due to the fact that the income and substitution effects of the inflation tax caused by consumption changes cancel each other thus eliminating any expected utility consequences from holding real balances. Hence, all that matters is the expected growth rate of the money stock.

With respect to the implied equilibrium characteristics of the model, note that the modified social planner's problem (P.1) clearly illustrates the potential inefficiency of a monetary equilibrium.<sup>5</sup> That is, if  $\psi_t$  is not equal to unity (monetary growth does not satisfy the Friedman, 1969, rule), then the resulting competitive equilibrium implies that the modified social planner problem places less weight on the utility of consumption vis-à-vis leisure than do agents. Consequently, the competitive equilibrium is not Pareto optimal.

The necessary conditions for equilibrium [Eqs. (21) and (22)] also facilitate the understanding of the cyclical behavior of the economy. For instance, Eq. (22) demonstrates that if  $\psi_t$  is unchanging (because either money growth is constant or i.i.d.), then investment decisions are not directly affected by the money supply, but only indirectly through the labor decision because of the presence of  $\psi_t$  in Eq. (21). It is for this reason that Cooley and Hansen (1989) discovered that, when the monetary growth rate was assumed to be constant, the cyclical properties of their model were the same as a nonmonetary real business cycle model. Specifically, in their analysis they assumed that utility was linear in leisure with this assumption being motivated by Hansen's (1985) indivisible labor model. This implies that the marginal utility of leisure is constant so that, from Eq. (21), the response of capital and labor to a realization of the technology shock is not affected by the value of  $\psi$ . In contrast, if the utility of leisure is logarithmic, then the left-hand side of Eq. (21) is a nonlinear function of labor. Consequently, as  $\psi$  changes (due to variations in the monetary growth rate), the response of capital and labor to a technology shock realization is altered. That is, since in this case the implicit function for labor defined by Eq. (21) is different than that for capital, the constants (notably  $\psi$ ) which enter into this function imply different elasticities for capital and labor with respect to a realization of  $z_t$ .

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<sup>5</sup> An anonymous referee pointed out that the social planner problem (P.1) had a nice interpretation in terms of the efficiency of equilibrium. Also, the same referee illustrated, through a numerical analysis, the importance of the assumption of divisible versus indivisible labor for Cooley and Hansen's (1989) results.



## References

- Baxter, M., 1988, Approximating suboptimal dynamic equilibria: A Euler equation approach, Reproduced (University of Rochester, Rochester, NY).
- Becker, R.A., 1985, Capital income and perfect foresight, *Journal of Public Economics* 26, 147–167.
- Bizer, D. and K. Judd, 1988, Capital accumulation, risk, and uncertain taxation, Reproduced (University of Chicago, Chicago, IL).
- Coleman, W., 1988, Money, interest, and capital in a cash-in-advance economy, Reproduced (Board of Governors of the Federal Reserve System, Washington, DC).
- Cooley, T.F. and G.D. Hansen, 1989, The inflation tax in a real business cycle model, *American Economic Review* 79, 748.
- Friedman, M., 1969, The optimum quantity of money, in: *The optimum quantity of money and other essays* (Aldine, Chicago, IL) 1–50.
- Hansen, G.D., 1985, Indivisible labor and the business cycle, *Journal of Monetary Economics* 16, 309–328.
- Hansen, G.D. and T.J. Sargent, 1988, Straight time and over time in equilibrium, *Journal of Monetary Economics* 21, 281–308.
- Kydland, F.E., 1989, The role of money in a business cycle model, Institute for Empirical Macroeconomics discussion paper 23 (Federal Reserve Bank of Minneapolis and University of Minnesota, Minneapolis, MN).
- Kydland, F.E. and E.C. Prescott, 1982, Time to build and aggregate fluctuations, *Econometrica* 91, 1345–1370.
- Lucas, R.E., Jr. and N.L. Stokey, 1987, Money and interest in a cash-in-advance economy, *Econometrica* 55, 491–514.