SECOND-DEGREE PRICE
DISCRIMINATION EXAMPLE

Let \( C \equiv (q, T) \) denote a generic contract, and let \( u(C, \theta) \) denote the utility from the contract \( C \) for a buyer of type \( \theta \), that is, her willingness to pay minus what she has to pay:

\[
u(C, \theta) \equiv v_\theta(q) - T, \quad \text{where } \theta = H, L.
\]

(1)

Finally, let \( C_L \equiv (q_L, T_L) \) and \( C_H \equiv (q_H, T_H) \) denote a pair of contracts.

1 Benchmark analysis: No asymmetric information

If there is no asymmetric information, \( P \) will be able to act as a perfectly discriminating monopolist because, in the adverse selection game, he is the “Stackelberg leader in contracts”: he makes a take-it-or-leave-it offer to the agent \( A \), hence he has all the bargaining power subject only to any information advantage \( A \) may have. So, in the absence of any information advantage, \( P \) offers the contract \( \hat{C}_L \) to the low-valuing buyer and the contract \( \hat{C}_H \) to the high-valuing buyer, where

\[
\begin{align*}
\hat{C}_L &\equiv (\hat{q}_L, \hat{T}_L) \equiv (1, v_L(1)) \\
\hat{C}_H &\equiv (\hat{q}_H, \hat{T}_H) \equiv (2, v_H(2)).
\end{align*}
\]

Call this the first-best outcome for the principal since he cannot hope for more profits than this. In the figure below, \( \hat{T}_L = A \) and \( \hat{T}_H = A + B + C \). Notice that, in the first-best outcome, the monopolist efficiently supplies each type, which is no surprise given that he can fully appropriate the surplus from any quantity he decides to sell.

2 Analysis with asymmetric information

Now \( P \)’s optimization problem becomes:

\[
\max_{\hat{C}_L, \hat{C}_H} E\pi \equiv \lambda T_L + (1 - \lambda) T_H \quad \text{subject to}
\]

\[
\begin{align*}
\mu(C_L, L) &\geq 0 & (IR_L) \\
\mu(C_H, H) &\geq 0 & (IR_H) \\
\mu(C_L, L) &\geq \mu(C_H, L) & (IC_L) \\
\mu(C_H, H) &\geq \mu(C_L, H) & (IC_H)
\end{align*}
\]

Even when there is no asymmetric information, \( P \)’s contracts must satisfy the individual rationality (IR) or participation constraints. But with asymmetric information, his contracts must also satisfy the incentive compatibility (IC)
con-straints. Thus the above two IC constraints are the new con-

straints that asymmetric information imposes. P needs the
agent A to self-select from the menu of contracts he offers be-
cause he cannot identify the type of the buyer by just looking
at her when she walks into his store.

Notice that the first-best contracts \( \hat{C}_L \) and \( \hat{C}_H \) satisfy the
IR constraints, indeed \( IR_L \) is binding for \( \hat{C}_L \) and \( IR_H \) is
binding for \( \hat{C}_H \) because the principal fully appropriates under
these contracts. But the first-best contracts are not incentive
compatible. In particular, the high-valuing buyer \( H \) strictly
prefers \( \hat{C}_L \) to \( \hat{C}_H \) since \( \epsilon > 0 \). Further, provided P sets \( \epsilon \) suffi-
ciently small, the above pair of contracts are a profitable devi-
ation for P: they do not affect his profit from selling to L, but
they increase P’s profit from selling to \( H \) by \( v_H(2) - v_H(q) - \epsilon \).

For example, as illustrated below, if \( T = v_L(q) \equiv A \) (the
highest tariff P could charge for \( q \) and still satisfy \( IR_L \)), then
\( T_H = A + B + C - \epsilon \). So his profit from selling to \( H \) increases by
\( B + C - \epsilon \). Notice that this pair of contracts leaves the high-

valuing buyer with consumer’s surplus equal to the shaded
area in the figure. The principal must give H at least this
much consumer’s surplus to satisfy \( IC_H \), otherwise the H type
would pretend she’s a low type in order to obtain the contract
\( C_L \).

Case 2: \( q = 2 \). In this case, relative to pooling, the fol-

lowing pair of self-selection contracts represent a profitable


definition for P. Let

\[
C_L = \hat{C} \quad \text{and} \quad C_H = (2, T + v_H(2) - v_H(q) - \epsilon),
\]

where \( \epsilon > 0 \). It is easy to check that this pair of contracts sat-
sifies the IR and IC constraints; indeed the high type strictly
prefers \( C_H \) to \( C_L \) since \( \epsilon > 0 \). Further, provided P sets \( \epsilon \) suffi-
ciently small, the above pair of contracts are a profitable devi-
ation for P: they do not affect his profit from selling to L, but
they increase P’s profit from selling to \( H \) by \( v_H(2) - v_H(q) - \epsilon \).

Step 1: No pooling equilibrium

We first show that the equilibrium will not involve pooling, that is, P will not offer the same contract \( \hat{C} = (q, T) \) to both
types. There are 2 cases to consider:

Case 1: \( q < 2 \). In this case, relative to pooling, the follow-
ing pair of self-selection contracts represent a profitable devi-


deviation for $P$. Let
\[ C_L = (1, T) \quad \text{and} \quad C_H = (2, T + v_H(2) - v_H(1) - \epsilon), \]
where $\epsilon > 0$. Again it is easy to check that this pair of contracts satisfies the IR and IC constraints; indeed the high type strictly prefers $C_H$ to $C_L$ since $\epsilon > 0$. Further, provided $P$ sets $\epsilon$ sufficiently small, the above pair of contracts are a profitable deviation for $P$: they do not change his profit from selling to $L$, but they increase P’s profit from selling to $H$ by $v_H(2) - v_H(1) - \epsilon$.

**Step 2: $IR_L$ and $IC_H$ are binding**

So far we know the equilibrium contract will be separating, that is, P will offer a (non-degenerate) pair of screening contracts to $A$. What will they be?

Observe that either $IR_L$ or $IC_L$ must bind. Otherwise $P$ would have a profitable deviation: increase $T_L$ by an epsilon, $\epsilon$. This deviation will increase his profit from the low type and still satisfy all constraints provided he chooses $\epsilon$ sufficiently small. Similarly, either $IR_H$ or $IC_H$ must bind. We first show that to solve his optimization problem, $P$ will choose his contracts so that $IR_L$ and $IC_H$ will be binding.

To show $IC_H$ will bind, suppose the contrary, that in the solution the $P$’s problem $IC_H$ is not binding, that is:
\[ v_H(C_H) - T_H > v_H(C_L) - T_L. \]
We will show a contradiction. From the above we know that $IR_H$ will be binding, that is,
\[ v_H(C_H) - T_H = 0. \]
Substituting into the inequality we conclude
\[ v_H(C_L) - T_L < 0. \]
But $v_L(C_L) < v_H(C_L)$, hence $v_L(C_L) - T_L < 0$, contradicting $IR_L$.

Now to show $IR_L$ will bind, suppose the contrary that it is not binding. Then we know $IC_L$ will bind, that is
\[ v_L(q_L) - T_L = v_L(q_H) - T_H, \]
or in other words
\[ v_L(q_H) - v_L(q_L) = T_H - T_L. \]
We will show a contradiction. Since $IC_H$ is binding, $T_H - T_L = v_H(q_H) - v_H(q_L)$, hence
\[ v_L(q_H) - v_L(q_L) = v_H(q_H) - v_H(q_L). \]
But since $H$’s demand curve lies strictly above $L$’s demand curve, this is possible only if $q_H = q_L$, that is if there is pooling. But we have already seen that the solution to $P$’s problem does not involve pooling, so we have arrived at a contradiction. To illustrate, in the figure below, $v_L(q_H) - v_L(q_L) = A$ while $v_H(q_H) - v_H(q_L) = A + B$.

**Step 3: The high-valuing buyer is efficiently supplied, but the low-valuing buyer is under-supplied**

We can now conclude our analysis. Since $IR_L$ is binding and $IC_H$ is binding:
\[ T_L = v_L(q_L) \quad \text{and} \quad T_H(q_H) = v_L(q_L) + v_H(q_H) - v_H(q_L). \]
Hence P’s optimization problem becomes:

$$\max_{q_H,q_L} \lambda v_L(q_L) + (1 - \lambda)\left[v_L(q_L) + v_H(q_H) - v_H(q_L)\right].$$

The first-order conditions for an interior solution are:

$$v'_H(q_H) = 0$$

$$\lambda v'_L(q_L) + (1 - \lambda)\left[v'_L(q_L) - v'_H(q_L)\right] = 0.$$

The first condition implies $q_H^* = 2$, so the high valuing buyer is efficiently supplied. But the second condition implies

$$v'_L(q_L) = (1 - \lambda)v'_H(q_L) > 0,$$

so the low-valuing buyer is under-supplied. In particular, using our assumption that $\lambda = 2/3$, the second condition implies $q_L^* = 1/2$.

Putting it together, the solution to P’s problem is:

$$C_L^* \equiv (q_L^*, T_L^*) = (1/2, 3/8) \quad \text{and} \quad C_H^* \equiv (q_H^*, T_H^*) = (2, 3/2).$$

This solution is illustrated below, with $T_L^* = A$ in the figure and $T_H^* = A + B + C$. Notice the consumer’s surplus of the high type equals the shaded area. As already mentioned, the principal must give H this much consumer’s surplus to keep him from envying $C_L^*$. In this equilibrium the monopolist is offering neither a quantity discount nor a quantity premium since the price per unit that L is paying (namely 3/4) is the same as the price per unit that H is paying. But this is not a general result. Depending on the height of H’s demand curve relative to L’s, H may end up paying less per unit than L (a quantity discount) or more per unit (a quantity premium).¹

¹Note: In the current example, even though both types end up paying the same amount per unit, the seller is nevertheless earning more money acting as a second-degree price discriminator than he would earn as a simple nondiscriminating monopolist who sets his price at 3/4.
The intuition for the under-supply result is shown graphically below. Starting from the efficient first-best contracts, notice that if P sells a bit less than 1 unit to L, his revenue from selling to L decreases by the area A — which is very tiny if \( q_L \) is very close to 1 — but gains the area B from selling to H, which is not tiny. Putting it together, his expected profit would increase by \((1 - \lambda)B - \lambda A\). The idea is that by decrease \( q_L \), P relaxes the incentive compatibility constraint \( IC_H \), allowing him to charge H a higher tariff \( T_H \) for \( q_H = 2 \) units. Relative to \( \hat{C}_L \), the contract \( C_L \) with \( q_L < 1 \) is less attractive to H, that’s why \( IC_H \) is relaxed. The first order condition for \( q_L \) tells P to keep reducing \( q_L \) until the expected loss from selling a bit less to L equals the expected gain from increasing the tariff to H.

To conclude our analysis, let’s check for a corner solution. If P only serves one market, he will clearly only sell to H, offering \( C_L = (0,0) \) and \( C_H = \hat{C}_H \). The advantage of only serving H is that the principal does not have to leave H with any consumer’s surplus: by offering \( C_L = (0,0) \), he relaxes the incentive compatibility constraint \( IC_H \). It is easy to check that with \( \lambda = 2/3 \), as we have assumed, there will not be a corner solution: P’s expected profit is greater if he serves both types. But if the probability of a low type, \( \lambda \), were sufficiently small (less than 1/2), there would be a corner solution; the principal would write off buyer L and prefer to only sell to H.