1. The hold-up problem We begin with a simple but educational dynamic game. In the 1920's, General Motors purchased its auto bodies from an independent firm, Fisher Body. GM decided to build a new plant and, in order to improve the reliability of supply and also to decrease shipping costs, GM asked Fisher Body to build a new auto body plant adjacent to GM's new assembly plant. Fisher refused. Let's see why.

Suppose the new investment cost $2 (in millions) and would create potential gains from trade of $4 between GM and Fisher (not including the investment cost). Notice that once the investment has been made, it is a sunk cost, so the $4 in GFT after the investment has been made is up for grabs. GM, player 2, can be a Tough Bargainer (TB) or a Fair Bargainer (FB), as illustrated below.

```
    Invest
     0, 0
   /     \
  /       \
 0, 0

TB -2, 4
```

a. Find the SPNE by backward induction. Use your solution to explain the "hold-up problem" that Fisher Body faced.

b. The possibility of an underdevelopment trap One solution to the hold-up problem is vertical integration (e.g., eventually GM purchased Fisher Body). But sometimes this is impossible. Suppose now that player 1 is a rich country considering making direct foreign investment in an underdeveloped country (player 2). Use the hold-up problem to explain why the poor country may remain underdeveloped.

2. Generalized backward induction Consider the following 2-stage game with 2 players. In the first stage they play the Prisoners' Dilemma game (PD). Then, in the second stage they play the matrix game A if the outcome in the first stage is (F,F), while they play the matrix game B if the outcome in the first stage is anything else (i.e., (F, M), (M, F), or (M, M)). Find all the SPNE of this 2-stage game using generalized backward induction. Assume each player's discount factor $\delta$ equals 1.

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3. Suppose Game B above is the strategic form of a static game in which player 1 moves first. Draw the game tree for Game B including player 2's information set.

4. The Tragedy of the Commons This is an example of the "tragedy of the commons." The "tragedy" is that common property tends to be over-utilized. For example, common fishing grounds are over-fished, common grazing lands are over-grazed, common roads are over-congested. The reason is that any one user imposes a negative real externality on the other users, which he does not take into account.

$I$ fishermen fish on the same lake. Fisherman $i$ fishes $\ell_i$ hours. The total number of fish caught depends on $L = \sum_i \ell_i$ and is
given by \[ Q(L) = 1000L - L^2. \]

Consequently, the **average product** per hour (i.e., fish caught per hour) if fishermen spend \( L \) hours fishing is

\[ AP(L) \equiv \frac{Q(L)}{L} = 1000 - L. \]

The opportunity cost of each hour of fishing is \( c = 10 \) (using fish as numeraire). The lake is common property, so a fisherman who fishes for \( \ell_i \) hours will catch \( \ell_iAP(L) \) fish. Notice there is “congestion” on the lake in the sense that any additional fisherman decreases the total catch of all others on the lake, imposing a negative externality on them. In particular, if \( (\ell_1, \ldots, \ell_T) \) is the profile of hours spent fishing by people, any one individual \( i \) will get a payoff of:

\[ \pi_i(\ell_1, \ldots, \ell_T) = \ell_iAP(\ell_i + L_{-i}) - c\ell_i, \quad (1) \]

where \( L_{-i} \equiv \sum_{j \neq i} \ell_j \). Each fisherman \( i \) picks \( \ell_i \) to maximize \( \pi_i \) taking \( L_{-i} \) as given.

**a.** Find the hours \( \ell^* \) each individual fishes in the symmetric NE where each individual fishes the same amount. Notice that \( \ell^* \) will be a function of \( I \), i.e., \( \ell^* = \ell^*(I) \).

**b.** Show that the outcome is not efficient if \( I > 1 \). That is, it does not maximize the gains from trade, where the gains from trade from \( L \) hours spent fishing is given by

\[ G(L) \equiv Q(L) - cL. \]

Illustrate by calculating and graphing any one fisherman’s MPB, MSB, and M-EXT functions, given others fish \( L_{-i} \) hours.

What happens to the size of the inefficiency as \( I \to \infty? \) Explain the surprising conclusion.

**c. Pigovian taxation as a solution** Find the Pigovian tax per hour fishing that will lead to efficiency. Show that your tax brings each fisherman’s MPB in line with MSB.

d. **Privatization as a solution** What if the lake were owned by someone. Would the lake be efficiently fished? Explain.

5. **Rubinstein’s bargaining model** This well-known perfect information game is similar to the ultimatum game, except the 2 players’ bargaining power is more symmetric. It involves players making alternating offers.

Two players bargain over their shares of a pie of size 1. At date 1, player 1 offers a share \( x_1 \in [0,1] \) to player 2; player 2 accepts or refuses. If he accepts then he obtains \( 1 - x_1 \) to player 1. If he refuses, he makes offer \( x_2 \in [0,1] \) at date 2. If this offer is accepted by player 1 then player 1 gets \( x_2 \) in period 2 and player 2 gets \( 1 - x_2 \). If player 1 refuses the game ends and no one gets anything. The players are impatient to “eat” the pie, so the payoffs are \( \delta^{-1}x_1 \) and \( \delta^{-1}(1 - x_1) \) if they agree at date 1 on a share \( x_1 \) for player 1, where \( \delta \) is the **discount factor**, a number in \([0,1]\). Since this is not a finite game, we cannot illustrate it by a proper game tree, but the following “game tree” does help us visualize the structure of the game: Compared to the ultimatum game, in

\[
\begin{array}{c}
t=1 \\
1 \quad 2 \\
\delta x_1, (1-x_1), x_1 \\
1 \quad 2 \\
N_o \quad N_o \\
0, 0 \\
\end{array}
\]

the above game there are potentially 2 rounds of offers—not just one.

**a.** Solve for the (unique) SPNE by backward induction. Write down each player’s equilibrium strategy (i.e., complete contingent plan). Compare the outcome to the ultimatum game, where player 1 has all the bargaining power. In the current game, is there a “first mover advantage” or a “second mover advantage” (i.e., if you could
be either the player who moves first or moves second, which would you choose)?

b. The game also has many other NE. For example, find a NE in which player 1 gets the whole pie. Find another NE in which player 2 gets the whole pie. Show that both of these NE involve incredible threats, i.e., they are not subgame perfect, assuming $\delta \in (0,1)$.

In the full-blown Rubinstein model, the two players can alternate making offers indefinitely until one player accepts. As in our truncated version, impatience is the driving force that leads to agreement. In particular, Ariel Rubinstein showed that in his (untruncated) alternating-offers game the unique SPNE is that the offering player always offers a share $\frac{\delta}{1+\delta}$ to the responding player; the latter accepts any share at least equal to $\frac{\delta}{1+\delta}$ and rejects any lower share. Thus, the game ends immediately with player 1 getting $\frac{1}{1+\delta}$ of the pie and player 2 getting $\frac{\delta}{1+\delta}$. Notice that, although there is a first mover advantage if $\delta < 1$, it is only slight if $\delta$ is close to one.

c. Check that Rubinstein’s strategy profile is indeed a SPNE for his game. That is, check that it is a NE at each subgame. HINT: While there are an infinite number of subgames, they all fall into one of 2 types: the subgame beginning (i) when player $i$ must make an offer, and (ii) when player $i$ must respond to some offer of $x \in [0,1]$. [Do not try to prove uniqueness. It requires a special argument that we will not bother with.]