PRE-CONTRACTUAL INVESTMENT WITHOUT THE FEAR OF HOLDUPS: THE PERFECT COMPETITION CONNECTION

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Contents

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1. Introduction

Holdup problems are undoubtedly, empirically very important. But it is worth pointing out that many pre-contractual investments are undertaken without any fear of holdups. For example, many college students get their education *before* entering the job market, many new houses are built *before* being put up for sale. In both of these examples, the future seller does not fear holdups because he trusts, when the time comes, there will be strong ex post competition for his product (skill or house). In this paper we characterize what "strong ex post competition" means, and we given conditions under which such competition leads to efficient pre-contractual investments.

We build on two ambitious papers by Harold Cole, George Mailath, and Andrew Postlewaite (2001a, 2001b), hereafter referred to as CMP1 and CMP2. CMP1 studies the efficiency of pre-contractual investments in a two-stage game: investments are undertaken in stage 1, while core bargaining and contracting only happen in stage 2. The stage-2 game is an assignment model in which buyers and sellers match given their pre-contractual investments. CMP1 show that in their investment game three types of inefficient equilibria can occur:

- under-investment equilibria,
- over-investment equilibria,
- mismatch equilibria, as first identified by Felli and Roberts (2000).

The CMP1 model focuses on small-group interactions: the model contains only a finite number of individuals. In CMP2, the authors extend their model to one with a nonatomic continuum of individuals, to see what difference perfect competition makes in promoting efficiency. They find that Felli-Roberts-type inefficient equilibria do not survive in the continuum, but the other two types of inefficiencies do survive. They present an interesting example that illustrates the point.

CMP2 view perfect competition as essentially automatic in the continuum. After all, how can a nonatomic individual affect equilibrium prices by his tiny investment decision? A sharply contrasting view of perfect competition is presented in a pair of papers by Neil Gretsky, Joseph Ostroy, and William Zame (1992 and 1999), henceforth GOZ1 and GOZ2. GOZ1 characterizes the core allocations in a static assignment model with a nonatomic continuum of agents. Then, in GOZ2, the authors characterize when core bargaining leads to a perfectly competitive outcome. In this sequel, they show that perfect competition is generic in their continuum assignment model, but far from automatic.

The observation that perfect competition is not automatic in large economies has an illustrious history, dating back to Edgeworth, the father of the core. He discovered (by way of his master-servant example) that even in economies with many buyers and sellers — indeed, arbitrarily many — any one buyer or seller may still possess significant monopoly power, that is, the power to affect equilibrium prices. To use his language, core bargaining may not be "determinate": even with many competing agents, there may remain significant bilateral monopoly surpluses to haggle over.

In the GOZ2 approach to perfect competition, each individual in the continuum is viewed as an infinitesimal — the limit of smaller and smaller groups with positive mass. From this perspective it is not difficult to understand that, even in the continuum, individuals may matter. For example, the GOZ2 approach to perfect competition easily identifies the continuum version of Edgeworth's master-servant example as imperfectly competitive — although such examples are exceptional in their static assignment model. By contrast, CMP2 view each nonatomic individual as a mere point on the real line. This viewpoint does not leave room for any exceptions: all nonatomic continuum models are viewed as perfectly competitive.

Here we analyze a 2-stage investment game similar to that analyzed in CMP2. But we follow the GOZ2 approach to perfect competition, to see what difference it makes. The model and analysis will be at the same level of generality as GOZ2; the supermodular model studied in CMP2 will be a special case. In contrast to CMP2, we find: (1) All three types of inefficient equilibria identified in CMP1 survive in the continuum—including Felli-Roberts-type mismatch equilibria. So there is no difference between the finite and continuum models in terms of the variety of possible outcomes that may occur, at least in the absence of perfect competition. But (2) if there is ex post perfect competition in the continuum, only underinvestment equilibria survive; by implication, CMP2's overinvestment example is not perfectly competitive in the sense of GOZ2. Recalling the GOZ2 genericity result, does this mean CMP2's overinvestment example is exceptional? In contrast to the GOZ2 static assignment model, we find (3) ex post perfect competition is not necessarily generic in the investment game; indeed ex post *imperfect* competition is generic in supermodular models with costly investment (which fits the CMP2 example), but ex post perfect competition is generic in a "housing model" related to the model studied in GOZ2.

The current paper was initially envisioned as a mere application of the results in GOZ2. But soon a rather important mistake in GOZ2's characterization of perfect competition was discovered. A substantial amount of the current paper involves correcting this mistake, and hence refining the GOZ2 characterization. The mistake is explained in Remark A.1 in the appendix; our suggested fix occupies much of Sections 3 and 5.

Interpreting the continuum. There is more or less a consensus now that continuum economies are to be interpreted as idealized representations of large but finite economies. It is also widely agreed that, for such an interpretation to be meaningful, the equilibrium concept being used should have some continuity properties with respect to the distribution of agents in the economy (e.g., Hildenbrand (1974), Dubey, Mas-Colell, and Shubik (1980), Champsaur and Laroque (1982), Mas-Colell and Vives (1989)). The GOZ2 approach to perfect competition follows this line. In particular, the reader will find that a perfectly competitive ex post population μ is defined as a distribution of individual characteristics with the property that the Walrasian equilibrium price correspondence is continuous at μ with respect to small perturbations in the distribution. Given this perspective, perfect competition is not automatic in the continuum because, as the master-servant example illustrates, the Walrasian price correspondence is not always continuous.

By contrast CMP2, to confirm their strong prior that all continuum economies should be perfectly competitive, introduce a special definition of feasible bargaining outcomes in the continuum (Definition 1 in CMP2) that is not even approximately representative of feasibility in some large but finite economies; hence a discontinuity at infinity is built into their analysis.

Two margins of analysis: The commodity margin and the individual margin. In finite economies, assuming commodities are divisible, each unit of any commodity is infinitesimal relative to the size of the individual, whence the bread-and-butter calculus at the commodity margin (e.g., the marginal conditions for optimal consumption by a price taker). By contrast, in economies with a nonatomic continuum of individuals, there are two infinitesimal margins, not just one. Just as each unit of a commodity is infinitesimal relative to the size of the individual, in the continuum each individual is infinitesimal relative to the size of the entire economy. Hence in the continuum there is the possibility of doing calculus over individuals that closely resembles the more familiar calculus over commodities. In particular, we will be interested in calculating the marginal contribution of each infinitesimal individual to society's gains from trade; this is literally a directional derivative on society's gains function. GOZ2 develops the basic calculus over individuals for the assignment model; Section 3 continues this development.

The full appropriation theme. In Section 5 the calculus over individuals, developed in Section 3, is applied to characterize perfect competition. GOZ2 show that perfect competition is intimately related to the differentiability of the gains function (the absence of kinks). In terms of the calculus over individuals, under perfect competition each individual's private benefit equals his full contribution to social gains. Section 5 strengthens this image; it shows that if there is strong ex post competition each individual fully appropriates his social contribution no matter what investment he undertakes, not just in equilibrium (Corollary 5.3 in Section 5).

In Section 6, the full appropriation property is used to address the efficiency of investment equilibria under strong ex post competition. The fact that perfect competition gives individuals good incentives because private benefit and social benefit are aligned (PB = SB) is a dominant theme in Makowski and Ostroy (1995). The advantage of the continuum is that, by using the calculus over individuals, PB = SB can be derived as a conclusion; by contrast, since the above paper worked with a finite number of individuals, the property was hypothesized. The above paper is generously cited in CMP2, indeed a limited differential version of full appropriation is proved in CMP2: if any individual changes his attribute by an infinitesimal amount, the derivative of the individual's PB will equal the derivative of SB (CMP2, Proposition 2). Notice this version emphasizes the commodity margin. Corollary 5.3 below will show that under perfect competition a much stronger version of full appropriation holds: an individual always fully appropriates, whether he changes his attribute infinitesimally or globally. In other words, full appropriation holds not just at the commodity margin but also at the individual margin.

Makowski and Ostroy (1995) offer a revision of the first welfare theorem that emphasizes two potential sources of market failure: failures of perfect competition and coordination failures. Both sources can operate in the investment model, even in the continuum: Example 4.5 below will illustrate that a fear of holdups (ex post imperfect competition) can lead to ex ante underinvestment; Example 6.1 will illustrate that coordination problems can lead to underinvestment even when there is strong ex post competition.

The basic model. Sections 2–4 below present our basic model, a two-stage investment and matching game as in CMP1 and CMP2. In stage one, individuals make pre-contractual investments. Then, in stage two, they bargain and match. The concept of an investment equilibrium ties the two stages together. In an investment equilibrium, each individual has rational conjectures about the outcome of his ex post core bargaining (contingent on his ex ante investment). Thus, although the second stage is modelled as a cooperative game (core bargaining), the game is solved by backward induction just like any purely-noncooperative two-stage game of perfect information: each individual uses his rational conjectures about stage 2 to guide his investment decision in stage 1. The hybrid noncooperative/cooperative game model we analyze has some resemblance to the biform game concept of Adam Brandenburger and Harborne Stuart (1994), except that in stage 1 we assume — as in CMP1 and CMP2 — that individuals have fixed (nonrandom) conjectures about the outcome of stage 2 core bargaining. By contrast, in a biform game each individual makes his stage-1 decision based on his confidence in the share of any bilateral monopoly surplus he will be able to obtain in stage 2. Since we are mainly concerned with core bargaining under perfect competition — where bargaining is *determinate*, so there are no residual bilateral monopoly surpluses to haggle over by persuasion, bluffing holding out, and so on — this difference is not significant for the current analysis.

The paper by Brandenburger and Stuart deserves a larger audience than it has yet received. They offer some powerful images regarding the comparative advantage of the core relative to noncooperative game theory as a way to model *free-form competition*, the idea being that noncooperative game analysis requires a rigid bargaining procedure to apply its analytical tools, but any such rigid game captures very little of the unstructured, fluid moves and countermoves that characterize real-world bargaining. By contrast, the core — since it abstracts away procedures to concentrate instead on the possibilities for agreement — is tailor-made to describe fluid free-form competition.

Here is a brief guide to the sequel. Sections 2 – 4 present the basic model. Edgeworth's master-servant example is presented in Section 4, along with examples of each of the three types of market failure identified in CMP1. This motivates our interest in the question

> *To what extent are these market failures due to an absence of perfect competition?*

Section 5 characterizes strong ex post competition, which prepares us to address the efficiency question in Section 6. Section 7 analyzes the genericity of perfect competition. Section 8 offers a number of asymptotic results which help confirm the meaningfulness of our analysis.

Section 9 compares the current model with one in which contracting precedes investment, essentially a model with complete contracts. Section 10 presents some special results that apply only to the supermodular assignment model. The paper concludes with a brief appendix discussing how our results can be applied to the revelation game studied in GOZ2.

2. The ex ante investment game

The following notation will be used throughout. Given any compact metric space X,

- $C(X)$ will denote the Banach space of all bounded, real-valued continuous functions on X, equipped with the supremum norm, i.e., $\|f\| = \sup_{x\in X} f(x)$, and
- $M(X)$ will denote the Banach space of all countably additive Borel measures on X, equipped with the total variation norm, i.e., $\|\mu\| = \sup \sum_i |\mu(E_i)|$ taken over all finite measurable partitions $\langle E_i \rangle$ of X.

Write $C_+(X)$ for the cone of nonnegative-valued functions and $M_+(X)$ for the cone of nonnegative-valued measures. Notice $\mu \in M_+(X) \Rightarrow \|\mu\| = \mu(X)$.

We analyze a two-stage investment and matching game. Let's begin with stage one.

The set of possible types is T, partitioned into a set of buyer types I and seller types J. We assume T is a compact metric space, with I and J disjoint closed subsets. The distribution of types in the economy is summarized by a measure $\mathcal{E} \in M_+(T)$. For any Borel subset $E \subset T$, $\mathcal{E}(E)$ equals the mass of individuals with types in E. Each individual must acquire (invest in) exactly one attribute, chosen from a set of possible attributes $\mathbf{A}$. We partition $\mathbf{A}$ into a set of buyer attributes $\mathbf{B}$ and seller attributes $\mathbf{S}$. $\mathbf{A}$ is a compact metric space with $\mathbf{B}$ and $\mathbf{S}$ disjoint closed subsets.

An individual's type determines his cost of acquiring an attribute. Let

$$c : \mathbf{A} \times T \longrightarrow \mathbb{R} \cup \{\infty\}.$$

Interpret $c(a,t)$ as the cost of acquiring attribute a for an individual of type t. In particular, $c(a,t) = \infty$ means that a is prohibitively costly for t. The *effective domain* of c is

$$\operatorname{dom} c = \{(a,t) \in \mathbf{A} \times T : c(a,t) < \infty\}.$$

We assume sellers find it prohibitively costly to acquire buyer attributes, and vice versa, hence

$$\operatorname{dom} c(\cdot, i) \subset B, \quad \forall i \in I \quad \text{and} \quad \operatorname{dom} c(\cdot, j) \subset \mathbf{S}, \quad \forall j \in J.$$

3. The ex post assignment game

Individuals' choices of attributes leads to an ex post distribution of attributes, a measure $\mu \in M_+(\mathbf{B} \cup \mathbf{S})$. For any Borel subset $E \subset \mathbf{B} \cup \mathbf{S}$, the number $\mu(E)$ equals the mass of individuals with attributes in E.

To allow for the possibility that not all individuals will be matched, it is convenient to add to the set of attributes a null attribute denoted $\mathbf{0}$, and to extend the metric on $\mathbf{A}$ to $\mathbf{A} \cup \{\mathbf{0}\}$ so that the null attribute is isolated (i.e., there is an open set containing $\mathbf{0}$ and no other attribute). Let $\mathbf{B}^0 = \mathbf{B} \cup \{\mathbf{0}\}$ and $\mathbf{S}^0 = \mathbf{S} \cup \{\mathbf{0}\}$. A feasible *assignment* for the ex post population μ is a measure $x \in M_+(\mathbf{B}^0 \times \mathbf{S}^0)$ satisfying $x(\mathbf{0}, \mathbf{0}) = 0$, $x(E, \mathbf{S}^0) = \mu(E)$ for all Borel subsets $E \subset \mathbf{B}$, and $x(\mathbf{B}^0, F) = \mu(F)$ for all Borel subsets $F \subset \mathbf{S}$. So null attributes are not assigned to one another; each buyer is either assigned to a seller or to the null attribute; and similarly each seller is either assigned to a buyer or to the null attribute. Interpret being assigned to the null attribute as not being matched. For example, $x(E, \mathbf{0})$ is the measure of buyers in E that are not matched with any seller.

Let $v \in \mathcal{C}_+(\mathbf{B}^0 \times \mathbf{S}^0)$ with $v(\mathbf{0}, \mathbf{0}) \equiv 0$. If $(b, s) \in \mathbf{B} \times \mathbf{S}$, interpret $v(b, s)$ as the value of a match between a buyer with attribute b and seller with attribute s. For any $s \in \mathbf{S}$, let $r(s) = v(\mathbf{0}, s)$. Interpret $r(s)$ as the outside option or reservation value of a seller s, the utility he obtains if not matched. For any $b \in \mathbf{B}$, define the outside option or reservation value of a buyer b similarly: $r(b) = v(b, \mathbf{0})$. It will be technically convenient to assume v is bounded above by some real number $\overline{V}$.

The (ex post) gains from the assignment x is $\int v \, \mathrm{d}x$. Hence the maximum attainable gains for the population μ is

$$g(\mu) = \sup \int v \, \mathrm{d}x \quad \text{s.t. } x \text{ is feasible for } \mu.$$

Any feasible x such that $g(\mu) = \int v \, \mathrm{d}x$ will be called an efficient assignment for μ. Viewed as a function of the ex post distribution of attributes, the gains function $g : M_+(\mathbf{A}) \to \mathbb{R}_+$ is homogeneous of degree 1 and concave. Hence for any μ there is a non-empty set of continuous functions $q \in C(\mathbf{A})$ satisfying the *subgradient inequality*

$$g(\mu + y) - g(\mu) \le qy, \quad \forall \mu + y \in M_+(\mathbf{A}),$$

where $qy \equiv \int q\,\mathrm{d}y$. Any such q is called a subgradient or support of g at μ; and the set of all such supports is called the subdifferential of g at μ, denoted $\partial g(\mu)$. Since g is homogeneous of degree 1, $q \in \partial g(\mu)$ is equivalent to $g(\mu) = q\mu$ and $g(\mu + y) \leq q\mu + qy$ for all $\mu + y \in M_+(\mathbf{A})$. [See GOZ1 or GOZ2.]

A price function is a $p \in C_+(\mathbf{S}^0)$ satisfying $p(\mathbf{0}) \equiv 0$. Given any price function p, define the indirect utility functions

$$\begin{aligned} v_b{}^*(p) &= \max_{s\in\mathbf{S}^0} v(b,s) - p(s), && \forall b \in \mathbf{B}, \\ v_s{}^*(p) &= \max\{r(s), p(s)\}, && \forall s \in \mathbf{S}. \end{aligned}$$

Also define the payoff functions:

$$\begin{aligned} \pi_b(b,s) &= v(b,s) - p(s), \\ \pi_s(b,s) &= \begin{cases} p(s) & \text{if } b \in \mathbf{B} \\ r(s) & \text{if } b = \mathbf{0}. \end{cases} \end{aligned}$$

An *(ex post) Walrasian equilibrium* for μ is a pair (x,p), where p is a price function and x is a feasible assignment for μ satisfying

(1) for each buyer $b \in \text{supp } \mu$ and $(b,s) \in \text{supp } x$: $\pi_b(b,s) = v_b{}^*(p)$;
(2) for each seller $s \in \text{supp } \mu$ and $(b,s) \in \text{supp } x$: $\pi_s(b,s) = v_s{}^*(p)$.

$P(\mu)$ will denote the set of all Walrasian equilibrium price functions for μ. An important property of the assignment model is *interchangeability*: if one pairs any price function $p \in P(\mu)$ with any efficient assignment x for μ, then (x,p) is Walrasian for μ (see GOZ2).

An imputation $q^* \in C(\text{supp } \mu)$ is in the (ex post) *core* of μ if $q^*\mu = g(\mu)$ and $q^*\mu' \geq g(\mu')$ for all coalitions $\mu' \in M_+(\mathbf{A})$ s.t. $\mu' \leq \mu$. In the assignment model, there is an intimate relation between the core of μ, the subdifferential of the gains function g at μ, and the set of Walrasian equilibria for μ. Remarkably, the three are essentially equivalent, as shown in GOZ1. For any $p \in P(\mu)$, let $v^*(p) \in C_+(\mathbf{A})$ satisfy

$$v^*(p)(a) = v_a{}^*(p), \quad \forall a \in \mathbf{A},$$

and let $v^*(p)\mid_A$ denote the restriction of $v^*(p)$ to the domain A.

Proposition 3.1 (GOZ1 characterization of the core)**.** *Let $A \equiv \text{supp } \mu$. The following statements are equivalent:*

(1) *q^* is in the core of μ;*
(2) *there exists a pair (x,p) that is Walrasian for μ with $q^* = v^*(p)\mid_A$.*
(3) *there exists a subgradient $q \in \partial g(\mu)$ with $q^* = q\mid_A$ and $p = q\mid_{\mathbf{S}}$.*

3.1. **Pricing untraded goods.** Given any ex post population $\mu \in M_+(\mathbf{A})$, let $A(\mu) = \mathbf{A} \cap \text{supp } \mu$, $B(\mu) = \mathbf{B} \cap \text{supp } \mu$, and $S(\mu) = \mathbf{S} \cap \text{supp } \mu$. Throughout the paper, whenever there is no chance of ambiguity about μ, we will simply write A, B, and S for these sets.

The set of goods that are *traded* in μ is

$$S^*(\mu) = \{s \in \text{supp } \mu : p(s) \geq r(s) \text{ for all } p \in P(\mu)\},$$

$S(\mu) - S^*(\mu)$ is the set of goods s that are available but *not traded* in μ, and $\mathbf{S} - S(\mu)$ is the set of goods *not available* in μ.

Even if the Walrasian prices of all traded goods are unique, there typically is a large range of possible prices for untraded and unavailable goods. In particular if $p \in P(\mu)$, then any $p' \in C_+(\mathbf{S}^0)$ such that $p'(s) = p(s)$ for all $s \in S^*(\mu)$, $p'(s) \geq p(s)$ for all $s \in \mathbf{S} - S(\mu)$, and

$$p'(s) \in \big[\sup_{b \in B(\mu)} v(b,s) - v_b{}^*(p),\ r(s)\big] \text{ for all } s \in S(\mu) - S^*(\mu)$$

is also in $P(\mu)$, where

$$\sup_{b \in B(\mu)} v(b,s) - v_b{}^*(p)$$

measures *buyers' reservation price* for s given p and μ, i.e., the most that the buyers in the support of μ would be willing to pay for s.

It will be convenient to introduce a set of normalize prices. Given any $p \in P(\mu)$, for each $s \in \mathbf{S}$ let

$$\hat{p}(p,\mu)(s) = \max\left\{r(s), \sup_{b \in B(\mu)} v(b,s), -v_b{}^*(p)\right\}.$$

By construction, $\hat{p}(p,\mu) \in P(\mu)$. This normalization of prices does not change the prices of traded goods, and it sets the price of each other good s at the maximum of the seller's reservation price $r(s)$ and buyers' reservation price. The set of normalized Walrasian prices for μ will be denoted

$$\hat{P}(\mu) \equiv \{\hat{p}(p,\mu) : p \in P(\mu)\}.$$

Relatedly, there also is a large range of possible subgradients: If $q \in \partial g(\mu)$, then any $q' \in C(\mathbf{A})$ satisfying $q' \geq q$ and $q'(a) = q(a)$ for all $a \in A(\mu)$ also is in $\partial g(\mu)$. We will adopt the "shrink fit" method in GOZ1 to introduce a normalized set of subgradients. Starting from any bounded measurable (not necessarily continuous) function q satisfying the subgradient inequality for g at μ, for each $s \in \mathbf{S}$ let

$$\hat{q}(s) = \max\left\{r(s), \sup_{b \in B(\mu)} v(b,s) - q(b)\right\}$$

and for each $b \in \mathbf{B}$ let

$$\hat{q}(b) = \max\left\{r(b), \sup_{s\in\mathbf{S}} v(b,s) - \hat{q}(s)\right\}.$$

By construction $\hat{q} \le q$ on $\mathbf{S}$ and $\hat{q} \ge q$ on $\mathbf{B}$, with equality on $A(\mu)$. GOZ1 show $\hat{q} \in \partial g(\mu)$. Since $\hat{q}$ depends on q and μ, we will denote it by $\hat{q}(q,\mu)$. The set of normalized subgradients will be denoted

$$\hat{\partial} g(\mu) \equiv \{\hat{q}(q,\mu) : q \in \partial g(\mu)\}.$$

There is an interesting dual relationship between the normalized prices and normalized subgradients.

Proposition 3.2 (dual normalizations)**.** *The following dual relationships hold:*

$$\hat{g}(\mu) = \{q = v^*(p) : p \in \hat{P}(\mu)\}$$

and

$$\hat{P}(\mu) = \{p = q \mid_{\mathbf{S}} : q \in \hat{g}(\mu)\}.$$

Proof. Given any $q \in \hat{g}(\mu)$, notice $\hat{q}(q,\mu) = q$. Hence for each $s \in \mathbf{S}$,

$$q(s) = \max\{r(s), \sup_{b\in B(\mu)} v(b,s) - q(b)\}.$$

Let $p = q \mid_{\mathbf{S}}$. Core equivalence along with the subgradient inequality imply $p \in P(\mu)$, and the above implies $p \in \hat{P}(\mu)$. Further, by construction of $\hat{q}(q,\mu)$, ${v_b}^*(p) = q(b)$ for all $b \in \mathbf{B}$ and ${v_s}^*(p) = q(s)$ for all $s \in \mathbf{S}$. Hence $q = v^*(p)$.

Conversely, given any $p \in \hat{P}(\mu)$, for each $s \in \mathbf{S}$,

$$p(s) = \max\{r(s), \sup_{b\in B(\mu)} v(b,s) - {v_b}^*(p)\}.$$

Let $q(a) = {v_a}^*(p)$ for all $a \in A(\mu)$ and $q(a) = \overline{V}$ for all $a \notin A(\mu)$. Core equivalence along with the large imputations to attributes outside $A(\mu)$ imply q satisfies the subgradient inequality. Hence we can use the shrink fit method on q to construct a $\hat{q} \in \partial\hat{g}(\mu)$. By construction $p = \hat{q} \mid_{\mathbf{S}}$. □

Let

$$\mathcal{P} = \bigcup_{\mu\in M_+(\mathbf{A})} \hat{P}(\mu) \quad \text{and} \quad \mathcal{Q} = \bigcup_{\mu\in M_+(\mathbf{A})} \hat{\partial} g(\mu).$$

Proposition 3.3. *$\mathcal{P}$ and $\mathcal{Q}$ are (norm) compact sets.*

Proof. The compactness of $\mathcal{P}$ is proved in Proposition 4 of GOZ2. Although we have adopted a slightly different normalization, the proof is basically the same. The compactness of $\mathcal{Q}$ is essentially Theorem 7 in GOZ1. We have just changed the order of "shrink fitting," letting sellers go before buyers. □

Note: Reversing the order of shrink fitting, letting buyers go first, yields another subgradient in $\partial g(\mu)$. More precisely, starting from any q as before, for each $b \in \mathbf{B}$ let

$$\check{q}(b) = \max\left\{ r(b), \sup_{s \in S(\mu)} v(b,s) - q(s) \right\}$$

and for each $s \in \mathbf{S}$ let

$$\check{q}(s) = \max\left\{ r(s), \sup_{b \in \mathbf{B}} v(b,s) - \check{q}(b) \right\}.$$

Unlike $\hat{q}$, the vector $\check{q}$ is smaller than q for buyers rather than sellers: $\check{q} \le q$ on $\mathbf{B}$ and $\check{q} \ge q$ on $\mathbf{S}$, with equality on $A(\mu)$. Since $\check{q}$ depends on q and μ, we will denote it by $\check{q}(q,\mu)$, and we will let $\check{\partial} g(\mu) = \{\check{q}(q,\mu) : q \in \partial g(\mu)\}$. As the reader may suspect, there must be an alternative normalization of prices that is dual to this alternative normalization of subgradients. For any $p \in P(\mu)$, let $\check{p}(p,\mu)(s) = \hat{p}(p,\mu)(s)$ for all $s \in S(\mu)$ and let

$$\check{p}(p,\mu)(s) = \max\{r(s), \sup_{b \in \mathbf{B}} v(b,s) - v_b{}^*(p)\}, \quad \forall s \in \mathbf{S} - S(\mu).$$

Let $\check{P}(\mu) = \{\check{p}(p,\mu) : p \in P(\mu)\}$. Analogous to the above duality result, we have:

Proposition 3.4 (alternative dual normalizations)**.** *The following dual relationships hold:*

$$\check{\partial} g(\mu) = \{v^*(p) : p \in \check{P}(\mu)\}$$

and

$$\check{P}(\mu) = \{q \,|_{\mathbf{S}} : q \in \check{\partial} g(\mu)\}.$$

Furthermore, $\check{\mathcal{Q}} \equiv \{q : q \in \check{\partial} g(\mu)$ *and* $\mu \in M_+(\mathbf{A})\}$ *and* $\check{\mathcal{P}} \equiv \{p : p \in \check{P}(\mu)$ *and* $\mu \in M_+(\mathbf{A})\}$ *are norm compact.*

Proof. The proof is analogous to that of Proposition 3.2 and Proposition 3.3. □

3.2. **The differentiability of the gains function.** There is an intimate connection between perfect competition in the population μ and the differentiability of the gains function g at μ.

The set of feasible directions starting from μ is

$$D = \{y \in M(\mathbf{A}) : \mu + ty \ge 0 \text{ for some } t > 0\}.$$

The directional derivative of g at μ in the direction y is

$$g'(\mu; y) \equiv \lim_{t \downarrow 0} \frac{g(\mu + y) - g(\mu)}{t}.$$

These directional derivatives exist, and they are connected to the subdifferential $\partial g(\mu)$ by the following inf formula.

Theorem 3.1 (inf formula for calculating directional derivatives)**.** *For any $y \in D$,*

$$g'(\mu; y) = \inf\{qy : q \in \partial g(\mu)\},$$

and the infimum is realized at some $q \in \partial g(\mu)$. Moreover, $g'(\mu; y)$ is a homogeneous of degree 1 and superadditive function of y.

Proof. Fix a $y \in D$. Let $q_n \in \hat{\partial} g(\mu + \frac{1}{n}y)$. Since $\mathcal{Q}$ is norm compact, the sequence $\langle q_n \rangle$ will have a subsequence converging to some $\tilde{q} \in \mathcal{Q}$. It is straightforward to verify that $\tilde{q} \in \partial g(\mu)$ (see the proof of Proposition 3 in GOZ2). The subgradient inequality implies for any $q \in \partial g(\mu)$ and any n,

$$qy \geq \frac{g(\mu + ty) - g(\mu)}{t} \geq q_n y,$$

hence we can conclude that $\tilde{q} = g'(\mu, y) = \operatorname{argmin}_{q \in \partial g(\mu)} qy$. The function $g'(\mu; y)$ is homogeneous of degree 1 in y since $\inf q(ty) = t \inf qy$; and it is superadditive since $\inf q(y + y') \geq \inf qy + \inf qy'$. □

For any ex post population $\mu \in M_+(\mathbf{A})$, the set of normalized Walrasian prices $\hat{P}(\mu)$ is a lattice, hence it contains a smallest and largest price vector (see GOZ2). Similarly, $\check{P}(\mu)$ is a lattice with a smallest and largest price vector. These facts permit a more specific characterization of the directional derivatives in some directions. In particular, define the set of directions

$$D(B-) = \{y \in D : y \leq 0 \text{ and } \operatorname{supp} y \subset B\}$$

and

$$D(\mathbf{B}+) = \{y \in D : y \geq 0 \text{ and } \operatorname{supp} y \subset \mathbf{B}\}.$$

Define $D(S-)$ and $D(\mathbf{S}+)$ analogously. Let $\underline{p}$ be the smallest price function in $\hat{P}(\mu)$, and let $\overline{p}$ be the largest price function in $\check{P}(\mu)$. Let $\underline{q} = v^*(\underline{p})$ and $\overline{q} = v^*(\overline{p})$.[1]

Corollary 3.1. *For any $y \in D(B-) \cup D(\mathbf{S}+)$,*

$$g'(\mu; y) = \underline{q}y,$$

[1] Because the order of shrink fitting is reversed, the largest price function in $\check{P}(\mu)$ is even larger than the largest price function in $\hat{P}(\mu)$.

and for any $y \in D(\mathbf{B}+) \cup D(S-)$*,*

$$g'(\mu; y) = \overline{q}y.$$

Proof. Since $\underline{p} \in \hat{P}(\mu)$ and $\underline{q} = v^*(\underline{p})$, duality implies $\underline{q} \in \hat{g}(\mu)$. By construction, for any $q \in \partial g(\mu)$, $\hat{q}(q,\mu) \le q$ on $\mathbf{S}$ and $\hat{q}(q,\mu) \ge q$ on $\mathbf{B}$, with equality on $A(\mu)$. So there is no $q \in \partial g(\mu)$ that is smaller than $\underline{q}$ on $\mathbf{S}$ or larger than $\underline{q}$ on $B(\mu)$. Hence the inf formula implies $g'(\mu; y) = \underline{q}y$ for all $y \in D(B-) \cup D(\mathbf{S}+)$.

Similarly, since $\overline{p} \in \check{P}(\mu)$ and $\overline{q} = v^*(\overline{p})$, duality implies $\overline{q} \in \check{\partial} g(\mu)$. By construction, for any $q \in \partial g(\mu)$, $\check{q}(q,\mu) \le q$ on $\mathbf{B}$ and $\check{q}(q,\mu) \ge q$ on $\mathbf{S}$, with equality on $A(\mu)$. So there is no $q \in \partial g(\mu)$ that is smaller than $\underline{q}$ on $\mathbf{B}$ or larger than $\overline{q}$ on $S(\mu)$. Hence the inf formula implies $g'(\mu; y) = \underline{q}y$ for all $y \in D(\mathbf{B}+) \cup D(S-)$. □

A more demanding notion of differentiability is Frechet differentiability. Let $D' \subset D$. The gains function g is *(Frechet) differentiable* at μ in the directions D' if there exists a $q \in \partial g(\mu)$ with the property that, for all $y \in D'$

$$g(\mu + y) - g(\mu) = qy + o(\|y\|),$$

or in other words

$$\lim_{\|y\| \to 0} \frac{g(\mu + y) - g(\mu) - qy}{\|y\|} = 0.$$

Such a q, if it exists, will be called a *derivative* of g at μ in the directions D'.

Note: In the special case when g is differentiable at μ in all directions D, our generalization of differentiability reduces to the usual definition (subject to respecting the restriction to nonnegative measures). As we shall see, typically g will not be differentiable in this strong sense, which motivates the above generalized notion of differentiability.

Theorem 3.2 (one-sided differentiability even without perfect competition)**.** *The gains function g is (Frechet) differentiable at μ in the directions $D(\mathbf{S}+) \cup D(B-)$ with derivative $\underline{q}$, and also in the directions $D(S-) \cup D(\mathbf{B}+)$ with derivative $\overline{q}$.*

Proof. We will demonstrate for the directions $D(B-)$, the other cases are treated similarly.

Consider the set of unit-normalized directions

$$\tilde{D} = \{\tilde{y} \in D(B-) : \|\tilde{y}\| = 1\}.$$

Notice the set of directions $y \in D(B-)$ satisfying $\|y\| < \delta$ is equivalent to the set of $y < \delta\tilde{y}$ for some $\tilde{y} \in \tilde{D}$. So Frechet differentiability is equivalent to the statement, for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|\frac{g(\mu + ty) - g(\mu)}{t} - \underline{q}y| < \epsilon \quad \forall y \in \tilde{D}, \quad \forall t < \delta.$$

For each $y \in \tilde{D}$ and $t \in (0, 1]$ let

$$f(y, t) = \frac{g(\mu + ty) - g(\mu)}{t} - \underline{q}y,$$

and let $f(y, 0) = 0$ for all $y \in \tilde{D}$. Corollary 3.1 implies $\lim_{t\downarrow 0} f(y, t) = 0$ for each $y \in \tilde{D}$; so $f(y, t)$ is continuous at $t = 0$ (as a function of t). Indeed, since g is weak* continuous on $M_+(\mathbf{A})$ (Proposition 3 in GOZ1), the function $f(y, t)$ is (jointly) weak* continuous on its entire domain $\tilde{D} \times [0, 1]$. Since $\tilde{D}$ is compact in the weak* topology (by Alaoglu's Theorem and the fact that any closed subset of a compact set is compact), $\tilde{D} \times [0, 1]$ is compact in the product topology. Hence $f(y, t)$ is uniformly continuous on this topology. Since on metric spaces the product topology agrees with the topology given by the product metric, we can conclude that for any $\epsilon > 0$ there is a $\delta > 0$ such that $|f(y, t)| < \epsilon$ for all $t < \delta$ and all $y \in \tilde{D}$. □

3.3. **An individual's contribution to the gains from trade: Calculus at the individual margin.** The Introduction already presented the idea that, in continuum economies, there are two infinitesimal margins of analysis, not just one. We are now ready to begin our analysis at the individual margin, to apply the calculus at this margin.

When there is a nonatomic continuum of agents, it is convenient to think of each individual as infinitesimal, the limit of a sequence of small groups of individuals. Thus we define the social contribution of adding to μ an (infinitesimal) individual with attribute a — the individual's "marginal product" — as the limit of the contribution from adding a small group of individuals with attribute a, as the size of the group goes to zero:

$$MP^+(a) \equiv \lim_{t\downarrow 0} \frac{g(\mu + t\delta_a) - g(\mu)}{t} \equiv g'(\mu; \delta_a),$$

where δ_a is the measure in $M_+(\mathbf{A})$ that puts unit mass on attribute a and zero mass on all other attribute. By Corollary 3.1, these marginal products exist and are given by

$$MP^+(b) = \overline{q}(b), \ \forall b \in \mathbf{B} \quad \text{and} \quad MP^+(s) = \underline{q}(s), \ \forall s \in \mathbf{S}.$$

Unless $\mu(a) > 0$ (so a is an atom), measuring the gains that would be lost from subtracting an infinitesimal individual with attribute $a \in A$

is a bit more delicate. Call $y \in M_+(\mathbf{A})$ a subpopulation of μ if $\|y\| > 0$ and $\mu - y \in M_+(\mathbf{A})$. Let $\langle y_a^n \rangle$ denote a sequence of such subpopulations satisfying

$$\text{supp } y_a^n \to \{a\} \quad \text{and} \quad \|y_a^n\| \to 0.$$

When n is large, the small group of individuals in the subpopulation y_a^n all have attributes very close to a. In accord with our image of an individual in the continuum as an infinitesimal, it is intuitively helpful to think of the sequence of small groups $\langle y_a^n \rangle$ as converging to an infinitesimal individual with attribute a. Thus we define the social contribution of an infinitesimal individual with attribute a — his marginal product — by

$$MP^-(a) \equiv \lim_{n \to \infty} \frac{g(\mu) - g(\mu - y_a^n)}{\| y_a^n \|},$$

assuming this limit exists and is the same for all sequences $\langle y_a^n \rangle$. The Frechet differentiability of g at μ in the directions $D(B-)$ and $D(S-)$ imply these marginal products exist and are given by:

$$MP^-(b) = \underline{q}(b), \ \forall b \in B \quad \text{and} \quad MP^-(s) = \overline{q}(s), \ \forall s \in S.$$

We will be particularly interested in what will happen if an individual deviates from some attribute $a \in A$ to an attribute $d \in \mathbf{A}$. We can think of this deviation as simultaneously subtracting an individual with attribute a from the population and adding one with attribute d. Since we view a deviating individual as the limit of a sequence of small deviating groups, observe that the population

$$\mu^n = \mu - y_a^n + \|y_a^n\| \delta_d$$

would result if each individual in the small group y_a^n deviated to d. This would change the total social gains per deviating individual by

$$\frac{g(\mu + \delta_{a,d}^n) - g(\mu)}{\|\delta_{a,d}^n\|},$$

where $\delta_{a,d}^n = \|y_a^n\| \delta_d - y_a^n$. Letting the size of the deviating group go to zero, we define the social effect of an infinitesimal individual deviating from a to d — the (social) marginal product of the deviation — as

$$MP(a, d) = \lim_{n \to \infty} \frac{g(\mu + \delta_{a,d}^n) - g(\mu)}{\|\delta_{a,d}^n\|}$$

whenever the limit exists and is the same for all sequences $\{\delta_{a,d}^n\}$. These marginal products exist and are given by an inf formula.

Corollary 3.2. *For any* $(a, d) \in A \times \mathbf{A}$ *and any sequence* $\langle \delta^n_{a,d} \rangle$*, the marginal product* $MP(a, d)$ *exists and is given by*

$$MP(a, d) = \lim_{n\to\infty} g'(\mu; y^n) = \inf_{q\in\partial g(\mu)} q(d) - q(a),$$

where $y^n = \delta^n_{a,d}/\|\delta^n_{a,d}\|$.

Proof. We first verify the second equality in the theorem, the inf formula. Let $q^n \in \partial g(\mu)$ satisfy $q^n y^n = g'(\mu; y^n)$. Consider the sequence of subgradients $\langle q^n \rangle$. If $d \in \mathbf{S}$ (resp. $\mathbf{B}$), we can assume without loss of generality that each $q^n \in \hat{\partial} g(\mu)$ (resp. $\check{\partial} g(\mu)$). The compactness of $\mathcal{Q}$ (resp. $\check{\mathcal{Q}}$) implies $\langle q^n \rangle$ has a subsequence, say $\langle q^{n_k} \rangle$, converging pointwise to some $\tilde{q} \in \partial g(\mu)$. Since $\langle y^n \rangle$ converges setwise to the set function $\delta_d - \delta_a$ and $\langle q^{n_k} \rangle$ converges pointwise to $\tilde{q}$, $q^{n_k} y^n$ converges to $\tilde{q}(d) - \tilde{q}(a)$. Since each q^{n_k} satisfies the inf formula for the direction y^{n_k}, we can also conclude that $\tilde{q}(d) - \tilde{q}(a) = \inf_{q\in\partial g(\mu)} q(d) - q(a)$.

To verify the first equality in the theorem, observe $MP(a, d) = \lim_{n\to\infty} \dfrac{g(\mu + t^n y^n) - g(\mu)}{t^n}$, where $t^n = \|\delta^n_{a,d}\|$, while $g'(\mu; y^n) = \lim_{t\downarrow 0} \dfrac{g(\mu + t y^n) - g(\mu)}{t}$. Hence, it easily follows that $MP(a, d) = \lim_{n\to\infty} g'(\mu; y^n)$. □

As the above may suggest, viewing each individual in a continuum economy as an infinitesimal, and viewing the social consequences of his actions as directional derivatives (in appropriately chosen directions) makes for a pleasing conceptual unity. Below we will see that it also provides a useful engine for competitive analysis.

4. Investment equilibrium

When is a given distribution of attributes μ an equilibrium? What would happen if an infinitesimal individual with attribute $a \in A$ switched his investment to some $d \in \mathbf{A}$? Let

$$\varrho : A \times \mathbf{A} \to C_+(\mathbf{S}^0),$$

where $\varrho(a, d) \in P(\mu)$ is interpreted as the conjecture of someone with attribute $a \in A$ about the Walrasian prices that will result if he switches to $d \in \mathbf{A}$, given the aggregate distribution of individuals' characteristics is μ. We insist that these conjectures are rational in the following sense. Let

$$\rho : M_+(\mathbf{A}) \to C_+(\mathbf{S}^0)$$

denote any price selection from the correspondence $\hat{P}$, hence $\rho(\mu)$ is a normalized Walrasian price function for μ. Since we regard an infinitesimal individual deviating as the limit of a sequence of small groups of individuals deviating, we will say that the conjectures ϱ are *rational* starting from μ if there exists a Walrasian price selection ρ such that $\varrho(a,a) = \rho(\mu)$ for every $a \in A$, while for every $(a,d) \in A \times \mathbf{A}$,

$$\varrho(a,d) = \lim_{n\to\infty} \rho(\mu + \delta^n_{a,d})$$

for some sequence $\langle \delta^n_{a,d} \rangle$. Interpret $\varrho(a,a)$ as the prices that will prevail if there is no deviation, while $\varrho(a,d)$ are the prices that will prevail if there is a deviation to d.

Note: A stronger and more appealing rationality condition would be that $\varrho(a,d) = \lim \rho(\mu + \delta^n_{a,d})$ for all sequences $\langle \delta^n_{a,d} \rangle$, not just some. This stronger rationality condition is satisfied by any rational conjectures if μ is perfectly competitive, since then $\hat{P}(\mu)$ is a singleton. Since we will mainly be concerned with perfectly competitive economies, the stronger condition can be substituted for the weaker wherever the latter appears in the paper, except in the proof of Theorem 9.1. (Even there the substitution may be possible, but I have not been able to prove this.)

Fix the ex ante population $\mathcal{E}$. An allocation of attributes is a measure $\nu \in M_+(\mathbf{A} \times T)$, with marginals $\nu_{\mathbf{A}}$ and ν_T. The allocation ν is *feasible* for $\mathcal{E}$ if $\nu_T = \mathcal{E}$. Let $\mu = \nu_{\mathbf{A}}$. A pair (ν, ϱ) is an *investment equilibrium* for $\mathcal{E}$ if ν is feasible, ϱ are rational conjectures starting from μ, and for all $(a,t) \in \text{supp } \nu$:

$$v_a{}^*(\varrho(a,a)) - c(a,t) \geq v_d{}^*(\varrho(a,d)) - c(d,t), \quad \forall d \in \mathbf{A},$$

so no individual has a profitable deviation.

The total cost of the attribute selections ν is $\int c \, \mathrm{d}\nu$. Hence the total surplus (or net gain) from ν is

$$G(\nu) = g(\nu_{\mathbf{A}}) - \int c \, \mathrm{d}\nu.$$

The allocation ν is *efficient* for $\mathcal{E}$ if it is feasible, and $G(\nu) \geq G(\nu')$ for all other feasible allocations ν'. We will be interested in the efficiency of investment equilibria, especially under perfect competition.

4.1. **Perfect competition is not automatic in the continuum.** It is tempting to think that whenever there is a nonatomic continuum of agents, each agent is necessarily a perfect competitor since he cannot affect the macro-economic data of the economy, e.g., the distribution of agents' characteristics μ. This is the thought pursued in CMP2. Unfortunately, accepting this point of view, one runs the danger of making continuum analysis totally unconnected with the analysis of large but finite economies, where any one agent can affect the macroeconomic data of the economy.

An alternative point of view, taken in GOZ2 and followed here, is to insist that continuum analysis be representative of what happens in large but finite economies. Viewing an individual in the continuum as the limit of small groups of individuals — so an "infinitesimal individual" rather than just a point of measure zero on the real line — is the way GOZ2 implements this desirable feature.

We illustrate the danger in taking perfect competition for granted in the continuum using a well known example: Edgeworth's master-servant example, also known in modern core analysis as a glove-market example. The example shows that an arbitrarily large but finite assignment economy need not be perfectly competitive. Consequently, if the continuum is to be representative of a large but finite economy, we cannot insist that continuum economies are necessarily perfectly competitive.

Example 4.1 (Edgeworth's master-servant example)**.** Let $\mathbf{B} = \{b_0, b_1\}$ and $\mathbf{S} = \{s_0, s_1\}$, where $b_0 = s_0 = 0$ and $b_1 = s_1 = 1$. Assume $v(b, s) = bs$ for all $(b, s) \in B \times S$, while $v(b, \mathbf{0}) = v(\mathbf{0}, s) = 0$. Interpret any buyer b_1 as a master and any seller s_1 as a servant; while the attributes b_0 and s_0 are dummies that cannot generate any value. Notice each master and servant has a reservation value of 0; so value is only created by matching.

Consider a population μ in which $\mu(b_0) = \mu(s_0) = 0$ and $\mu(b_1) = \mu(s_1) = N \geq 1$. So there are just as many masters as servants. For this population there is a continuum of possible Walrasian equilibrium price vectors, namely, any $p \equiv (p(s_0), p(s_1))$ with $p(s_0) = 0$ and $p(s_1) \in [0, 1]$. Any price of servants $p(s_1) \equiv p_1 \in [0, 1]$ determines a core imputation, a particular division of the total gains $g(\mu) = N$. If $p_1 = 1$, the servants get all the surplus; if $p_1 = 0$, the masters get it all. If $p_1 \in (0, 1)$, then each side gets some of the surplus.

Suppose first that the economy is finite and μ is the counting measure. Choose any ${p_1}^* \in [0, 1]$ as the Walrasian equilibrium price for a servant. Notice if even one servant withdraws from the economy (chooses the dummy attribute s_0), the equilibrium price will jump to 1; and if even one master withdraws, it will drop to 0. So, no matter which ${p_1}^*$ one chooses, and no matter how large N is, some agents can affect the equilibrium price a lot. Hence this finite economy is not perfectly competitive, regardless of how large N is.

Consider now the continuum version of Edgeworth's example in which there is a nonatomic continuum of masters having measure N and a corresponding continuum of servants. If we take the point of view that in the continuum any one individual cannot affect prices, we would have to conclude there is a discontinuity at infinity, which would make the analysis of the continuum version of the example rather uninformative since it would not be asymptotically meaningful.

The alternative point of view adopted here is to view an individual as the limit of small groups. To see what difference this makes, notice if a small group of servants deviates to s_0, the equilibrium price will jump to 1 even in the continuum; similarly if a small group of masters deviates to b_0, it will drop to 0 even in the continuum—no matter how small the group is, as long as it has positive measure.

Let $\varrho(a, d)(s_1)$ be agents' conjecture about what will happen to the price ${p_1}^*$ if an individual with attribute a deviates to attribute d. How can we pin down these conjectures? The naive point of view would insist that starting from any ${p_1}^* \in [0, 1]$,

$$\varrho(a, d)(s_1) = {p_1}^*, \quad \forall a \in A, \quad \forall d \in \mathbf{A}.$$

Instead, following the approach in GOZ2, we insist that

$$\varrho(a, d)(s_1) = \lim_{n\to\infty} p^n(s_1), \quad \text{where } p^n \in P(\mu + d^n_{a,d}),$$

leading to the (unique) prediction that the price will jump to 1 if an individual servant deviates — where an individual servant deviating is the limit of a sequence of small groups of servants deviating $\langle \delta^n_{s_1,s_0} \rangle$,— and the price will drop to 0 if an individual master deviates — where an individual master deviating is the limit if a sequence of small groups of masters deviating $\langle \delta^n_{b_1,b_0} \rangle$. That is, the GOZ2 approach pins down (rational) conjectures to be

$$\varrho(s_1, s_0)(s_1) = 1 \quad \text{and} \quad \varrho(b_1, b_0)(s_1) = 0,$$

starting from any ${p_1}^* \in [0, 1]$. ◇

4.2. **Some inefficient equilibria.** In the holdup literature, attention is often limited to the special case of our model in which $\mathbf{A} \subset R_+$ (so attributes are one dimensional), $r(a) = 0$ for all $a \in \mathbf{A}$ (so value is not created unless one is matched), and for any attributes $b, b' \in \mathbf{B}$ and $s, s' \in \mathbf{S}$,

$$v(b, s) + v(b', s') > v(b, s') + v(s', b)$$

whenever $b' > b$, $s' > s$ (so v is strictly supermodular). It is also assumed that v is strictly increasing in both arguments on R^2_{++}, and $v(b, s) = 0$ iff $b \in \{\mathbf{0}, 0\}$ or $s \in \{\mathbf{0}, 0\}$. We will call this special case the *supermodular model.* CMP1, CMP2, and Felli and Roberts limit their attention to this model.

For the supermodular model with a finite number of individuals, CMP1 catalog three types of inefficient investment equilibria: equilibria with:

- underinvestment equilibria,
- overinvestment equilibria,
- investor mismatch equilibria, as first identified by Felli and Roberts (2000).

Extending the CMP1 model to a setting with a nonatomic continuum of individuals, CMP2 show that in their continuum model only the first two sorts of inefficiencies survive; equilibria with investor mismatches do not survive. The CMP2 model views perfect competition as automatic when agents are nonatomic. By contrast, using the GOZ2 rather than the CMP2 notion of perfect competition—hence allowing for the possibility that not all continuum economies are perfectly competitive,—we now illustrate that all three sorts of inefficiencies survive even in the continuum. Indeed, in the absence of perfect competition in the sense of GOZ2, there is little difference between finite and continuum economies.

Example 4.2 (an underinvestment equilbrium)**.** This is a continuum version of an example in CMP1. Consider the supermodular model in which $\mathbf{B} = \{b_1, b_2\}$, $\mathbf{S} = \{s_1, s_2\}$, and $v(b, s) = bs$. Assume $b_1 = s_1 = 1$ and $b_2 = s_2 = 2$. There is only one buyer type i with $c(1, i) = 0$ and $c(2, i) = 1.25$. There also is only one seller type j with $c(\cdot, j) = c(\cdot, i)$. Finally, assume there is an equal mass of the two types, so $\mathcal{E}(i) = \mathcal{E}(j) = N$, where $N \geq 1$.

Let $p \equiv (p(s_1), p(s_2)) = (.5, 1.5)$. There is an investment equilibrium (ν, ϱ) in which all individuals choose attribute 1, so ν satisfies $\nu(b_1, i) = \nu(s_1, j) = N$. The allocation ν is supported as an equilibrium by the rational conjectures $\varrho(b_1, b_1) = \varrho(s_1, s_1) = p$, while

$$\varrho(b_1, b_2) = (.5, 2.5) \quad \text{and} \quad \varrho(s_1, s_2) = p.$$

Since $p(s_1) = .5$, in equilibrium each matched couple evenly splits the ex post value $v(b_1, s_1) = 1$. If a buyer deviates to b_2, he conjectures that $p(s_1)$ will not be affected, so he will be able to appropriate the entire increase in value, namely $v(b_2, s_1) - v(b_1, s_1) = 2 - 1 = 1$. Similarly, if a seller deviates to s_2, he conjectures he will be able to appropriate the entire increase in value since $p(s_2) - p(s_1) = 1$.

The reader may wonder why $\varrho(b_1, b_2) = (.5, 2.5)$ rather than $(.5, 1.5)$. Notice a buyer who deviates to $b_2 = 2$, if he could match with a seller with attribute $s_2 = 2$, could generate value $v(b_2, s_2) = 4$, so his gross surplus would be $v(b_2, s_2) - p(s_2) = 4 - p(s_2)$, compared to his gross surplus matching with a seller s_1, namely $v(b_2, s_1) - p(s_1) = 2 - .5 = 1.5$. Thus $p(s_2) = 2.5$ is the minimum price that will deter the deviating buyer from wanting a match with an unavailable seller.

In equilibrium the net payoffs are

$$v_{b_1}{}^*(p) - c(b_1, i) = .5 \quad \text{and} \quad v_{s_1}{}^*(p) - c(s_1, j) = .5,$$

and no buyer or seller has a profitable deviation. For example, if a buyer deviated to b_2, his net payoff would fall to $1.5 - c(b_2, i) = .25$. The total net surplus in equilibrium is $G(\nu) = N$.

The equilibrium allocation ν involves inefficient underinvestment. If all individuals simultaneously deviated to attribute 2, leading to the feasible allocation ν' satisfying $\nu'(b_2, i) = \nu'(s_2, j) = N$, total net surplus would increase to $G(\nu') = 1.5N$. ◇

Example 4.3 (an overinvestment equilibrium)**.** This is a continuum version of another example in CMP1. It is identical to Example 1 except $c(2, i) = c(2, j) = 1.75$.

Let $\tilde{p} = (0, 2)$. There is an investment equilibrium $(\tilde{\nu}, \tilde{\varrho})$ in which all individuals choose attribute 2, so $\tilde{\nu}$ satisfies $\tilde{\nu}(b_2, i) = \tilde{\nu}(s_2, j) = N$. The allocation $\tilde{\nu}$ is supported as an equilibrium by the rational conjectures $\tilde{\varrho}(b_2, b_2) = \tilde{\varrho}(s_2, s_2) = \tilde{p}$, while

$$\tilde{\varrho}(b_2, b_1) = (2, 2) \quad \text{and} \quad \tilde{\varrho}(s_2, s_1) = (0, 2).$$

Since $\tilde{p}(s_2) = 2$, in equilibrium each matched buyer and seller evenly splits the value $v(b_2, s_2) = 4$. If a buyer deviates to b_1, he conjectures that he will have to give his trading partner the entire resulting ex post value $v(b_1, b_2) = 2$. Similarly, if a seller deviates to s_1, he conjectures that he will have to give his trading partner the entire resulting ex post value. In equilibrium the net payoffs are

$$\tilde{v}_{b_2}{}^*(\tilde{p}) - c(b_2, i) = .25 \quad \text{and} \quad \tilde{v}_{s_2}{}^*(\tilde{p}) - c(s_2, j) = .25,$$

and no one buyer or seller has a profitable deviation. In equilibrium the total net surplus is $G(\tilde{\nu}) = .5N$.

This equilibrium involves inefficient overinvestment. If all individuals simultaneously deviated to attribute 1, leading to the feasible allocation ν satisfying $\nu(b_1, i) = \nu(s_1, j) = N$, the total surplus would increase to $G(\nu) = N$. ◇

Example 4.4 (a mismatch equilibrium)**.** This is a simple example of a Felli-Roberts-type inefficiency.

Assume $v(b, s) = bs$ and there is only one-sided investment by sellers. Buyers have only one possible attribute $\mathbf{B} = \{b\}$, where $b = 4$; and there is one type of buyer, with $c(4, i) = 0$. For sellers $\mathbf{S} = \{0, 4\}$; and there are two types of sellers, H and L, with $c(0, H) = c(0, L) = 0$,

$c(4, H) = 15$, $c(4, L) = 1$. So type L is the low-cost producer of attribute 4. Assume $\mathcal{E}(i) = \mathcal{E}(H) = \mathcal{E}(L) = N$, where $N \geq 1$.

Let $s_0 = 0$, $s_1 = 4$, and $p \equiv (p(s_0), p(s_1)) = (0, 16)$. There is an efficient investment equilibrium (ν, ϱ) in which only the low-cost sellers invest, hence $\nu(s_1, L) = \nu(s_0, H) = N$, supported by the rational conjectures that if no one deviates prices will equal p, while if there is a deviation

$$\varrho(s_0, s_1) = (0, 0) \quad \text{and} \quad \varrho(s_1, s_0) = (0, 16).$$

In this equilibrium the total net surplus is $G(\nu) = 15N$.

But there also is an inefficient investment equilibrium (ν', ϱ) in which only the high-cost sellers invest, hence $\nu'(s_0, L) = \nu'(s_1, H) = N$, supported by the same conjectures. In this equilibrium the total net surplus drops to $G(\nu') = N$. There are also many other possible equilibria. Indeed, any mix between low and high-cost sellers investing in attribute 4, with the total mass of investing sellers equal to N, is a possible equilibrium. ◇

Provided N is an integer, the above three examples can be interpreted as happening in either an economy with a finite number of individuals (with ν the counting measure) or in an economy with a nonatomic continuum of individuals. Thus, in terms of the types of possible inefficiencies, there is no difference between the continuum and finite assignment models. To reassure the reader, we point out that all three sorts of inefficient equilibria can also be illustrated using examples with a nonatomic continuum of types, not just using examples with a finite number of types. Here is one more example, this one is very much in the spirit of the holdup literature.

Example 4.5 (fear of holdups)**.** Here is a simple example of underinvestment due to a rational fear of holdups.

Assume a supermodular model with $v(b, s) = bs$. For buyers $\mathbf{B} = \{b\} = \{1\}$, a singleton, so buyers cannot invest; and there is only one type of buyer, with $c(1, i) = 0$. For sellers $\mathbf{S} = \{s_1, s_2\} \equiv \{1, 2\}$; and there is one type of seller with $c(1, j) = 0$ and $c(2, j) \in (.5, 1)$. Assume $\mathcal{E}(i) = \mathcal{E}(j) = N > 0$.

There is an investment equilibrium in which all sellers choose s_1. Like the master-servant example, any price of s_1, say p_1, in the interval $[0, 1]$ is market clearing. Consider $p_1 = .5$ (split the surplus). No seller deviates from $s = 1$ to $s = 2$ because he fears a holdup, conjecturing whatever buyer he matches with will still insist on splitting the surplus, hence

$$\varrho(s_1, s_2)(s_2) = v(1, 2)/2 = 1,$$

which implies any seller's conjectured profit from deviating $1 - c(2, j)$ is less than .5, his profit from not deviating. No seller invests enough because he believes he will not be able to appropriate enough of the extra ex post surplus that more investment can generate. If $N = 1$ and the economy is finite (so $\mathcal{E}$ is the counting measure), this is a standard holdup example. The point is that the economy need not be finite ($\mathcal{E}$ can be an atomless measure) and the same holdup problem arises because the ex post population is imperfectly competition. To complete the description of the investment equilibrium, set the ex post Walrasian price of s_2 sufficiently high to choke off any buyer demand for the unavailable good, e.g., $p(s_2) = 2$; and notice that $\varrho(1, 2)(2) = 1$ implies $\varrho(1, 2)(1) = 0$, hence the insistence by s_2's trading partner on splitting the surplus creates a beneficial pecuniary externality for all other buyers. So, in the absence of perfect competition, one tough-bargaining buyer can benefit many less tough buyers!

$\diamond$

One of our motivations is to discover to what extent the inefficiencies illustrated above are due to an absence of ex post perfect competition. First we define and characterize such competition.

5. Characterizing perfect competition in the continuum assignment model

Throughout this section, our discussion will be relative to a fixed ex post population $\mu \in M_+(\mathbf{A})$. The central theme of GOZ2 is that there is an intimate relation between perfect competition at μ and each individual fully appropriating his social contribution. We will extend this theme from what they call "perfect competition" to what we will call "strong perfect competition."

5.1. Characterizing a perfectly competitive μ.

We will say that individuals face perfectly elastic demands and supplies in μ if no small group can influence equilibrium prices by changing their attributes to some others in supp μ. More precisely, individuals face *perfectly elastic demands and supplies* (PEDS) in μ if for any Walrasian price selection ρ and any sequence of populations $\mu^n \to \mu$ with supp $\mu^n \subset$ supp μ,

$$\rho(\mu^n) \to \rho(\mu).$$ [2]

Call μ *perfectly competitive* if individuals faces perfectly elastic demands and supplies in μ.

[2] Convergence will always mean in the norm topology, unless otherwise specified.

Let $\mathcal{K}(\mu)$ denote the set of imputations in the (ex post) core of μ; and for any $A' \subset \mathbf{A}$, let $D(A') = \{y \in D : \text{supp } \mu \subset A'\}$. The following characterization is proved in GOZ2.

Proposition 5.1 (GOZ2 characterization of perfect competition)**.** *The following are equivalent:*

(1) *all individuals face perfectly elastic demands and supplies in μ;*
(2) *$\hat{P}(\mu)$ is a singleton, p;*
(3) *$\mathcal{K}(\mu)$ is a singleton, q^* (so core bargaining is determinate);*
(4) *there is a $q^* \in \mathcal{K}(\mu)$ such that*

$$q^*(a) = MP^-(a) = MP^+(a), \quad \forall a \in \text{supp } \mu.$$

(5) *g is differentiable at μ in the directions $D(A)$ with derivative q equal to any element of $\partial g(\mu)$*
(6) *every $q \in \partial g(\mu)$ satisfies $q\mid_A = q^*$.*

To put 4–6 into perspective, the subdifferential inequality implies for any $q \in \partial g(\mu)$,

$$MP^-(a) \leq q(a) \leq MP^+(a), \quad \forall a \in \mathbf{A}.$$

If both weak inequalities are equalities, we will say $q(a)$ equals a's marginal product, denoted $q(a) = MP(a)$. Hence, since 6 tells us that $q^* = q\mid_A$, Statement 4 tells us that under perfect competition, for any $q \in \partial g(\mu)$,

$$q(a) = MP(a), \quad \forall a \in A.$$

That is, under perfect competition all individuals in supp μ are rewarded with their marginal products. Graphically this means that the gains function g is smooth at μ (no kinks) at least on the restricted domain A, hence g is differentiable (not just subdifferentiable) on this domain. Further, since $\int q \, \mathrm{d}\mu = g(a)$ for any $q \in \partial g(\mu)$, under perfect competition there is adding-up at μ in the sense that the sum of individuals' social contributions adds up to (accounts for) the total gains:

$$\int MP(a) \, \mathrm{d}\mu(a) = g(\mu).$$

Turning to the pricing connection, since the core coincides with the set of Walrasian allocations for μ, Statements 2 tells us that the unique core imputation can be realized by Walrasian price taking:

$$MP(a) = {v_a}^*(p), \quad \forall a \in A.$$

If μ is perfectly competitive we can pin down individuals' rational conjectures, at least for deviations inside supp μ.

Corollary 5.1 (rational conjectures inside the support)**.** *If μ is perfectly competitive and the conjectures ϱ are rational starting from μ, then*

$$\varrho(a, d) = p, \quad \forall a, d \in \text{supp } \mu,$$

where p is the unique price function in $\hat{P}(\mu)$. That is, no individual can rationally expect to influence Walrasian prices p by deviating inside supp μ.

Proof. Since ϱ is rational, there is a Walrasian price selection ρ such that, for every $(a, d) \in A \times \mathbf{A}$ and some sequence $\langle \delta^n_{a,d} \rangle$,

$$\varrho(a, d) = \lim_{n\to\infty} \rho(\mu + \delta^n_{a,d}).$$

Let $\mu^n = \mu + \delta^n_{a,d}$ and assume $d \in A$. Since $\mu^n \to \mu$, PEDS implies

$$\lim_{n\to\infty} \rho(\mu^n) = p.$$

□

The following example should help fix ideas.

Example 5.1 (A perfectly competitive μ)**.** Consider the supermodular model in which $\mathbf{B} = \mathbf{S} = \{1, 2\}$ and $v(b, s) = bs$. In the ex post population μ all buyers and sellers have attribute 1, and there are twice as many buyers as sellers. So, letting $b_1 = s_1 = 1$, $b_2 = s_2 = 2$, and $\mu = (\mu(b_1), \mu(s_1), \mu(b_2), \mu(s_2))$,

$$\mu = (2N, N, 0, 0)$$

where $N \geq 1$.

Applying Proposition 5.1, the population μ is perfectly competitive because $\hat{P}(\mu)$ is a singleton, containing only

$$p \equiv (p(s_1), p(s_2)) = (1, 2).$$

The fact that $p(s_1) = v(b_1, s_1) = 1$ (so sellers get all the surplus) follows from the fact that there are more buyers than sellers, so intense competition for a match among the many unmatched buyers forces the price of a match with a seller s_1 all the way up to $v(b_1, s_1)$. The fact that $p(s_2) = 2$ follows from $p \in \hat{P}(\mu)$: A buyer with attribute 1 could generate value $v(b_1, s_2) = 2$ if matched with a seller s_2, so $p(s_2) = 2$ is the minimum price needed to choke off demand for good s_2.

Another way to check that the economy is perfectly competitive is to verify that individuals face PEDS in μ. Any population μ^n with supp $\mu^n \subset$ supp μ will be of the form

$$\mu^n = (2N + \alpha, N + \beta, 0, 0).$$

If α and β are small, the fact that there are more buyers than sellers in μ^n implies p will also be the unique Walrasian price vector in $\hat{P}(\mu^n)$, so any small deviation inside supp μ will not affect prices.

Yet a third way to check that the population is perfectly competitive is to verify it has a core utility vector in which each individual fully appropriates his social contribution. Consider the core utilities resulting from prices p, that is,

$$v^*(p) \mid_{\text{supp } \mu} \equiv (q^*(b_1), q^*(s_1)) = (0, 1).$$

Any small mass of buyers b_1 contributes nothing to social gains — whether such a mass is subtracted from or add to μ — since there are many unmatched buyers. Hence

$$MP^-(b_1) = MP^+(b_1) = {v_{b_1}}^*(p) = 0.$$

But any small mass of sellers s_1 contributes $v(b_1, s_1) = 1$ to social gains — whether such a mass is subtracted from or added to μ: If subtracted, the mass of unmatched buyers would just decrease; if added, any new seller s_1 could match with a previously unmatched buyer b_1. Hence

$$MP^-(s_1) = MP^+(s_1) = {v_{s_1}}^*(p) = 1.$$

So there is full appropriation in the core utilities resulting from prices p. $\diamond$

5.1.1. *The three principles.* There are three simple principles that can serve as a guide to understanding the theory of perfect competition. They can be roughly expressed as follows. In a perfectly competitive economy everyone has many outside options, so none of his trading partners is in a position to press him for a better deal; the rivalry among potential trading partners makes the terms of trade determinate, there are no bilateral monopoly surpluses to bargain over; this same rivalry allows everyone to fully appropriate his social contribution, giving the perfect competitor good incentives to innovate.

The three principles can be visualized as the three vertices of a triangle labelled PS (the availability of perfect substitutes), PEDS (everyone faces perfectly elastic demands and supplies), and FA (everyone fully appropriates his social contribution). Perfect competition (PC) is in the center. Most of the results of perfect competition theory are fairly intuitive applications of these ideas. For example, Statements 1 - 3 in Proposition 5.1 may be viewed as expressions of the second principle, while Statement 4 is a version of full appropriation. Stronger versions, covering deviations outside the support of μ, will be given in Theorem 5.1 and Corollary 5.3 below. Theorem 5.2 will be an expression of the first principle.

The three principles can be visualized as the three vertices of a triangle labelled PS (the availability of perfect substitutes), PEDS (everyone faces perfectly elastic demands and supplies), and FA (everyone fully appropriates his social contribution). Perfect competition (PC) is in the center.

5.2. **Strong perfect competition and deviations outside the support of μ.** What will be the outcome of core bargaining, and hence to Walrasian prices, if there is a deviation to a $d \notin \text{supp } \mu$? To answer this question we must extend the concept of perfect competition to cover such deviations.

We will call μ strongly (perfectly) competitive if all individuals face perfectly elastic demands and supplies even for deviations outside of supp μ. More precisely, we say that individuals face *strong perfectly elastic demands and supplies* (strong PEDS) in μ if for any Walrasian price selection ρ and any sequence of populations $\mu^n \to \mu$ with supp $\mu^n \subset B \cup \mathbf{S}$,

$$\rho(\mu^n) \to \rho(\mu),$$

while for any ρ and any sequences $\mu^n \to \mu$ with supp $\mu^n \subset \mathbf{B} \cup S$,

$$\rho(\mu^n) \mid_S \to \rho(\mu) \mid_S .$$

That is, any sequence of price functions $\langle \rho(\mu^n) \rangle$ converges in norm to the (unique) price function $\rho(\mu)$ even if μ^n includes sellers with attributes outside supp μ, and the restriction of any such sequence to the prices of the commodities in S converges if μ^n includes buyers with attributes outside supp μ.

We will call μ *strongly (perfectly) competitive* if individuals face strong PEDS in μ. Strong PEDS implies PEDS since the latter restricts itself to sequences μ^n satisfying supp $\mu^n \subset A \equiv \text{supp } \mu$. Hence strong perfect competition implies perfect competition. Below we will show that the converse also holds. Hence strong competition comes "for free" in any perfectly competitive continuum economy.

Notice the definition of strong PEDS leaves open the possibility that, after a buyer deviation, the reservation prices of some unavailable goods $s \in \mathbf{S} - S$ may change. The continuation of Example 5.1 below will illustrate why this is necessary: Perfect competition does not imply for all sequences with supp $\mu^n \subset \mathbf{B} \cup S$,

$$\rho(\mu^n)(s) \to \rho(\mu)(s), \quad \forall s \in \mathbf{S} - S.$$

Unlike seller deviations, buyer deviations cannot change the set of available goods from S. Hence, in some sense, that a buyer deviation may affect the reservation prices of some unavailable goods is without behavioral significance. But it does have an interesting interpretive significance; it helps to explain why there may be underinvestment equilibria even under perfect competition. (See the discussion of reservation price externalities in Remark 6.1.)

The following extension of the GOZ2 characterization is the main result of this section.

Theorem 5.1 (characterizing strong perfect competition)**.** *The following statements are equivalent:*

(1) μ *is perfectly competitive;*
(2) μ *is strongly perfectly competitive;*
(3) g *is differentiable at* μ *in the directions* $D(A) \cup D(\mathbf{S}+)$ *with derivative* $\underline{q}$, *and in the directions* $D(A) \cup D(\mathbf{B}+)$ *with derivative* $\overline{q}$, *where* $\overline{q}\mid_A = \underline{q}\mid_A$;
(4) $p \in \hat{P}(\mu)$ *implies*

$$\begin{aligned}
&{v_a}^*(p) = MP(a), && \forall a \in A,\\
&{v_s}^*(p) = MP^+(s), && \forall s \in \mathbf{S} - S\\
&{v_b}^*(p, S) = MP^+(b), && \forall b \in \mathbf{B} - B\\
&MP(a, d) = MP^+(d) - MP^-(a), && \forall (a, d) \in A \times \mathbf{A}.
\end{aligned}$$

To prove the theorem, we first prove three lemmas.

Lemma 5.1. *If* μ *is perfectly competitive, it is strongly perfectly competitive.*

Proof. Let p be the unique price function in $\hat{P}(\mu)$. Assume $\langle \mu^n \rangle$ approach μ, and $p^n \in \hat{P}(\mu^n)$ for each n. We must show that (i) $p^n \to p$ on S if supp $\mu^n \subset \mathbf{B} \cup S$ for all n, and (ii) $p^n \to p$ on all of $\mathbf{S}$ if supp $\mu^n \subset B \cup \mathbf{S}$ for all n.

Since $\mathcal{P}$ is compact, there is a subsequence of $\langle p^n \rangle$, say $\langle p^{n_k} \rangle$, that has a limit point $p' \in \mathcal{P}$. It is readily verified that $p' \in P(\mu)$. Taking this for granted, let $B^n = \{b \in \text{supp } \mu^n\}$. Since $p^n \in \hat{P}(\mu^n)$,

$$p^n(s) = \max \left\{ r(s), \max_{b \in B^n} v(b, s) - {v_b}^*(p^n) \right\}, \quad \forall s \in \mathbf{S}.$$

Since $\mathbf{B}$ is compact, we can assume without loss of generality that the sequence of supports, $\langle \text{supp } \mu^{n_k} \rangle$, converges to some $B' \cup S'$.

If supp $\mu^n \subset \mathbf{B} \cup S$ for all n, then $\mu^n \to \mu$ implies $B \subset B'$, hence

$$p'(s) \geq \max \left\{ r(s), \max_{b \in B} v(b, s) - {v_b}^*(p') \right\}, \quad \forall s \in \mathbf{S}.$$

Since $p' \in P(\mu)$, the above weak inequality must be an equality on S. So $p' = p$ on S.

Suppose instead that supp $\mu^n \subset B \cup \mathbf{S}$ for all n. Then $B' = B$, hence the above weak inequality becomes an equality for all $s \in \mathbf{S}$. In other words, $p' = p$ as required. □

Lemma 5.2. *If μ is perfectly competitive, then g is (Frechet) differentiable at μ in the directions $D(A) \cup D(\mathbf{B}+)$ with derivative $\overline{q}$, and in the directions $D(A) \cup D(\mathbf{S}+)$ with derivative $\underline{q}$, where $\overline{q} \mid_A = \underline{q} \mid_A$.*

Proof. If μ is perfectly competitive, then the normalized price functions are unique and coincide on supp μ: $\hat{P}(\mu) = \{p\}$ and $\check{P}(\mu) = \{p'\}$, where $p \mid_A = p' \mid_A$. Hence $\underline{q} = v^*(p)$ and $\overline{q} = v^*(p')$, and $\overline{q} \mid_A = \underline{q} \mid_A$. The lemma now follows Theorem 3.2 and Proposition 5.1. □

Lemma 5.3. *If μ is perfectly competitive and $p \in \hat{P}(\mu)$, then*

(a) $v_a{}^*(p) = MP(a), \quad \forall a \in A,$
(b) $v_s{}^*(p) = MP^+(s), \quad \forall s \in \mathbf{S} - S,$
(c) $v_b{}^*(p, S) = MP^+(b), \quad \forall b \in \mathbf{B} - B,$
(d) $MP(a,d) = MP^+(d) - MP^-(a), \quad \forall (a,d) \in A \times \mathbf{A}.$

Proof. Recall that $MP^+(s) = \underline{q}(s) = v_s{}^*(\underline{p})$ for all $s \in \mathbf{S}$, and $MP^-(s) = \overline{q}(s) = v_s{}^*(\overline{p})$ for all $s \in S$. Since $\underline{p}$ and $\overline{p}$ are equal on supp μ, and since perfect competition implies $\underline{p} = p$, we conclude that $MP(s) = v_s{}^*(p)$ for all $s \in S$ and $MP^+(s) = v_s{}^*(\overline{p})$ for all $s \in \mathbf{S} - S$. The verification of statement (a) for buyers and statement (c) is analogous. Finally, to verify statement (d), observe since μ is perfectly competitive, the core utility $q(a)$ is unique for all $a \in A$. Hence for any $d \in \mathbf{A}$, $\inf_{q \in \partial g(\mu)} q_d - q(a)$ is given by the subgradient that minimizes q_d, which is $\overline{q}$ (resp $\underline{q}$) if $d \in \mathbf{S}$ (resp. $d \in \mathbf{B}$). Hence $MP(a,d) = MP^+(d) - MP^-(a)$ (Corollary 3.2). □

Proof of Theorem 5.1. From the above lemmas, 1 ⇒ 2, 3, and 4. Conversely, 2 ⇒ 1 is obvious, and 3 or 4 ⇒ 1 by Proposition 5.1. □

If μ is strongly competitive, we can pin down individuals' rational conjectures even for deviations outside supp μ.

Corollary 5.2 (rational conjectures outside the support)**.** *If μ is perfectly competitive and the conjectures ϱ are rational starting from μ, then for every $(a,d) \in A \times \mathbf{S}$,*

$$\varrho(a,d) = p,$$

and for every $(a,d) \in A \times \mathbf{B}$, $\varrho(a,d) \in P(\mu)$ satisifes

$$\varrho(a,d)(s) = p(s), \quad \forall s \in S,$$

where p is the unique price function in $\hat{P}(\mu)$.

Proof. Since ϱ is rational, there is a Walrasian price selection from the correspondence $\hat{P}(\cdot)$, say ρ, such that, for every $(a, d) \in A \times \mathbf{A}$ and some sequence $\langle \delta^n_{a,d} \rangle$,

$$\varrho(a, d) = \lim_{n \to \infty} \rho(\mu + \delta^n_{a,d}).$$

Let $\mu^n = \mu + \delta^n_{a,d}$. Since $\mu^n \to \mu$, strong PEDS implies for any $(a, d) \in A \times \mathbf{S}$,

$$\lim_{n \to \infty} \rho(\mu^n) = p,$$

and for any $(a, d) \in A \times \mathbf{B}$,

$$\lim_{n \to \infty} \rho(\mu^n)(s) = p(s), \quad \forall s \in S.$$

□

An immediate consequence of the corollary is that, if μ is perfectly competitive, each individual rationally conjectures that his private benefit from any deviation will equal the full social benefit.

Corollary 5.3 (private benefit = social benefit)**.** *Suppose (ν, ϱ) is a perfectly competitive investment equilibrium. Then everyone (rationally) conjectures he will appropriate his full social contribution after any deviation:*

$$v_d{}^*(\varrho(a, d)) = MP^+(d), \quad \forall (a, d) \in A \times \mathbf{A}.$$

Hence everyone (rationally) conjectures that his private benefit from any deviation will equal the social benefit, that is,

$$\begin{aligned} v_d{}^*(\varrho(a, d)) - v_a{}^*(p) &= MP^+(d) - MP^-(a) \\ &= MP(a, d), \quad \forall (a, d) \in A \times \mathbf{A}. \end{aligned}$$

Proof. This follows immediately from Theorem 5.1 and Corollary 5.2.
□

Example 5.1 (continued)**.** Let's now consider deviations outside the support of μ, beginning with seller deviations. Any population μ^n with supp $\mu^n \subset B \cup \mathbf{S}$ will be of the form

$$\mu^n = (2N + \alpha, N + \beta, 0, \gamma).$$

If α, β, and γ are small, $p = (1, 2)$ remains the unique price vector in $\hat{P}(\mu^n)$ — in accord with strong PEDS. The fact that there are more buyers than sellers in μ^n implies any buyer b_1 will have to pay $p(s_2) = v(b_1, s_2) = 2$ for a match with a seller s_2 — the entire value created.

In terms of full appropriation, if a small mass of sellers with attribute s_2 is added to μ, social gains would increase by $v(b_1, s_2) = 2$ per capita. Hence in accord with Corollary 5.3, any deviating seller will obtain his full social contribution:

$$v_{s_2}{}^*(p) = MP^+(s_2) = 2.$$

Hence, since $MP(s_1) = 1$, any deviating seller's private benefit will equal the full social benefit:

$$v_{s_2}{}^*(p) - v_{s_1}{}^*(p) = 2 - 1 = MP^+(s_1) - MP(s_2) = MP(s_1, s_2).$$

Turning to buyer deviations, any population μ^n with supp $\mu^n \subset \mathbf{B} \cup S$ will be of the form

$$\mu^n = (2N + \alpha, N + \beta, \gamma, 0).$$

If α, β, and γ are small, the unique price vector in $\hat{P}(\mu^n)$ will be

$$p^n = (1, 4).$$

In accord with strong PEDS, the prices of all available goods S will not change after a small deviation (here $S = \{s_1\}$). But notice that buyers' reservation price for the unavailable good s_2 will increase from 2 to 4. The higher price of good s_2 reflects the fact that buyers b_2 have a higher reservation price than buyers b_1: Any buyer b_2 could generate value $v(b_2, s_2) = 4$ by matching with a seller s_2, so a price of 4 is the minimum price now needed to choke off demand for (the unavailable) good s_2.

In terms of full appropriation, if a small mass of buyers with attribute b_2 is added to μ, social gains would increase by $v(b_2, s_2) - v(b_1, s_1) = 2-1 = 1$ per capita. Hence, in accord with Corollary 5.3, any deviating buyer will obtain his full social contribution:

$$v_{b_2}{}^*(\varrho(b_1, b_2)) = v_{b_2}{}^*(1, 4) = 1 = MP^+(b_2).$$

Hence, since $MP(b_1) = 0$, any deviating buyer's private benefit will equal the full social benefit:

$$v_{b_2}{}^*(\varrho(b_1, b_2)) - v_{b_1}{}^*(p) = 1 - 0 = MP^+(b_2) - MP(b_1) = MP(b_1, b_2).^3$$

◇

[3] This is a nice illustration of the difference between "competitive conjectures" in the naive sense (that a deviating individual cannot affect any prices) and in our sense. The definition of strong PEDS only says that a deviating buyer cannot affect the prices in S. But a deviating buyer can affect other prices; in particular, notice $\varrho(b_1, b_2) \neq p$. Nevertheless the conjecture $\varrho(b_1, b_2) = (1, 4)$ results from perfect (i.e., highly intense) competition. If a small group of buyers deviates to b_2, intense competition among sellers for a match with one of these high-attribute buyers would make the terms of trade completely determinate: $p(s_1)$ would remain at 1, so each

If μ is strongly competitive, agents' conjectures about the prices of both traded goods *and* potentially traded goods are determinate (unique). Ex post competition for matches among buyers and among sellers is sufficiently intense to eliminate any monopoly bargaining power. In particular, there is no possibility of holdups. Indeed, since any deviating agent rationally conjectures that his private benefit from any deviation will equal the full social benefit, we suspect that perfect competition will give agents good incentives to invest. This suspicion will be examined in Section 6.

5.3. **A marginal trader is necessary for ex post perfect competition.** This subsection develops a necessary condition for perfect competition in assignment models. The condition will prove useful for identifying *imperfectly* competitive populations.

There are *no gains from trade* in μ or, synonymously, μ is trivial, if

$$g(\mu) - \int r d\mu = 0.$$

That is, there are no gains from matching individuals rather than leaving everyone unmatched. Any μ with no gains from trade will be (trivially) perfectly competitive since $p \in \hat{P}(\mu)$ implies $p(s) = r(s)$ for all $s \in S$, so $\hat{P}(\mu)$ is a singleton. Another way to see it is to observe that, in any population with no gains from trade, the unique core utilities are $q^*(a) = r(a)$ for all $a \in A$.

If μ is nontrivial, a necessary condition for perfect competition is the presence of a marginal trader. To introduce this concept, given any (x, p) that is Walrasian for μ, define the set of individuals in productive matches in x (i.e., matches that generate positive gains from trade) as

$$\begin{aligned} A(x_+) = \{s : (b, s) \in \text{supp } x \text{ and } v(b, s) - r(b) - r(s) > 0\} \\ \cup \{b : (b, s) \in \text{supp } x \text{ and } v(b, s) - r(b) - r(s) > 0\}. \end{aligned}$$

The set of individuals who are unmatched or in unproductive matches in x is

$$\begin{aligned} A(x_0) = \{s : (b, s) \in \text{supp } x \text{ and } v(b, s) - r(b) - r(s) = 0\} \\ \cup \{b : (b, s) \in \text{supp } x \text{ and } v(b, s) - r(b) - r(s) = 0\}, \end{aligned}$$

where $r(\mathbf{0}) \equiv 0$.

Roughly, a marginal trader is someone who is indifferent between being matched and not being matched. More precisely, there is a *marginal trader* in (x, p) if for any $\epsilon > 0$, at least one of the following

deviating buyer would be able to fully appropriate the increase in ex post surplus $v(b_2, s_1) - v(b_1, s_1)$.

three conditions holds: (1) There is an individual $a \in A(x_+)$ such that ${v_a}^*(p) - r(s) < \epsilon$ (i.e., some individual in a productive match is almost indifferent between being matched and not being matched); (2a) there is a seller $s \in A(x_0)$ and a buyer $b \in A(x_+)$ such that ${v_b}^*(p) - [v(b,s) - r(s)] < \epsilon$ (i.e., there is a seller that is within ϵ of being a perfect substitute for a productive seller); or (2b) there is a buyer $b \in A(x_0)$ and a seller $s \in A(x_+)$ such that ${v_s}^*(p) - [v(b,s) - r(b)] < \epsilon$. There is a *marginal trader* in μ if, for every (x,p) that is Walrasian for μ, there is a marginal trader in (x,p).

For example, the master-servant example has no marginal trader: $A(x_0) = \emptyset$ since everyone is in a productive match; and when $p(s_1) = v(b_1, s_1)/2$ every individual in $A(x_+)$ enjoys positive gains from trade bounded away from zero, so for this Walrasian selection there is no marginal trader in $A(x_+)$ either. By contrast, if there are more servants than masters, then $p(s_1) = 0$ and all servants in $A(x_+)$ and in $A(x_0)$ are marginal traders. Indeed a sufficient (but not necessary) condition for the presence of a marginal trader is $A(x_+) \cap A(x_0) \neq \emptyset$ since interchangeability implies any individual a in this intersection must satisfy ${v_a}^*(p) - r(a) = 0$ for any $p \in P(\mu)$. The following proposition is due to Joe Ostroy (private communication).

Theorem 5.2. *If μ is perfectly competitive and nontrivial, there must be a marginal trader in μ.*

Proof. Suppose μ is perfectly competitive and nontrivial. Let (x,p) be any Walrasian equilibrium for μ. If for every $\epsilon > 0$ there is an individual $a \in A(x_+)$ such that ${v_a}^*(p) - r(a) < \epsilon$, there is nothing left to prove. Hence suppose the contrary, that there is an $\epsilon > 0$ such that ${v_a}^*(p) - r(a) \geq \epsilon$ for all $a \in A(x_+)$. That is, there is no marginal trader with an attribute in $A(x_+)$. Notice in this case optimization given p implies the attributes in $A(x_+)$ are disjoint from those in $A(x_0)$. Choose any one a in the former set and consider a sequence $\langle y_a^n \rangle$ of groups with attributes close to a and mass approaching zero. Let $\mu^n = \mu - y_a^n$. In each population μ^n, a mass $\|y_a^n\|$ of individuals in $A(x_+)$ now find themselves either unmatched or in matches with individuals in $A(x_0)$. Let $\epsilon' \geq 0$ satisfy ${v_s}^*(p) - [v(b,s) - r(b)] \geq \epsilon'$ for all $s \in A(x_+)$ and $b \in A(x_0)$ and also satisfy ${v_b}^*(p) - [v(b,s) - r(s)] \geq \epsilon'$ for all $b \in A(x_+)$ and $s \in A(x_0)$. Hence in μ^n there is a set of individuals of mass $\|y_a^n\|$ whose gains from trade has fallen by at least ϵ' per capita. Since μ is perfectly competitive, for any $p^n \in P(\mu^n)$, $p^n\ |_S \to p\ |_S$, hence ${v_a}^*(p^n, S) \to {v_a}^*(p, S)$ for all $a \in \text{supp}\ \mu^n$, which implies $\epsilon' = 0$. That is, there must be a marginal trader in $A(x_0)$. □

6. The efficiency of investment equilibria under perfect competition

Throughout this section, let (ν, ϱ) be an investment equilibrium, and let $\mu = \nu_{\mathbf{A}}$. We will call ν *perfectly competitive* if the ex post population μ is perfectly competitive. We will be interested in the efficiency of investment equilibria under perfect competition. Are the three types of inefficient equilibria illustrated in Section 4 all due to the absence of perfect competition?

6.1. **One-person deviations from ν.** Let us call ν *locally efficient* if

$$MP(a, d) \leq c(d, t) - c(a, t), \qquad \forall (a, d) \in A \times d,$$

so the net marginal benefit of any deviation from ν is nonpositive. In other words, no one-person deviation can infinitesimally increase total surplus.

Full appropriation gives individuals good incentives to invest, at least locally.

Proposition 6.1 (local efficiency)**.** *If (ν, ϱ) is a perfectly competitive investment equilibrium, ν is locally efficient.*

Proof. Since ν is an equilibrium allocation, for any $(a, t) \in \text{supp}\ \nu$,

$$v_a{}^*(\varrho(a, a)) - c(a, t) \geq v_d{}^*(\varrho(a, d)) - c(d, t), \ \forall d \in \mathbf{A}.$$

Since μ is perfectly competitive,

$$v_d{}^*(\varrho(a, d)) - v_a{}^*(\varrho(a, a)) = MP(a, d).$$

Combining yields the conclusion. □

Since no one-person deviation can increase total surplus, the only sorts of inefficiencies that can arise under perfect competition must involve coordination failures.

6.2. **Multi-person deviations from ν.** Can any multi-person deviation from ν increase total surplus? We will use the first theorem of welfare economics to analyze this question. The first theorem tells us that if people act as price-takers and markets are complete, any Walrasian outcome is Pareto efficient. More generally, if markets are incomplete, any Walrasian outcome is efficient relative to the set of available markets. This generalization will be useful below because investment equilibria may not be Pareto efficient.

Let $\hat{S} \subset \mathbf{S}$ denote the set of available markets, and let $\hat{S}^0 = \hat{S} \cup \{\mathbf{0}\}$. In the special case $\hat{S} = \mathbf{S}$, markets are complete. Call the allocation ν

feasible relative to $\hat{S}$ if it is feasible and $\mathbf{S} \cap \operatorname{supp} \nu_{\mathbf{A}} \subset \hat{S}$. As an application of the first welfare theorem, we will prove that any Walrasian equilibrium relative to $\hat{S}$ is efficient relative to $\hat{S}$. In preparation, we first introduce the idea of an interim assignment.

Let us call an attribute-type pair (a, t) a *characteristic*. To distinguish a distribution of characteristics $\nu \in M_+(\mathbf{A} \times T)$ from a distribution of attributes $\mu \in M_+(\mathbf{A})$, let's call the former (latter) an interim (ex post) population. Given any interim population ν, a feasible *interim assignment* for ν is a measure $\chi \in M_+(\mathbf{B}^0 \times I \times \mathbf{S}^0 \times J)$ satisfying $\chi(\mathbf{0}, I, \mathbf{0}, J) = 0$, $\chi(E, \mathbf{S}^0, J) = \nu(E)$ for all Borel $E \subset \mathbf{B} \times I$, and $\chi(\mathbf{B}^0, I, F) = \nu(F)$ for all Borel $F \subset \mathbf{S} \times J$. So, null characteristics are not assigned to one another; each buyer $(b, i) \in \operatorname{supp} \nu$ is either assigned to a seller $(s, j) \in \operatorname{supp} \nu$ or to a null characteristic $(\mathbf{0}, j)$; and similarly each seller is either assigned to a buyer or to a null characteristic. Interpret being assigned to a null characteristic as not being matched. Hence $\chi(E, \mathbf{0}, J)$ is the measure of buyers in E that are not matched with any seller. Given any interim population ν and any feasible interim assignment χ for ν, the implied assignment for the ex post population $\mu = \nu_{\mathbf{A}}$ is the measure x given by the marginal of χ on $\mathbf{B}^0 \times \mathbf{S}^0$.

An *(ex ante) Walrasian equilibrium relative to* $\hat{S}$ is a triple (ν, x, p) such that ν is feasible relative to $\hat{S}$, x is the marginal on $\mathbf{B}^0 \times \mathbf{S}^0$ of some χ that is feasible for ν, $p \in C_+(\mathbf{S}^0)$, and

(1) for each buyer $(b, i) \in \operatorname{supp} \nu$ and $(b, i, s, j) \in \operatorname{supp} \chi$:

$$\pi_b(b, s) - c(b, i) \geq \pi_b(b', s') - c(b', i) \quad \forall b' \in \mathbf{B}, \quad \forall s' \in \hat{S}^0.$$

(2) for each seller $(s, j) \in \operatorname{supp} \nu$ and $(b, i, s, j) \in \operatorname{supp} \chi$:

$$\pi_s(b, s) - c(s, j) \geq \pi_s(b', s') - c(s', j), \quad \forall s' \in \hat{S}, \quad \forall b' \in \mathbf{B}^0.$$

So, in an ex ante Walrasian equilibrium, each buyer and seller chooses his attribute and match to maximize his net private benefit subject to the set of available markets.

Lemma 6.1 (relative efficiency)**.** *If (ν, x, p) is ex ante Walrasian relative to $\hat{S}$, then ν is efficient relative to $\hat{S}$, that is,*

$$G(\nu) \geq G(\nu')$$

for all allocations ν' that are feasible relative to $\hat{S}$.

Proof. Let ν' be any feasible allocation relative to $\hat{S}$, and χ' any feasible interim assignment for ν'. For any $i \in I$, let $Q(i) = v_b{}^*(p) - c(b, i)$ for any $(b,i) \in \text{supp}\ \nu$; and for any $j \in J$, let $Q(j) = v_s{}^*(p) - c(s,j)$ for any $(s,j) \in \text{supp}\ \nu$. Optimization implies

$$\int v(b,s) - c(b,i) - c(s,j)\,\mathrm{d}\chi'(b,i,s,j) \leq \int Q(i) + Q(j)\,\mathrm{d}\chi'(b,i,s,j) = \int Q\,\mathrm{d}\mathcal{E},$$

where $c(\mathbf{0},i) \equiv c(\mathbf{0},j) \equiv Q(\mathbf{0}) = 0$. Since $\int Q\,\mathrm{d}\mathcal{E}$ is the net gain realized in the Walrasian equilibrium, there are no attribute choices ν' relative to $\hat{S}$ that lead to a larger net gain. □

Theorem 6.1 (perfect competition implies relative efficiency)**.** *Let (ν, ϱ) be any perfectly competitive investment equilibrium. Let $\mu = \nu_{\mathbf{A}}$ and $S = S(\mu)$. Then there exists an (x,p) such that (ν, x, p) is ex ante Walrasian relative to S. Hence ν is efficient relative to S.*

Proof. Let (x,p) be any ex post Walrasian equilibrium for μ with $p \in \hat{P}(\mu)$. Since μ is perfectly competitive, $\varrho(s,s') = p$ for all $s \in S$ and $s' \in \mathbf{S}$. Hence since (ν, ϱ) is an investment equilibrium, for any $(s,j) \in \text{supp}\ \nu$ and $(b,s) \in \text{supp}\ x$:

$$\pi_s(b,s) - c(s,j) \geq \pi_{s'}(b',s') - c(s',j), \qquad \forall s' \in \mathbf{S}, \quad \forall b' \in \mathbf{B}^0,$$

a fortiori for all $s' \in S^0$. Similarly, ex post perfect competition implies $\varrho(b,b')(s) = p(s)$ for all $b \in B$, $b' \in \mathbf{B}$, and $s \in S$. Hence, since (ν, ϱ) is an investment equilibrium, for any $(b,i) \in \text{supp}\ \nu$ and $(b,s) \in \text{supp}\ x$:

$$\pi_b(b,s) - c(b,i) \geq \pi_{b'}(b',s') - c(b',i), \qquad \forall b' \in \mathbf{B}, \quad \forall s' \in S^0.$$

□

The theorem implies that in any perfectly competitive investment equilibrium, there cannot exist any Felli-Roberts-type investor mismatches because the the ex post distribution of attributes μ will be produced in a cost-minimizing way.

Corollary 6.1 (no investor mismatches under perfect competition)**.** *Let (ν, ϱ) be any perfectly competitive investment equilibrium, and let ν' be any other feasible allocation satisfying $\nu'_{\mathbf{A}} = \nu_{\mathbf{A}}$. Then*

$$\int c\ d\nu' \geq \int c\ d\nu.$$

Proof. Set $\mu = \nu_{\mathbf{A}}$. The relative efficiency of ν implies $G(\nu') \equiv g(\mu) - \int c\,\mathrm{d}\nu' \leq G(\nu) \equiv g(\mu) - \int c\,\mathrm{d}\,\mathrm{d}\nu$. Hence $\int c\,\mathrm{d}\nu' \geq \int c\,\mathrm{d}\nu$. □

The following corollary strengthens Theorem 6.1 by showing any Pareto improvement relative to an investment equilibrium must involve both buyers and sellers deviating outside the support of μ.

Corollary 6.2 (Both sides must go outside the support). *If (ν, ϱ) is a perfectly competitive investment equilibrium, there does not exist a feasible allocation ν' such that $G(\nu') > G(\nu)$ if either $B(\mu') \subset B(\mu)$ or $S(\mu') \subset S(\mu)$, where $\mu = \nu_{\mathbf{A}}$ and $\mu' = \nu'_{\mathbf{A}}$.*

Proof. If $S(\mu') \subset S(\mu)$, the corollary follows immediately from Theorem 6.1. Let ν' be any allocation feasible for $\mathcal{E}$ with $B(\mu') \subset B(\mu)$, and let χ' be any efficient interim assignment for ν'. Let $p \in \hat{P}(\mu)$, and define the payoff function $Q(\cdot)$ as in the proof of Lemma 6.1. Since (ν, ϱ) is a perfectly competitive investment equilibrium for $\mathcal{E}$, for each $j \in \text{supp}\ \mathcal{E}$ and each $s \in \mathbf{S}$,

$$p(s) - c(s, j) \leq Q(j)$$

because seller j conjectures he cannot affect prices p by his choice of attribute s. Analogously, for each buyer $i \in \text{supp}\ \mathcal{E}$ and each $(b, i, s, j) \in \text{supp}\ \chi'$

$$v(b, s) - c(b, i) - p(s) \leq Q(i)$$

since i conjectures prices will remain at p if he does not deviate outside supp μ. Hence

$$\int v(b, s) - c(b, i) - c(s, j)\, \mathrm{d}\chi' \leq \int Q(i) + Q(j)\, \mathrm{d}\chi' = G(\nu).$$

□

Remark 6.1 (efficient market-making). The proof of Corollary 6.2 has some interest in itself. An investment equilibrium can be viewed as the outcome of a two-stage game involving market-making in stage 1: Given sellers' stage-1 investment decisions, only markets for goods $S(\mu)$ will be available to buyers in stage 2. Thus, as CMP1 and CMP2 point out, an investment equilibrium is intimately connected to the literature on competitive equilibria with endogenous market-making, notably Hart (1980), Makowski (1980), and Makowski and Ostroy (1995).

Using the language of Oliver Hart, competitive equilibria with endogenous market-making will be Pareto efficient if a *simultaneous reservation price property* holds: essentially, the prices in the open markets can be extended to a complete price system, one that prices even commodities that are not marketed and still clears all markets. The proof of Corollary 6.2 shows that under strong ex post competition, $p \in \hat{P}$ satisfies this property for sellers; but it may not satisfy this property for buyers. That is, even under perfect competition, only in special cases can we extend $p \in \hat{P}(\mu)$ to a market-clearing price system with complete markets. This gives one insight why perfectly competitive investment equilibria may not by Pareto efficient.

In Makowski (1980) the simultaneous reservation price property is called "no reservation price externalities." The fact that buyers can affect the reservation prices of some unavailable goods even under perfect competition (e.g., recall the definition of strong PEDS or Example 5.1) implies that buyers, if they could somehow communicate their strong desire for some commodity $s \in \mathbf{S} - S$, may be able to create a beneficial reservation price externality for potential sellers of this commodity, inducing the latter to efficiently move outside the support of μ. The fact that, even under perfect competition, there is room for reservation price externalities gives another insight why perfectly competitive investment equilibria may not by Pareto efficient.

When only agents on one side of the market can undertake precontractual investments, a stronger conclusion is possible. In this case perfect competition leads to (unconstrained) Pareto efficiency. Formally, we will say *only sellers can investment* if each buyer type i has one attribute $\omega_i \in \mathbf{B}$ such that $c(\omega_i, i) = 0$ while $c(b, i) = \infty$ for all $b \in \mathbf{B} - \{\omega_i\}$. Similarly, we will say that only buyers can investment if each seller type j has one attribute $\omega_j \in \mathbf{S}$ such that $c(\omega_j, j) = 0$ while $c(s, j) = \infty$ for all $s \in \mathbf{S} - \{\omega_j\}$. There is only *one-sided investment* if only buyers can invest or only sellers can invest.

Corollary 6.3 (Pareto efficiency with one-sided investment)**.** *If (ν, ϱ) is a perfectly competitive equilibrium with only one-sided investment, then ν is Pareto efficient, that is,*

$$G(\nu) \geq G(\nu')$$

for all feasible allocations ν'.

Proof. Let $\mu = \nu_{\mathbf{A}}$ and $\mu' = \nu'_{\mathbf{A}}$. Since only one side of the market can move outside the support of μ without suffering prohibitive costs, any candidate Pareto improvement ν' must satisfy $B(\mu') \subset B(\mu)$ if only sellers can invest or $S(\mu') \subset S(\mu)$ if only buyers can invest. Hence Corollary 6.2 implies $G(\nu') \leq G(\nu)$. □

Corollary 6.2 leaves open the possibility of Pareto inefficient investment equilibria. Such equilibria exist and are easy to construct, as the following supermodular example illustrates. As we will prove in Section 10, in the supermodular model all inefficient investment equilibria involve two-sided *underinvestment*, so both sides of the market must deviate to the right of supp μ to create a Pareto improvement.

Example 6.1 (An underinvestment equilibrium even under perfect competition)**.** Let's continue with the supermodular model in Example 5.1. We now specify individuals' investment costs. Assume there is only one buyer type i with $c(1,i) = 0$ and $c(2,i) = 1.25$. Similarly, there also is only one seller type j with $c(\cdot, j) = c(\cdot, i)$.

Recall there are twice as many buyers as sellers. So the ex ante population is

$$\mathcal{E} \equiv (\mathcal{E}(i), \mathcal{E}(j)) = (2N, N).$$

We have already seen that the ex post population $\mu = (2N, N, 0, 0)$ in which all individuals choose attribute 1 is perfectly competitive because of the intense competition for matches from the long side of the market (the buyers' side). We now show that there is an investment equilibrium (ν, ϱ) in which $\nu_{\mathbf{A}} = \mu$.

From our previous analysis we know that to support ν as an equilibrium, individuals must conjecture $p = (1, 2)$ if there is no deviation; and, if there is a deviation, rationality dictates $\varrho(s_1, s_2) = p$ while $\varrho(b_1, b_2) = (1, 4)$. It is straightforward to check that (ν, ϱ) is indeed an equilibrium: Since $v_{s_1}{}^*(p) = 1 > v_{s_2}{}^*(p) - c(s_2, j) = 2 - 1.25$, no seller has an incentive to deviate. And since $v_{b_1}{}^*(p) = 0 > v_{b_2}{}^*(p) - c(b_2, i) = 1 - 1.25$, no buyer has an incentive to deviate either.

But this equilibrium is not efficient: A coordinated deviation by a group of buyers to b_2 and and an equal number of sellers to s_2 would increase total surplus. In particular the allocation ν' such that $\nu'(b_1) = \nu'(b_2) = \nu'(s_2) = N$ leads to a net gain $G(\nu') = 1.5N > G(\nu) = N$. In the underinvestment equilibrium, because of the complementarity between buyers' and sellers' attributes, no buyer sees a profit to choosing b_2 when no seller chooses s_2, and vice versa — even though there is perfect competition in μ, hence any individual's private benefit from deviating equals the full social benefit. $\diamond$

For some applications, one may want to focus exclusively on the beneficial incentive effects of ex post perfect competition for solving holdup problems, and hence one may want to eliminate the annoyance of possible coordination problems. Fortunately this is possible. In Section 7 we introduce a special case of our investment model, the housing model, which generically delivers Pareto efficient investment equilibria — even with two-sided investment.

7. Genericity of perfect competition

GOZ2 proves that when attributes are endowed, not produced, perfect competition is generic: the set of perfectly competitive ex post populations is a dense G_δ nsubset of $M_+(\mathbf{A})$. In this section we examine to what extent this conclusion extends to the current model, in which the ex post population is chosen, not exogenously given.

To frame the discussion, let's begin with the following simple example.

Example 7.1 (Are master-servant examples nongeneric?)**.** Consider the supermodular model in which $\mathbf{B} = \mathbf{S} = \{0, 1\}$ and $v(b, s) = bs$. Let $b_0 = s_0 = 0$ and $b_1 = s_1 = 1$. Hence any ex post population μ can be represented as a vector

$$\mu = \Big(\mu(b_0), \mu(s_0), \mu(b_1), \mu(s_1)\Big).$$

Analogous to Example 4.1, the perfectly competitive populations include all μ in which there are more buyers than sellers with attribute 1 or vice versa, that is, all populations in which $\mu(b_1) > \mu(s_1)$ or $\mu(s_1) > \mu(b_1)$. Hence the perfectly competitive populations are open and dense in R^4_+, while the imperfectly competitive ones form a set of measure zero. This accords with the general genericity result in GOZ2.

But what attributes will be produced in an investment equilibrium? Assume that there is only one type of buyer i with $c(0, i) = 0$ and $c(1, i) = .25$. Similarly, there is only one type of seller j with $c(\cdot, j) = c(\cdot, i)$. The distribution of types in the ex ante population is described by the vector

$$\mathcal{E} = (\mathcal{E}(i), \mathcal{E}(j)) \gg 0.$$

We will show that for this population there is a continuum of possible investment equilibria (ν^α, ϱ) parameterized by α in the interval Ω, where

$$\Omega = [0, \min\{\mathcal{E}(i), \mathcal{E}(j\}].$$

For each $\alpha \in \Omega$, the ex post equilibrium distribution of attributes will be

$$\mu^\alpha = (\mathcal{E}(i) - \alpha, \mathcal{E}(j) - \alpha, \alpha, \alpha).$$

There is a trivial, $\alpha = 0$ equilibrium in which $\mu = (\mathcal{E}(i), \mathcal{E}(j), 0, 0)$: If no buyer chooses attribute b_1, it would be foolish for any seller to spend $c(1, j) > 0$ on acquiring attribute s_1, and vice versa. Similarly, there is never an investment equilibrium in which $\mu(s_1) \neq \mu(b_1)$ since someone with attribute 1 would remain unmatched and hence suffer a negative net payoff.

Now consider any population μ^α for $\alpha > 0$. This population is not perfectly competitive since $\hat{P}(\mu^\alpha)$ is not a singleton; indeed the reader can easily check that

$$\hat{P}(\mu^\alpha) = \{p \equiv (p(s_1), p(s_2) : p = (0, \gamma) \text{ and } \gamma \in [0,1]\}.$$

Equivalently, one can see that μ^α is not perfectly competitive since it does not satisfy PEDS: Just as in the master-servant example, the equilibrium price of s_1 would jump up to 1 if there were even a few more buyers of type b_1 (with positive mass) or a few less sellers of type s_1, and it would drop down to 0 if there were even a few less buyers of type b_1 or a few more sellers of type s_1. Consequently, in any investment equilibrium (ν^α, ϱ) for $\alpha > 0$, the only possible rational conjectures after a deviation are

$$\varrho(b_0, b_1) = \varrho(s_1, s_0) = (0, 1) \quad \text{and} \quad \varrho(b_1, b_0) = \varrho(s_0, s_1) = (0, 0).$$

The reader can easily verify that (ν^α, ϱ) is an investment equilibrium if, in addition to the above conjectures, everyone conjectures the equilibrium price will equal

$$p = (0, \gamma) \text{ for some } \gamma \in [.25, .75]$$

if there is no deviation from μ^α.

Summarizing, by focusing on the ex post populations that can arise in an equilibrium, we have found that the only perfectly competitive population is the trivial one with no gains from trade, μ^α with $\alpha = 0$. So while the perfectly competitive populations are open and dense in R^4_+, while the imperfectly competitive ones are of measure zero in R^4_+, when attention is restricted to the populations that can occur in an investment equilibrium, the opposite holds: the perfectly competitive populations are of measure zero in Ω, while the imperfectly competitive populations are open and dense in Ω.

In this example, individuals will rationally choose either to be in a master-servant-type economy with $\alpha > 0$, or in a trivial — but safe — economy without gains from trade. In particular, efficiency requires individuals to choose $\alpha = \min\{\mathcal{E}(i), \mathcal{E}(j)\} > 0$. Since this population is not perfectly competitive, to avoid hold up problems and achieve efficiency, each buyer b_1 and seller s_1 must bargaining vigorously for at least 25% of the value created in his match. Competition cannot efficiently solve the holdup problem in this example. Indeed, for some rational conjectures, efficiency cannot be achieved. For example, suppose sellers expect buyers to be tough bargainers, so they expect the price vector $p_0 = (0, 0)$ in any ex post population. Then, expecting a

100% holdup at the hands of buyers, no seller will invest in attribute s_1. $\diamond$

Let's now generalize the lesson of the example, checking for its genericity. Let $\mathcal{F}$ denote the family of all investment models, a family with the following exogenous parameters:

$$\langle I, J, \mathbf{B}, \mathbf{S}, c, v, \mathcal{E} \rangle.$$

We say that an investment model permits the *possibility of inaction* if there are two special attributes $0_b \in \mathbf{B}$ and $0_s \in \mathbf{S}$ with no cost but also not benefits from production, that is,

$$v(0_b, s) = v(b, 0_s) = c(0_b, i) = c(0_s, j) = 0, \quad \forall s, b, i, j.$$

The possibility of inaction ensures all individuals will have nonnegative payoffs in any investment equilibrium. Let us say there are *no outside options* relative to matching if $r(\cdot) \equiv 0$. Finally, let us say that *all productive investments are costly* if there is a constant $c_{\min} > 0$ such that for all $(b, i) \in \mathbf{B} \times I$ with $b \neq 0_b$, and for all $(s, j) \in \mathbf{S} \times J$ with $s \neq 0_s$,

$$c(b, i) \geq c_{\min} \quad \text{and} \quad c(s, j) \geq c_{\min}.$$

Interpret $c_{\min}$ as a fixed setup cost required to produce any nonzero, hence productive attribute. We emphasize that $c_{\min}$ can be very small. Let $\mathcal{F}^0$ denote the family of all investment models that have these three properties. Notice the economy in Example 7.1 belongs to $\mathcal{F}^0$. Indeed, all supermodular models, by definition, possess at least one of the three properties, namely $r(\cdot) \equiv 0$.

We will show that any nontrivial investment equilibrium for any economy in $\mathcal{F}^0$ will be imperfectly competitive. Thus the message of Example 7.1 extends far beyond a 1-type, 2-attribute supermodular example. Recall we call an investment equilibrium (ν, ϱ) *nontrivial* if the ex post gains exceed individuals' reservation values, that is, $g(\nu_{\mathbf{A}}) - \int r \, \mathrm{d}\nu_{\mathbf{A}} \geq 0$. Let PC denote the subset of $M_+(\mathbf{A})$ consisting of all perfectly competitive ex post populations. The complementary set we call the imperfectly competitive populations.

Proposition 7.1 (Impossibility of ex post perfect competition)**.** *For the family of investment models $\mathcal{F}^0$, any nontrivial investment equilibrium will be imperfectly competitive. That is, if (ν, ϱ) is a nontrivial investment equilibrium for any economy in $\mathcal{F}^0$, then $\nu_{\mathbf{A}} \notin PC$.*

Proof. Suppose (ν, ϱ) is any investment equilibrium for an economy in $\mathcal{F}^0$, the vector $p \equiv \varrho(a, a)$ is the conjectured equilibrium price function if there is no deviation from $\mu \equiv \nu_{\mathbf{A}}$, and (x, p) is an ex post Walrasian equilibrium for μ.

Assume there are positive gains from trade in μ. We will show there is no marginal trader in μ, hence μ is not perfectly competitive (Theorem 5.2). Let (b, s) be any couple in supp x satisfying $b \in B$, $s \in S$, and $v(b, s) > 0$. By assumption, the cost of producing the two attributes b and s was at least $c_{\min}$ per person, where $c_{\min} > 0$. Since these attributes were chosen in equilibrium and there was the possibility of inaction, $v_s{}^*(p) = p(s) \geq c_{\min} > 0$ and $v_b{}^*(p) = v(b, s) - p(s) \geq c_{\min} > 0$. Thus there is no marginal trader in $A(x_+)$: everyone in a productive match enjoys ex post gains bounded away from zero. Since the gains from matching with someone with the zero attribute is zero, there also is no marginal trader in $A(x_0)$. Hence there is no marginal trader in μ. □

Proposition 7.1 shows that the family $\mathcal{F}^0$ is most inhospitable to perfect competition. We emphasize that this family includes all supermodular models with the possibility of inaction and costly productive investment; so supermodular models with costly investment will also be inhospitable. But if we move outside the restrictive confines of models in which there are no outside options relative to matching, i.e., if we relax the strong assumption that $r(\cdot) \equiv 0$, the ex post competitive forces become much stronger, as the next example will illustrate. The example is framed inside a "housing model," another special case of our general investment model, one closely related to the model studied in GOZ1 and GOZ2. Unlike the supermodular model, the housing model is sufficiently rich to permit individuals to have outside options relative to matching.

In the *housing model*, we add to the data of the ex ante economy a compact metric space of housing types H, and we define the set of possible seller attributes as

$$\mathbf{S} = \{s = (h, r) : h \in H \text{ and } r \in [0, \overline{V}]\} \cup \{0_s\}.$$

A seller with the zero attribute, 0_s, is interpret as someone who has decided not to build a house (inaction). More interesting, a seller with attribute $s = (h, r)$ is interpreted as a seller who has built a house of type h and has reservation value r for his house. Sometimes we will denote a seller $s = (h, r)$ more compactly by s_{hr}. We take the distance between any two sellers $s = (h, r)$ and $s' = (h', r')$ as the sum of the distance between h and h' and the (Euclidean) distance between r and r'; and we assume the zero attribute 0_s is isolated (there is an open set containing 0_s and no other attribute).

A seller's type j specifies what house in H he can produce and his reservation value for his house, hence the set of possible seller types in the housing model is

$$J = \{j = (h, r) : h \in H \text{ and } r \in [0, \overline{V}]\}.$$

A seller of type $j = (h, r)$ can either choose to remain inactive (produce the zero attribute) or to build a house of type h which he will value at r. To keep the model simple, we do not permit a type j seller to choose which type house $h \in H$ he will build.

Let $c^H \in C_+(H)$. If a seller of type $j = (h, r) \equiv j_{hr}$ decides to produce a house h, his production cost will be $c^H(h)$, while inaction 0_s is costless. Hence in the housing model

$$c(s, j_{hr}) = \begin{cases} 0 & \text{if } s = 0_s \\ c^H(h) & \text{if } s = (h, r) \\ \infty & \text{otherwise.} \end{cases}$$

Each buyer has no reservation value for his attribute (so $r(b) = 0$ for all $b \in \mathbf{B}$) and only cares about the house he acquires, not his seller's reservation value (so there is a function $v^H : \mathbf{B} \times H \to \mathbb{R}_+$ such that $b \in \mathbf{B}$ and $s = (h, r) \Rightarrow v(b, s) = v^H(b, h)$).

Summarizing, the family of housing models, denoted $\mathcal{F}^{\text{hm}}$, has the following parameters:

$$\langle H, I, J, \mathbf{B}, \mathbf{S}, c^H, c, v^H, v, \mathcal{E} \rangle.$$

So in a housing model H, c^H, and v^H are added as data; J and $\mathbf{S}$ are built upon H as above; while c and v are built upon c^H and v^H as above.

The housing model still permits ex post populations in which each seller has zero reservation value for his house. (Let the ex ante population $\mathcal{E}$ have the property $j \in \operatorname{supp} \mathcal{E}$ implies $j = (h, 0)$.) But a typical ex post population will include some sellers with positive reservation values for their houses, and indeed sellers with heterogeneous reservation values. The interpretation is that, while buyers view all houses of any given type h as perfect substitutes, a seller may view his house h as special.

Thus, relative to the family $\mathcal{F}^0$, the housing model adds an entire extra dimension. To illustrate, suppose $H = [0, 1]$ and $\overline{V} = 1$, then the commodity space in a model without reservation values is $[0, 1] \subset \mathbb{R}$, while the commodity space in the housing model is $[0, 1] \times [0, 1] \subset \mathbb{R} \times \mathbb{R}$. The consequence is that the housing model is much more hospitable to perfect competition, even when all productive investments are costly.

Example 7.2 (Enough outside options lead to perfect competition)**.** Consider the following housing model variant of Example 7.1. Analogous to Example 7.1, there is only one type of house $H = \{1\}$,

$v^H(b,h) = bh$, there are two buyer attributes $\mathbf{B} = \{0,1\}$, and one type of buyer i with $c(0,i) = 0$ and $c(1,i) = .25$. Let $\overline{V} = 2$; hence, since there is only one type of house, we can identify a seller with his reservation value and set $J = [0,2]$. Analogous to Example 7.1, assume $c^H(h) = .25$. So all sellers have the same cost of producing a house, but a seller of type j will have a reservation value of j for his house.

Example 7.1 can be viewed as analyzing this economy for an ex ante population $\mathcal{E}'$ in which all $j \in \text{supp } \mathcal{E}'$ have reservation value 0. Now we will analyze the economy for an ex ante population $\mathcal{E}$ in which sellers' reservation values are uniformly distributed on [0,2]. We take the mass of sellers as $\mathcal{E}(J) = 2$, so we can imagine there is one seller of each type j. The mass of buyers is $\mathcal{E}(i) \equiv \beta$, a fixed nonnegative number. We will see that, as long as $\beta \geq .25$, any investment equilibrium in this economy will be perfectly competitive. The intuition is that, in any investment equilibrium, each seller with reservation value $r \geq .25$ will build a house even if he does not expect to be matched — because he has a personal value for his house exceeding his cost of production $c^H = .25$. Thus for any $\beta \geq .25$, in any investment equilibrium, there will be a marginal seller whose presence guarantees the equilibrium price function will be unique.

Let $b_0 = 0$ and $b_1 = 1$; and set $s_{hr} \equiv s_r$. If $\beta \geq .75$, in the unique investment equilibrium a mass $\mu(b_1) = .75$ of buyers invest in attribute b_1 while the remainder choose b_0. Each seller builds a house. Then in the ex post assignment game all sellers j with reservation values $j \in [0.75]$ match with a buyer b_1 (sell their houses), while all sellers with $j \in (.75, 2]$ remain unmatched (enjoy living in their houses). The unique price function in $\hat{P}(\mu)$ that supports the ex post equilibrium is $p(0_s) = 0$,

$$p(s_r) = .75 \text{ for all } r \leq .75,$$

and $p(s_r) = r$ for all $r > .75$. All individuals rationally conjecturing that prices will remain at p if there is any deviation. The equilibrium is perfectly competitive because there is a marginal trader, the seller of type $j = .75$, whose presence ensures p is the only price function in $\hat{P}(\mu)$. By contrast, recall in the equilibrium of Example 7.1 there were no marginal traders, hence $\hat{P}(\mu)$ contained a continuum of possible equilibrium price functions.

If $.25 \leq \beta < .75$, the unique investment equilibrium is similar except that all buyers invest in attribute b_1. Again each seller builds a house. Then in the ex post assignment game all sellers with reservation values $j \in [0, \beta]$ match with a buyer b_1, while all other sellers remain unmatched. The unique price function in $\hat{P}(\mu)$ which supports this ex post equilibrium is $p(0_s) = 0$,

$$p(s_r) = \beta \text{ for all } r \leq \beta,$$

while $p(s_r) = r$ for all $r > \beta$. In the investment equilibrium all individuals rationally conjecture that prices will remain at p if there is any deviation. Again the equilibrium is perfectly competitive because of the presence of a marginal trader; this time it's the seller with reservation value β.

By contrast, if $\beta < .25$, some but not all investment equilibria will be perfectly competitive. In equilibrium all buyers invest in attribute b_1, but only a mass β of all sellers with reservation values less than .25 build a house, the remainder choose inaction. Then in the ex post assignment game all sellers with reservation values less than .25 match with a buyer b_1, while all other sellers remain unmatched. The only price function in $\hat{P}(\mu)$ that supports any of these investment equilibria is $p(0_s) = 0$,

$$p(s_r) = .25 \text{ for all } r \leq .25,$$

while $p(s_r) = r$ for all $r > .25$.

The investment equilibria when $\beta < .25$ need not be perfectly competitive, it all depends on which sellers with reservation values less than .25 decide to build a house (enter the market). If the sellers with reservation values in $[.25 - \beta, .25)$ enter — which we will call the *regular* case, — $\hat{P}(\mu) = \{p\}$, a singleton. The equilibrium is perfectly competitive because of the presence of a marginal trader; this time it's the seller with reservation value .25. But if, for example, the sellers with reservation values in $[0, \beta]$ enter, then any price $\hat{p}(s_r) \in [\beta, .25]$ is ex post Walrasian for the commodities s_r with $r < .25$ (with the prices of all other commodities remaining at their p levels). So competition will not uniquely determine ex post prices — even though $p(s_r) = .25$ is the only price for a traded house that is consistent with the existence of an investment equilibrium. The intuition is that, in this latter entry scenario, even the entrant with the highest outside option (among those with $r < .25$) gets more ex post utility from being matched with a buyer at $p(s_r) = .25$ than from not being matched and hence obtaining his outside option utility level β. The difference, $.25 - \beta$, measures the ex post bilateral monopoly surplus to bargain over, reflected in the multiplicity of possible ex post Walrasian prices. Notice if the set of entrants with reservation values $r < .25$ is not connected to the set of sellers with $r \in [.25, \overline{V}]$, entrants run the risk of being held up because there is no marginal seller whose presence pins down ex post prices. Thus there is a reason to expect the "regular case" to prevail.

Summarizing, even when $\beta < .25$, all regular investment equilibria are perfectly competitive. $\diamond$

Compared to Example 7.1, the introduction of reservation values opens the possibility for ex post perfect competition. Let's now generalize and consider the family of all housing models $\mathcal{F}^{\mathrm{hm}}$, checking for the genericity of the example. Unlike Example 7.2, the set H may now contain a continuum of housing types, not just one. But analogous to Example 7.2, any seller of type j_{hr} will still produce a house of type h provided $c^H(h) < r$ — even if he does not expect to be matched with any buyer.

An ex ante population $\mathcal{E}$ satisfies the *no gaps* condition if, for every $h \in H$, the set

$$\{r : j \equiv (h,r) \in \text{supp } \mathcal{E}\} = [0, \overline{V}].$$

Call an investment equilibrium (ν, ϱ) *regular* if, whenever $j_{hr}, j_{hr'} \in$ supp $\mathcal{E}$ and $r < r'$, then $s_{hr} \in$ supp μ implies $s_{hr'} \in$ supp μ, where $\mu \equiv \nu_{\mathbf{A}}$. Generalizing Example 7.2, we will show the no gaps condition guarantees any regular investment equilibrium will be perfectly competitive. No-gaps-type conditions have been previously used in GOZ2, CMP2, and Kamecke (1990) to ensure the unicity of the core in continuum assignment models.

Theorem 7.1 (no gaps is sufficient for ex post perfect competition)**.** *In the housing model, if the ex ante population $\mathcal{E}$ satisfies the no gaps condition, every regular investment equilibrium for $\mathcal{E}$ will be perfectly competitive.*

Proof. Suppose $\mathcal{E}$ satisfies the no gaps condition, and (ν, ϱ) is any regular investment equilibrium for $\mathcal{E}$. Let $\mu = \nu_{\mathbf{A}}$, and let x be an efficient assignment for μ. We will show that $\hat{P}(\mu)$ is a singleton, hence μ is perfectly competitive.

If a commodity $s \in S(\mu)$ is not traded in x (in the sense that s is not matched by x with any buyer in $B(\mu)$), then $p \in \hat{P}(\mu)$ implies $p(s) = r(s)$. More interesting, consider now commodities $s \in S(\mu)$ that are traded in x, in the sense that $(b, s) \in$ supp x for some $b \in B(\mu)$. For any $p \in \hat{P}(\mu)$, interchangeability implies (x, p) is ex post Walrasian for μ. Hence optimization by buyers implies all commodities $s = s_{hr}$ that are traded must sell for the same price, say p_h. We claim $p_h = {p_h}^*$, where

$${p_h}^* = \sup\{r : s_{hr} \in S(\mu) \text{ and } s_{hr} \text{ is traded in } x\}.$$

If $p_h < {p_h}^*$, there would be sellers $s_{hr} \in S(\mu)$ who trade in x, yet do not want to trade under p since $p_h < r < {p_h}^*$. If $p_h > {p_h}^*$, no gaps plus regularity imply there would be sellers $s_{hr} \in S(\mu)$ who do not trade in x yet want to trade under p since $p_h > r \geq p_h{}^*$. Combining, we conclude that $p \in \hat{P}(\mu)$ implies p is unique, hence μ is perfectly competitive. □

Theorem 7.2 (Genericity of ex post perfect competition)**.** *For the housing model, the set of ex ante populations $\mathcal{E}$ satisfying the no gaps condition is a dense G_δ subset of $M_+(T)$ in both the norm and weak* topologies.*

Proof. Let $\alpha \in M_+(T)$ satisfy supp $\alpha = J$ and $\|\alpha\| = 1$. The measures $\epsilon\alpha$ satisfy the no gaps condition for all $\epsilon > 0$. Hence for any $\mathcal{E} \in M_+(T)$, the ex ante population $\mathcal{E}+\epsilon\alpha$ satisfies the no gaps condition. Since ϵ can be arbitrarily small, it follows that economies satisfying the condition are norm-dense in $M_+(T)$. A fortiori, they are also weak* dense.

Let $B(h, r, 1/n)$ be a weak*-open (hence norm-open) ball in $H \times [0, \overline{V}]$ with center (h, r) and radius $1/n$. For any population $\mathcal{E} + \epsilon\alpha$, let $O(\mathcal{E} + \epsilon\alpha, 1/n)$ be a weak* open set in $M_+(T)$ that contains the population $\mathcal{E} + \epsilon\alpha$ and has the property that for any economy E in this set

$$B(h, r, 1/n) \cap \{(h, r) \in \text{supp } E\} \neq \emptyset, \quad \forall (h, r) \in J.$$

Roughly speaking, all populations in this open set have at most $\frac{1}{n}$-sized gaps, hence they are almost perfectly competitive. Such an open set exists, otherwise there would be a sequence of populations $\langle E_n \rangle$ approaching $\mathcal{E} + \epsilon\mathbf{A}$ in the weak* topology, yet the support of E_n contains no seller types in the ball $B(h, r, 1/n)$ even though the support of $\mathcal{E} + \epsilon\mathbf{A}$ includes all types in this ball. Let

$$W_n = \bigcup_{\mathcal{E} \in M_+(T), \epsilon > 0} O(\mathcal{E} + \epsilon\alpha, 1/n).$$

For each n, the set W_n is open and dense in $M_+(T)$ (in either the weak* or norm topologies). Hence Baire's Theorem implies

$$\bigcap_{n=1}^{\infty} W_n$$

is dense. This set includes only economies with no gaps. □

Thus a dramatic reversal occurs relative to models without reservation values: in the housing model, whether or not productive investments are costly, all regular investment equilibria in a generic set of ex ante populations will be perfectly competitive.

The reader may have noticed that the regular equilibria in Example 7.2 are not only perfectly competitive, they are also Pareto efficient.

This is a general phenomenon, as we are about to prove. Thus, as mentioned in Section 6, the housing model is convenient for applications in which one wants to avoid (at least generically) inefficient equilibria due to complementary problems.

Theorem 7.3 (Genericity of efficient investment equilibria)**.** *In the housing model, any regular investment equilibrium for any population $\mathcal{E}$ satisfying the no gaps condition will be Pareto efficient. That is, if (ν, ϱ) is any such equilibrium and ν' is feasible for $\mathcal{E}$, then $G(\nu) \geq G(\nu')$.*

Proof. Let (ν, ϱ) be any regular investment equilibrium for $\mathcal{E}$, let $\mu = \nu_{\mathbf{A}}$, and let (x, p) be any ex post equilibrium for μ with $p = \varrho(a, a)$, hence $p \in \hat{P}(\mu)$. For any type of house h, let $p_h = \inf_r p(s_{hr})$. Recall from the proof of Theorem 7.1 that if houses of type h are traded in x, they will all sell for p_h.

The no gaps condition implies the set of sellers of houses of type h in the support of μ is

$$S_h = \{s_{hr} : c_h \leq r \leq \overline{V}\}.$$

As in the proof of Theorem 7.1, if h is traded in x, let

$${p_h}^* = \sup\{r : s_{hr} \in S_h \text{ and } s_{hr} \text{ is traded in } x\} = p_h.$$

If $S_h = \emptyset$, set ${p_h}^* = \overline{V}$; while if $S_h \neq \emptyset$ and h is not traded in x set

$${p_h}^* = \inf\{r : s_{hr} \in S_h\} = c_h.$$

Define the price function p^{**} with domain $\mathbf{S}^0$ by $p^{**}(s_{hr}) = {p_h}^*$ for all $s_{hr} \in \mathbf{S}$ and $p^{**}(0_s) = p^{**}(\mathbf{0}) = 0$. We will show that (ν, x, p^{**}) is ex ante Walrasian relative to $\mathbf{S}$ (complete markets).

We first show $p^* \in C_+(H)$, hence $p^{**} \in C_+(\mathbf{S}^0)$. In preparation observe that for any h traded in x,

$$p_h = {p_h}^* \geq c_h$$

while for any h that is not traded in x,

$$p_h \leq {p_h}^* = c_h.$$

[In particular, our normalization implies $p_h = \sup_{b \in B} v^H(b, h) - {v_b}^*(p)$.] Now suppose $\langle h_n \rangle$ approach h_0. If each h_n and h_0 are traded in x, then $p(h_n) \to p(h_0)$ implies ${p_{h_n}}^* \to {p_{h_0}}^*$. If each h_n and h_0 are not traded in x, then $c_{h_n} \to c_{h_0}$ (which follows from the continuity of c_h) implies ${p_{h_n}}^* \to {p_{h_0}}^*$. If each h_n is not traded in x but h is, then $p_{h_n} \leq {p_{h_n}}^* = c_{h_n}$ implies $p_{h_0} \leq c_{h_0}$. But since h is traded, $p_{h_0} \geq c_{h_0}$, hence $p_{h_0} = {p_{h_0}}^* = c_{h_0} = \lim c_{h_n} = \lim {p_{h_n}}^*$. Finally, if each h_n is traded but h is not, then $p_{h_n} = {p_{h_n}}^* \geq c_{h_n}$ for all n implies $p_{h_0} = \lim {p_{h_n}}^* \geq c_{h_0}$. But since h is not traded, $p_{h_0} = c_{h_0}$. Hence $\lim {p_{h_n}}^* = c_{h_0} = {p_{h_0}}^*$.

Since (ν, ϱ) is a perfectly competitive investment equilibrium, for any $j \in \text{supp } \mathcal{E}$ and $(s, j) \in \text{supp } \nu$,

$$v_s{}^*(p) - c(s, j) \geq v_d{}^*(p) - c(d, j), \quad \forall d \in \mathbf{S}.$$

The same inequality holds if p is replaced by p^{**}: Although $p^{**} \geq p$ (with equality for commodities traded in x), no seller will want to deviate from inactivity to production because $p^{**}(s_{hr}) \in \{p_h, c_h\}$ for any house h.

Similarly, since (ν, ϱ) is an investment equilibrium, for any $i \in \text{supp } \mathcal{E}$ and $(b, i) \in \text{supp } \nu$,

$$v_b{}^*(p) - c(b, i) \geq v_d{}^*(\varrho(b, d)) - c(d, i), \quad \forall d \in \mathbf{B}.$$

The inequality continues to hold if p is replaced by p^{**} since the latter equals the former for 0_s, $\mathbf{0}$, and all commodities s_{hr} that are traded in x in the sense that $(b, s_{hr}) \in \text{supp } x$ for some $b \in B$. Now replace $\varrho(b, d)$ by p^{**}. We claim the inequality still holds. For any house h that is not traded in x and any commodity $s = s_{hr}$, $\varrho(b, d)(s) \leq \overline{V} = p^{**}(s)$, hence

$$v(d, s) - \varrho(b, d)(s) \geq v(d, s) - p^{**}(s).$$

On the other hand, for any h that is traded in x, $\min_r \varrho(b, d)(s_{hr}) \leq p_h = p_h{}^*$, hence $\max_r\{v(d, s_{hr}) - \varrho(b, d)(s_{hr})\} \geq v^H(d, h) - p_h{}^* = v(d, s_{hr}) - p^{**}(s_{hr})$. So

$$v_b{}^*(p) - c(b, i) \geq v_d{}^*(p^{**}) - c(d, i), \quad \forall d \in \mathbf{B}.$$

□

8. Some asymptotic results

The distribution of attributes in a finite population can be summarized by a counting measure $m \in M_+(\mathbf{A})$. Such a measure has finite support, $m(a)$ equals the number of individuals in the population with each attribute $a \in \mathbf{A}$, and $m(\mathbf{A}) \equiv \|m\|$ equals the number of individuals in the population. It will be helpful to normalize finite populations to have unit mass, which leads to the measure $m/\|m\| \in M_+(\mathbf{A})$. Viewed abstractly, the measure m can be interpreted as representing either a finite population or a population with a nonatomic continuum of individuals with finite support. Similarly, the measure $m/\|m\|$ can be interpreted as representing a normalized finite population or a population with a nonatomic continuum of individuals.

A key observation is that $P(m) = P(m/\|m\|)$, that is, the Walrasian equilibrium prices for the population m and the normalized population are identical, regardless of whether one interprets them as representing finite or continuum populations. Hence, since in assignment models the core always coincides with the set of Walrasian equilibria — regardless of the number of agents, — the economies m and $m/\|m\|$ will have identical cores.

Now let $\langle m^n \rangle$ denote a sequence of finite populations with $\|m^n\| \to \infty$ (the finite economies are getting larger) and

$$\text{supp } m^n \subseteq \text{supp } m^{n+1} \subseteq \text{supp } m^{n+2} \subseteq \cdots$$

(the supports are weakly increasing). An important special case involves replication, where supp m^n is identical for all n, only the number of buyers and sellers with each attribute increases with n. We will call the sequence of populations $\langle m^n \rangle$ *asymptotically perfectly competitive* if the sequence $\langle m^n/\|m^n\| \rangle$ converges in norm to some perfectly competitive population μ. If $\langle m^n \rangle$ is asymptotically perfectly competitive, then any finite population m^n with n sufficiently large will be nearly perfectly competitive: no individual in m^n, by deviating, will be able to influence equilibrium prices by more than ϵ. To see this, the following restatement of strong PEDS is useful.

Theorem 8.1 (approximately strong PEDS)**.** *If μ is perfectly competitive and $p \in \hat{P}(\mu)$, then for any $\epsilon > 0$ there exists a norm-open set $O \subset M_+(\mathbf{A})$ such that for any $\mu' \in O$ and any $p' \in \hat{P}(\mu')$, if* supp $\mu' \subset B \cup \mathbf{S}$ *then*

$$\|p' - p\| < \epsilon,$$

while if supp $\mu' \subset \mathbf{B} \cup S$ *then*

$$\|p' \mid_S - p \mid_S\| < \epsilon.$$

Proof. Suppose supp $\mu' \in B \cup \mathbf{S}$. If the claim in the theorem were false, there would be an $\epsilon > 0$, a sequence $\mu^n \to \mu$ (in norm), and a selection $p^n \in \hat{P}(\mu^n)$ such that $\|p^n - p\| > \epsilon$ for all n, contradicting the definition of strong PEDS. The proof is similar if supp $\mu' \in \mathbf{B} \cup S$ except one focuses on the price functions on the restricted domain S. $\square$

To apply the theorem, fix an $\epsilon > 0$, and let O be the open set of populations described in the theorem. Let $\langle m^n \rangle$ be asymptotically perfectly competitive, and let $\langle \mu^n \rangle$ be the corresponding sequence of normalized economies, where $\mu^n \to \mu$. Choose any attribute a contained in supp m^n. If an individual with attribute a deviates to some $d \in \mathbf{A}$, the distribution of attributes in the finite population m^n would change to

$$m^n_{a,d} = m^n - \delta_a + \delta_d,$$

and the corresponding normalized population would change to $\mu^n_{a,d} \equiv m^n_{a,d}/\|m^n\|$. Notice for any $(a,d) \in A(m^n) \times \mathbf{A}$ satisfying $d \neq a$,

$$\|\mu^n_{a,d} - \mu^n\| = 2/\|m^n\|.$$

Hence the triangle inequality implies

$$\|\mu^n_{a,d} - \mu\| \leq \|\mu^n_{a,d} + \mu^n\| + \|\mu^n - \mu\| = 2/\|m^n\| + \|\mu^n - \mu\|.$$

Since $\mu^n \to \mu$ and $\|m^n\| \to \infty$, it follows that there is an integer N sufficiently large that for all $n > N$,

$$\{\mu^n_{a,d} : (a,d) \in A(\mu) \times \mathbf{A}\} \subset O.$$

Since $\hat{P}(\mu^n) = \hat{P}(m^n)$, it follows that any deviating individual in any (finite) economy m^n with $n > N$ cannot affect equilibrium prices by more than ϵ (with the behaviorally-insignificant caveat that a deviating buyer may be able to affect the reservation prices for some unavailable goods).

Our conclusion can be re-phrased in terms of the benefit to any individual in deviating from price-taking behavior in an investment equilibrium. Recall that in a continuum economy, if (μ, ϱ) is a perfectly competitive investment equilibrium, each infinitesimal individual will rationally act as a price-taker since he cannot influence equilibrium prices by any deviation (with the behaviorally-insignificant caveat mentioned above). By contrast, in a finite economy that is not perfectly competitive, each individual may be able to influence prices. Since the individual is discrete (not infinitesimal), instead of ϱ, his rational conjectures are summarized by some price selection ρ from the Walrasian correspondence $\hat{P}(\cdot)$, where $\rho(m^n_{a,d})$ is interpreted as the individual's price conjecture if he deviates from a to d in the population m^n. If $\langle m^n \rangle$ is asymptotically perfectly competitive, any individual in the finite economy m^n cannot increase his utility by more than ϵ by deviating from price-taking behavior, provided n is sufficiently large.

Theorem 8.2 (ϵ-incentive to deviate from price-taking)**.** *Let $\langle m^n \rangle$ be asymptotically perfectly competitive, and let ρ be any selection from the Walrasian price correspondence $\hat{P}(\cdot)$. Then for any $\delta > 0$ there exists an N such that for all $n > N$,*

$$|v_d{}^*(\rho(m^n_{a,d})) - v_d{}^*(\rho(m^n))| < \delta, \quad \forall (a,d) \in A(m^n) \times \mathbf{A}.$$

Proof. Suppose $d \in \mathbf{S}$. Let $p^n = \rho(m^n)$, $p^n_{a,d} = \rho(m^n_{a,d})$, and define the function $f(d, p^n_{a,d}, p^n) = v_d{}^*(p^n_{a,d}) - v_d{}^*(p^n)$. Notice f is continuous on $\mathbf{A} \times \mathcal{P} \times \mathcal{P}$, a compact set; hence f is uniformly continuous. It follows there is an $\epsilon > 0$ such that $\|p^n_{a,d} - p^n\| < \epsilon$ implies $v_d{}^*(p^n_{a,d}) - v_d{}^*(p^n) < \delta$ for all $(a,d) \in A(m^n) \times \mathbf{A}$ and all n. In turn, Theorem 8.1 implies there is an open set of populations O containing μ such that if the normalized populations μ^n and $\{\mu^n_{a,d} : (a,d) \in A(\mu^n) \times \mathbf{A}\}$ are all in O, the distance $\|p^n_{a,d} - p^n\| < \epsilon$ for all $(a,d) \in A(\mu^n) \times \mathbf{A}$. (Recall $\hat{P}(m^n) = \hat{P}(\mu^n)$ and $\hat{P}(m^n_{a,d}) = \hat{P}(\mu^n_{a,d})$.) Finally in turn, the discussion preceding the current proposition implies that there is an integer N, such that all the above populations are in O provided $n > N$.

If $d \in \mathbf{B}$, the proof is analogous except one lets $f(d, p^n_{a,d}, p^n) = v_d{}^*(p^n_{a,d}, S) - v_d{}^*(p^n, S)$. Since a deviation to an attribute in $\mathbf{B}$ cannot increase the set of marketed commodities, and since $S(m^n) \subset S(\mu) \equiv S$, the restriction of trading to commodities S will not be binding in the sense that $v_d{}^*(p^n_{a,d}, S) = v_d{}^*(p^n_{a,d})$ and $v_d{}^*(p^n, S) = v_d{}^*(p^n)$. □

To apply the theorem, suppose (x^n, p^n) is an ex post Walrasian equilibrium for the finite economy m^n, where $p^n = \rho(m^n)$. For any deviation $d \in \mathbf{A}$ by any individual $a \in \text{supp } m^n$, the individual conjectures prices will change to $\rho(m^n_{a,d})$. The theorem tells us that if n is sufficiently large, $|v_d{}^*(\rho(m^n_{a,d})) - v_d{}^*(p^n)| < \epsilon$ for all possible deviations d by the individual. Hence each individual has at most an ϵ incentive to deviate from price-taking behavior, that is, to deviate from the (perhaps irrational) conjecture that prices will remain at p^n.

Remark 8.1. Theorem 8.2 is stronger than the asymptotic result in GOZ2. Theorem 5 in GOZ2 only proves that most (rather than all) individuals in supp m^n will have no more than an ϵ incentive to deviate from price-taking behavior if m^n is nearly perfectly competitive.

We conclude this section by examining the appropriability properties of populations that are almost perfectly competitive. We will see that if $\langle m^n \rangle$ is asymptotically perfectly competitive, any individual in the finite economy m^n will be within ϵ of appropriating his full social contribution, provided n is sufficiently large.

In any finite economy m, the infinitesimal marginal products are replaced by discrete marginal products, to respect the discreteness (indivisibility) of individuals in finite economies. In particular, the social contribution of any individual with attribute $a \in \text{supp } m$ is given by the discrete directional derivative

$$mp^-(a \mid m) = g(m) - g(m - \delta_a), \quad \forall a \in \text{supp } m.$$

Similarly, the social contribution of adding an individual to m with attribute $a \in \mathbf{A}$ is given by the discrete directional derivative

$$mp^+(a \mid m) = g(m + \delta_a) - g(m), \quad \forall a \in \mathbf{A}.$$

Finally, the social contribution of any individual deviating from attribute $a \in \operatorname{supp} m$ to attribute $d \in \mathbf{A}$ is given by the discrete directional derivative

$$mp(a, d \mid m) = g(m - \delta_a + \delta_d) - g(m), \quad \forall a \in \operatorname{supp} m, \quad \forall d \in \mathbf{A}.$$

The following theorem gives formulas for calculating individuals' discrete marginal products. The formula for $mp^-(b \mid m)$ comes from Lemma 8.15 in Roth and Sotomayor, who credit the formula to Demange and Leonard. The formulas for $mp^+(a \mid m)$ appear related to independent work of Mo summarized by Roth and Sotomayor in Section 8.5 of their book. The formulas use the fact that the set of Walrasian price vectors form a lattice, a property that applies to both the finite and continuum versions of the assignment model. As usual, in the theorem below $\underline{p}$ and $\overline{p}$ denote, respectively, the smallest price function in $\hat{P}(m)$ and largest price function in $\check{P}(m)$. Notice $\underline{p}$ and $\overline{p}$ coincide on $S(m)$, the set of commodities available in m.

Theorem 8.3 (formulas for discrete derivatives)**.**

$$mp^-(b \mid m) = v_b{}^*(\underline{p}), \quad \forall b \in \operatorname{supp} m,$$

and

$$mp^-(s \mid m) = v_s{}^*(\overline{p}), \quad \forall s \in \operatorname{supp} m.$$

Similarly,

$$mp^+(s \mid m) = v_s{}^*(\underline{p}), \quad \forall s \in \mathbf{S},$$

and

$$mp^+(b \mid m) = v_b{}^*(\overline{p}), \quad \forall b \in \mathbf{B}.$$

Further, $\underline{p} \in P(m - \delta_b)$ for all $b \in \operatorname{supp} m$, $\overline{p} \in P(m - \delta_s)$ for all $s \in \operatorname{supp} m$, $\underline{p} \in P(m + \delta_s)$ for all $s \in \mathbf{S}$, and $\overline{p} \in P(m + \delta_b)$ for all $b \in \mathbf{B}$.

Proof. To verify the formula for $mp^-(b \mid m)$, see the proof of Lemma 8.15 in Roth and Sotomayor, which is based on Demange's proof. The proof also implies $\underline{p} \in P(m - \delta_b)$. The proof of the formula for $mp^-(s \mid m)$ and the fact that $\overline{p} \in P(m - \delta_s)$ is analogous.

Here is a Demange-like proof of the formula for $mp^+(s)$. Consider adding a new seller s to the population m. We can suppose without loss of generality that $s \notin \operatorname{supp} m$ (re-labelling attributes if necessary). Let x be any efficient assignment for m. Construct a graph whose vertices are $\mathbf{B} \cup \mathbf{S}$. There are two kinds of arcs. If b is matched to s', there is an arc from b to s'. If $b \in \operatorname{supp} m$, $v(b, s') - \underline{p}(s') = v_b{}^*(\underline{p})$, and b is not matched to s', there is an arc from s' to b. Our normalization of prices (that is, $\underline{p} \in \hat{P}(m)$) implies that if $\underline{p}(s) > r(s)$, there is an oriented path c starting from s and ending either at a matched seller $s_k \in \operatorname{supp} m$

for whom $v_{s_k}{}^*(\underline{p}) = r(s_k)$ or an unmatched buyer $b_{k+1} \in \text{supp } m$, that is, either

$$c = (s, b_1, s_1, b_2, s_2, \ldots, b_k, s_k) \text{ or } c = (s, b_1, s_, b_2, s_2, \ldots, b_k, s_k, b_{k+1})$$

(otherwise the minimality of $\underline{p}$ would be contradicted since one could decrease by an epsilon all the prices of the commodities on the arc starting from s and still support x as an equilibrium).

If $\underline{p}(s) = r(s)$, let x' be the assignment for $m + \delta_s$ that is identical to x except s is added to the population but is not matched. Clearly $\underline{p}$ will support x' as an ex post (hence efficient) Walrasian equilibrium, and $mp^+(s \mid m) = v_s{}^*(\underline{p}) = \underline{p}(s) = r(s)$. On the other hand, if $\underline{p}(s) > r(s)$, then $\underline{p}$ will support as an ex post (hence efficient) Walrasian equilibrium the assignment x' that matches s to b_1, s_1 to b_2, …, and s_{k-1} to b_k and leaves s_k unmatched or matches s_k with b_{k+1} (otherwise x' is identical to x). In this case $mp^+(s \mid m) = v_s{}^*(\underline{p}) = \underline{p}(s)$. In either case, notice $\underline{p} \in P(m + \delta_s)$.

Here is the analogous proof of the formula for $mp^+(b \mid m)$. Consider adding a new buyer b to the population m, and suppose without loss of generality that attribute $b \notin \text{supp } m$. Since $\overline{p} \in \check{P}(m)$, the prices of all unavailable goods are high enough to deter any new buyer from wanting any $s \notin \text{supp } m$; hence $v_b{}^*(\overline{p}) > r(b)$ implies $v_b{}^*(\overline{p}) = v(b, s) - \overline{p}(s)$ for some $s \in \text{supp } m$. Again let x be any efficient assignment for m, and construct a graph with two kinds of arcs. If b' is matched to s, there is an arc from b' to s. If $s \in \text{supp } m$, $v(b', s) - \overline{p}(s) = v_{b'}{}^*(\overline{p})$, and b' is not matched to s, there is an arc from b' to s. By construction, if $v_b{}^*(\overline{p}) > r(b)$, there is an oriented path c starting from b and ending either at a matched buyer $b_k \in \text{supp } m$ for whom $v_{b_k}{}^*(\overline{p}) = r(b_k)$ or a seller $s_{k+1} \in \text{supp } m$ who is not matched in x, that is, either

$$c = (b, s_1, b_1, s_2, b_2, \ldots, s_k, b_k) \text{ or } c = (b, s_1, b_1, s_2, b_2, \ldots, s_k, b_k, s_{k+1})$$

(otherwise the maximality of $\overline{p}$ would be contradicted since one could increase by an epsilon all the prices of the commodities on the arc starting from b and still support x as an equilibrium).

If $v_b{}^*(\overline{p}) = r(b)$, proceed as in the proof of the previous formula to conclude $mp^+(b \mid m) = v_b{}^*(\overline{p}) = r(b)$. If $v_b{}^*(\overline{p}) > r(b)$, consider the assignment x' that matches b to s_1, b_1 to s_2, …, and matches b_{k-1} to s_k while leaving b_k unmatched or matches b_k with s_{k+1}. The construction of the path implies $(x', \overline{p})$ is Walrasian for $m + \delta_b$. Hence $mp^+(b \mid m) = v_b{}^*(\overline{p})$. In either case, notice $\overline{p} \in P(m + \delta_b)$. □

Corollary 8.1. *For any* $(a, d) \in B(m) \times \mathbf{B}$*,*

$$mp(a, d \mid m) = v_d{}^*(\overline{p}_a) - v_a{}^*(\overline{p}_a)$$

while for any $(a, d) \in S(m) \times \mathbf{S}$,

$$mp(a, d \mid m) = {v_d}^*(\underline{p}_a) - {v_a}^*(\underline{p}_a),$$

where $\underline{p}_a$ *is the smallest price in* $\hat{P}(m - \delta_a)$, *and* $\overline{p}_a$ *is the largest price function in* $\check{P}(m - \delta_a)$.

Proof. Notice for any $(a, d) \in A(m) \times \mathbf{A}$,

$$\begin{aligned} mp(a, d \mid m) &\equiv g(m - \delta_a + \delta_d) - g(m) \\ &= [g(m - \delta_a + \delta_d) - g(m - \delta_a)] - [g(m - \delta_a + \delta_a) - g(m - \delta_a)] \\ &= mp^+(d \mid m - \delta_a) - mp^+(a \mid m - \delta_a). \end{aligned}$$

Now the corollary follows immediately from the previous result. □

An interesting fact is that the discrete and infinitesimal marginal products coincide in any finite assignment game.

Corollary 8.2 (equality between discrete and directional derivatives)**.** *For any* $a \in \operatorname{supp} m$,

$$mp^-(a \mid m) = MP^-(a \mid m),$$

and for any $a \in \mathbf{A}$,

$$mp^+(a \mid m) = MP^+(a \mid m).$$

Further, for any (a, d) *in either* $B(m) \times \mathbf{B}$ *or* $S(m) \times \mathbf{S}$,

$$mp(a, d \mid m) = MP(a, d \mid m).$$

Proof. The first assertion follows immediately from the fact that both the discrete and the infinitesimal marginal product for any buyer (rep. seller) in the support of m equals ${v_b}^*(\underline{p})$ (resp. ${v_s}^*(\overline{p})$). Similarly for the second assertion.

To prove the third assertion, notice first that the discrete derivative converges to the directional derivative as one replicates a finite economy.[4] That is, setting $y = \delta_d + \delta_a$,

$$\begin{aligned} \lim_{r\to\infty} mp(a, d \mid rm) &\equiv \lim_{r\to\infty} g(rm + y) - g(rm) \\ &= \lim_{r\to\infty} \frac{g(m + \frac{1}{r}y) - g(m)}{\frac{1}{r}} \\ &= g'(m; y) \equiv MP(a, d \mid m), \end{aligned}$$

where the second equality uses the homogeneity of g.

Further, for any r,

$$mp(a, d \mid rm) \equiv g(rm + y) - g(rm) = mp(a, d \mid m).$$

[4]Thanks to Joe Ostroy for this tip.

We verify for $(a,d) \in S(m) \times \mathbf{S}$, the argument when $(a,d) \in B(m) \times \mathbf{B}$ is similar. Let p be the smallest price function in $\tilde{P}(m-\delta_a)$. Theorem 8.3 implies $p \in P(m)$ and $p \in P(m-\delta_a+\delta_d)$. Consequently, for any replica r,

$$p \in P(rm+y) \quad \text{and} \quad p \in P(rm),$$

where the first statement follows from the fact that $p \in P((r-1)m)$ and $p \in P(m-\delta_a+\delta_d)$. Hence $g(rm+y) - g(rm) = {v_d}^*(p) - {v_a}^*(p)$. Combining we conclude

$$mp(a,d \mid m) = MP(a,d \mid m).$$

□

Perfect competition in a finite economy m — just as in a continuum economy μ — occurs when the set of normalized Walrasian prices is a singleton, hence when everyone can fully approprate (see GOZ2). More generally, a large but finite economy that is almost perfectly competitive will exhibit almost full appropriation.

Theorem 8.4 (ϵ-full appropriation)**.** *Suppose $\langle m^n \rangle$ is asymptotically perfectly competitive, and let $p^n \in \hat{P}(m^n)$. Then for any $\epsilon > 0$ there exists an N such that for all $n > N$:*

(1) $|{v_a}^*(p^n) - mp^-(a \mid m^n)| < \epsilon, \quad \forall a \in S(m^n),$
(2) $|{v_s}^*(p^n) - mp^+(s \mid m^n)| < \epsilon, \quad \forall s \in \mathbf{S},$
(3) $|{v_b}^*(p^n, S) - mp^+(b \mid m^n)| < \epsilon, \quad \forall b \in \mathbf{B},$
(4) $|[{v_d}^*(p^n) - {v_a}^*(p^n)] - mp(a,d \mid m^n)| < \epsilon, \quad \forall (a,d) \in S(m) \times \mathbf{S},$
(5) $|[{v_d}^*(p^n, S) - {v_a}^*(p^n, S)] - mp(a,d \mid m^n)| < \epsilon, \quad \forall (a,d) \in B(m) \times \mathbf{B}.$

Proof. To prove the first statement, recall Theorem 8.3 implies for any n:

$$mp^-(b \mid m^n) = {v_b}^*(\underline{p}^n), \quad \forall b \in \text{supp } m^n,$$
$$mp^-(s \mid m^n) = {v_s}^*(\overline{p}^n), \quad \forall s \in \text{supp } m^n,$$

where $\overline{p}^n$ and $\underline{p}^n$ denote respectively the largest and smallest price vectors in $\hat{P}(m^n)$. Let $f(a,p',p^n) = {v_a}^*(p') - v_a(p^n)$, and observe f is continuous on the compact set $\mathbf{A} \times \mathcal{P} \times \mathcal{P}$, hence it is uniformly continuous. It follows there is an $\epsilon > 0$ such that $\|p' - p^n\| < \epsilon$ implies ${v_a}^*(p') - {v_a}^*(p^n) < \delta$ for all $a \in A(m^n)$ and all n. Let O be the open set in Theorem 8.1, and let N' be sufficiently large that $m^n \in O$ for all $n > N'$. Then the first statement in the theorem holds provided $N \geq N'$.

Theorem 8.3 implies

$$mp^+(s \mid m^n) = {v_b}^*(\underline{p}^n), \quad \forall s \in \mathbf{S},$$

where $\overline{p}^n$ and $\underline{p}^n$ are as defined above. Hence the above argument shows the second statement in the theorem holds provided $N \geq N'$. A similar argument shows the third statement also holds provided $N \geq$ some N''. Hence any $N \geq \max\{N', N''\}$ satisfies the first three statements in the theorem.

For any $(a, d) \in S(m^n) \times \mathbf{S}$,

$$mp(a, d \mid m^n) = v_d{}^*(\underline{p}^n_a) - v_a{}^*(\underline{p}^n_a),$$

where $\underline{p}^n_a$ denotes the smallest price vectors in $\hat{P}(m^n - \delta_a)$. Let $f(a, d, p', p^n) = [v_d{}^*(p') - v_a{}^*(p')] - [v_d{}^*(p^n) - v_a{}^*(p^n)]$. Arguing as above, one shows that there is an N''' such that $f(a, d, \underline{p}^n_a, p^n) < \delta$ for all $n > N'''$ and all $(a, d) \in S(m^n) \times \mathbf{S}$. Similarly, for any $(a, d) \in B(m^n) \times \mathbf{B}$,

$$mp(a, d \mid m^n) = v_d{}^*(\overline{p}^n_a, S) - v_a{}^*(\overline{p}^n_a, S),$$

where $\overline{p}^n_a$ denotes the largest price function in $\hat{P}(m^n - \delta_a)$. Letting $f'(a, d, p', p^n) = [v_d{}^*(p', S) - v_a{}^*(p', S)] - [v_d{}^*(p^n, S) - v_a{}^*(p^n, S)]$, and proceeding as above, one shows that there is an N'''' such that $f'(a, d, \overline{p}^n_a, p^n) < \delta$ for all $n > N''''$ and all $(a, d) \in B(m^n) \times \mathbf{B}$. So, combining, if $N \geq \max\{N', N'', N''', N''''\}$, all statements in the theorem hold. □

The difference between small and large economies that we want to highlight has to do with the likelihood of ex post perfect competition. We illustrate with the housing model. When there are only a few sellers, there will typically be large gaps between different sellers' reservation values for any type of house. But as the number of sellers increases, the diversity of sellers' outside options will also increase, typically (generically) filling in all "gaps," leading to perfect competition in the limit, and almost perfect competition (small gaps) before the limit is reached.

9. Comparisons with ex ante contracting

If individuals could contract before investing, they would play an ex ante rather than ex post assignment game. Interestingly, these two games are isomorphic. Paralleling the data in the ex post assignment game, in the ex ante game the data are:

- AGENTS:: the set of buyer types I and the set of seller types J,
- A POPULATION MEASURE:: $\mathcal{E} \in M_+(I \cup J)$,
- A VALUE FUNCTION:: $V : I^0 \times J^0 \to \mathbb{R}$,

where, if a buyer of type $i \in I$ and a seller of type $j \in J$ match, the value they can create is given by

$$V(i, j) = \max_{b \in \mathbf{B},\ s \in \mathbf{S}} v(b, s) - c(b, i) - c(s, j);$$

while if a buyer i remains unmatched, the value he can create is given by

$$V(i,\overline{\mathbf{0}}) = \max_{b\in\mathbf{B}} v(b,\mathbf{0}) - c(b,i),$$

where $\overline{\mathbf{0}}$ denotes the "null type" (paralleling the null attribute $\mathbf{0}$); and if a seller j remains unmatched

$$V(\overline{\mathbf{0}},j) = \max_{s\in\mathbf{S}} v(\mathbf{0},s) - c(s,j),$$

letting $I^0 \equiv I \cup \{\overline{\mathbf{0}}\}$, $J^0 \equiv J \cup \{\overline{\mathbf{0}}\}$, and $V(\overline{\mathbf{0}},\overline{\mathbf{0}}) \equiv 0$.

For the remainder of this section assume that $\operatorname{dom} c(\cdot,i) = \mathbf{B}$ for all buyers $i \in I$, $\operatorname{dom} c(\cdot,j) = \mathbf{S}$ for all sellers $j \in J$, and c is jointly continuous on its effective domain. Also assume there is the possibility of inaction. Then, just as v is continuous and nonnegative, V is continuous and nonnegative.[5]

The maximum net gains for the ex ante population $\mathcal{E}$ is

$$G^*(\mathcal{E}) = \sup\ G(\nu) \text{ s.t. } \nu \text{ is feasible for } \mathcal{E}.$$

Like the gains function g, the net gains function G^* is concave and homogeneous of degree one, hence it is subdifferentiable. Let $Q^* \in C(\text{supp } \mathcal{E})$. Q^* is in the *ex ante core* of $\mathcal{E}$ if $\int Q^* \, \mathrm{d}\mathcal{E} = G^*(\mathcal{E})$, and for any coalition $\mathcal{E}' \in M_+(T)$ satisfying $\mathcal{E}' \leq \mathcal{E}$, $\int Q^* \, \mathrm{d}\mathcal{E}' \geq G^*(\mathcal{E}')$. As an expression of core equivalence, any core imputation Q^* can be realized as an ex ante Walrasian equilibrium (ν, x, p) with *complete* markets.

Lemma 9.1. *Suppose Q^* is in the ex ante core of $\mathcal{E}$, ν is efficient for $\mathcal{E}$, and χ is efficient for ν (i.e., χ is feasible for ν and $\int v\, dx = g(\mu)$, where $\mu = \nu_{\mathbf{A}}$ and $x = \chi_{\mathbf{B}^0\times\mathbf{S}^0}$.). Then there exists a p such that (ν, x, p) is ex ante Walrasian for $\mathcal{E}$ with complete markets, and*

$$Q^*(t) = \sup_{a\in\mathbf{A}} v_a{}^*(p) - c(a,t), \quad \forall t \in \text{supp } \mathcal{E}.$$

Proof. Let $p(\mathbf{0}) = 0$ and, for each $s \in \mathbf{S}$, let

$$p(s) = \max\left\{ r(s), \sup_{b\in\mathbf{B},\ i\in\text{supp } \mathcal{E}} v(b,s) - c(b,i) - Q^*(i) \right\}.$$

By construction $p \in C_+(\mathbf{S}^0)$. Further, since Q^* is in the ex ante core, $Q^*(t) = v_a{}^*(p) - c(a,t)$ for all $t \in \text{supp } \mathcal{E}$. □

In an investment equilibrium, individuals must invest prior to bargaining and contracting; core bargaining only occurs after the distribution of attributes μ has been chosen. Core equivalence implies any ex post core imputation $q^* \in C(\text{supp } \mu)$ can be realized as an ex post

[5] Weaker assumptions on c could be substituted, if needed for some applications.

Walrasian equilibrium (x, p). Since investment decisions are based on individuals' conjectures about ex post bargaining, even if these conjectures are rational, investment equilibria need not be ex ante efficient.

Nevertheless, in this section we will show that any Pareto efficient allocation ν and any ex ante core imputation Q^* can be realized in an investment equilibrium, (ν, ϱ), hence with ex post rather than ex ante contracting. We emphasize that the conjectures ϱ that support ν, while rational, may depend on individuals' relative (ex post) bargaining abilities since ν may not be perfectly competitive. That is, while ν is efficient, it may be a particular configuration of ex post bargaining skills rather than intense ex post competition for matches that resolves holdup problems in the equilibrium (ν, ϱ). Thus realizing efficiency without ex ante contracting may be fragile. This holds even if the ex ante population $\mathcal{E}$ is perfectly competitive in the sense of having a unique ex ante core, a unique normalized ex ante Walrasian price function, etc. Using Oliver Williamson's terminology, there may be a "fundamental transformation" if investment precedes bargaining, from ex ante perfect competition to ex post monopoly power. But, in keeping with our theme, we also emphasize that such a fundamental transformation need not occur — provided there are enough outside options ex post to ensure ex post perfect competition.

Theorem 9.1. *Let (ν, x, p) be any ex ante Walrasian equilibrium with complete markets. There exist rational conjectures ϱ such that (ν, ϱ) is an investment equilibrium with the property*

$$\varrho(a, a)(s) = p(s), \quad \forall s \in S(\nu_{\mathbf{A}}),$$

where $a \in \text{supp } \nu_{\mathbf{A}}$. (That is, the same prices p prevail in the investment equilibrium as in the Walrasian equilibrium, at least for the produced goods; hence the same profile of net utilities are realized in the two equilibria.)

Proof. Let $\mu = \nu_{\mathbf{A}}$. Since (ν, x, p) is ex ante Walrasian, (x, p) is ex post Walrasian for μ. Denote the normalized Walrasian price function as $p^* \equiv \hat{p}(p, \mu)$. It follows that (x, p^*) also is ex post Walrasian for μ. Let Q^* denote the ex ante core utilities realized in the ex ante Walrasian equilibrium, so $Q^*(t) \equiv {v_a}^*(p) - c(a, t)$ for all $(a, t) \in \text{supp } \nu$; and let q^* denote the ex post core utilities realized in the ex post Walrasian equilibrium, so $q^*(a) = {v_a}^*(p) = {v_a}^*(p^*)$ for all $a \in \text{supp } \mu$. Notice

$$Q^*(t) = q^*(a) - c(a, t), \quad \forall (a, t) \in \text{supp } \nu,$$

so Q^* is realized in (x, p^*).

We will show that Q^* is also realized in an investment equilibrium. Let ρ be any Walrasian price selection satisfying $\rho(\mu) = p^*$; $\rho(\mu + \delta^n_{s,s'})$ equals the smallest price function in $\hat{P}(\mu+\delta^n_{s,s'})$ after any deviation by a group of sellers (so for any $s \in S$ and $s' \in \mathbf{S}$), and $\rho(\mu+\delta^n_{b,b'})$ equals the largest price function in $\hat{P}(\mu + \delta^n_{b,b'})$ after any deviation by a group of buyers. Thus sellers conjecture that buyers will be tough bargainers if any group of sellers deviates, and buyers conjecture that sellers will be tough bargainers if any group of buyers deviates. We claim that these conjectures deter deviation, hence (ν, ϱ) is an investment equilibrium, where for any $(a, d) \in A \times \mathbf{S}$,

$$\varrho(a, d) = \lim_{n\to\infty} \rho(\mu + \delta^n_{a,d})$$

for some sequence $\langle \delta^n_{a,d} \rangle$ (to be specified shortly).

To check the claim, we need to verify that for all $(a, t) \in \operatorname{supp} \nu$ and all $d \in \mathbf{A}$:

$$v_a{}^*(p^*) - c(a, t) \geq v_d{}^*(\varrho(a, d)) - c(d, t).$$

Let's begin with the sellers' side. Suppose $(a, t) = (s, j) \in S \times J$ and $d \in \mathbf{S}$. We must show

$$v_s{}^*(p^*) - c(s, j) \geq v_d{}^*(p') - c(s', j),$$

where $p' = \lim p^n$ and $p^n \equiv \rho(\mu + \delta^n_{s,d})$.

First we specify the sequence of deviations $\langle \delta^n_{s,d} \rangle$ we will consider. Since $(s, j) \in \operatorname{supp} \nu$, there exists a sequence of subpopulations $\langle \nu^n \rangle$ such that so $\nu^n \in M_+(\mathbf{A} \times T)$, $\nu^n \leq \nu$, $\|\nu^n\| \to 0$, and $\operatorname{supp} \nu^n \to \{(s, j)\}$. The sequence can be thought of as converging to a single individual with characteristics (s, j). Let y^n denote the marginal of ν^n on $\mathbf{A}$. If the entire group of sellers in y^n deviates to d, the distribution of their attributes will change to $\sigma^n \equiv \|y^n\|\delta_d$, which will result in the ex post population $\mu^n \equiv \mu + \sigma^n - y^n$. We will focus on the sequence of deviations $\delta^n_{s,d} = \sigma^n - y^n$, resorting to a subsequence if necessary to ensure $\langle p^n \rangle$ converges.

Let β^n be the distribution of buyers who are matched with the sellers in σ^n in an efficient assignment for μ^n, and let $o^n \equiv \mu^n - \sigma^n - \beta^n$ be the distribution of all other individuals. Let q^n be the ex post core allocation resulting from prices $p^n \equiv \rho(\mu^n)$.

Since buyers are tough bargainers in μ^n, Theorem 3.2 implies

$$\int q^n \,\mathrm{d}\beta^n = g(\mu^n) - g(\mu^n - \beta^n) + o(\|\beta^n\|).$$

Hence, since $g(\mu^n) = \int q^n$,

$$\int q^n \,\mathrm{d}(\sigma^n + o^n) = g(\mu^n - \beta^n) - o(\|\beta^n\|).$$

Since all the individuals in $\mathcal{E}$ excluding the buyers in β^n could have formed a coalition in the ex ante game with the individuals in ν^n agreeing to produce d instead of y^n, the fact that Q^* is in the ex ante core implies

$$g(\mu^n - \beta^n) - \int q^* \,\mathrm{d}(y^n + o^n) \leq \int c(d, j) \,\mathrm{d}\nu^n_T(j) - \int c \,\mathrm{d}\nu^n,$$

where ν^n_T is the marginal of ν^n on T; hence

$$\int q^n \,\mathrm{d}(\sigma^n + o^n) \leq \int q^* \,\mathrm{d}(y^n + o^n) + \\ [\int c(d, j) \,\mathrm{d}\nu^n_T(j) - \int c \,\mathrm{d}\nu^n] - o(\|\beta^n\|)$$

Since q^n is in the (ex post) core of μ^n, others must do at least as well as in q^* since their trading is self-contained, that is,

$$\int q^n \,\mathrm{d}o^n \geq q^* o^n,$$

hence

$$\int q^n \,\mathrm{d}\sigma^n \;\leq\; \int q^* \,\mathrm{d}y^n \;+\; [\int c(d, t) \,\mathrm{d}\nu^n_T(t) \;-\; \int c \,\mathrm{d}\nu^n] \;-\; o(\|\beta^n\|).$$

Dividing both sides by $\|y^n\|$ and letting $n \to 0$, we conclude

$$v_s{}^*(p') - c(d, j) \leq v_s{}^*(p) - c(s, j),$$

as asserted. The demonstration for any $(b, i) \in \operatorname{supp} \nu$ is analogous. □

Example 9.1 ("Are master-servant examples nongeneric?" continued)**.** We continue our analysis of Example 7.1. In accord with the GOZ2 genericity result, typically there will be more buyers than sellers in the ex ante population or vice versa, that is, $\mathcal{E}(i) \neq \mathcal{E}(j)$. In these cases, ex ante core bargaining will lead to an efficient *and* perfectly competitive outcome. To illustrate, suppose $\mathcal{E}(i) > \mathcal{E}(j)$, then the unique ex ante core outcome for Example 7.1 is $Q^*(i) = 0$ and $Q^*(j) = .5$. Sellers get all the surplus because of the intense competition for matches by buyers.

But as we know from our previous analysis of this example, even if $\mathcal{E}(i) > \mathcal{E}(j)$, in all (nontrivial) ex post populations there will be a fundamental transformation, from ex ante perfect competition to ex post monopoly power. Nevertheless, we also know that there exist

Pareto efficient investment equilibria (ν, ϱ), including one leading to final utilities Q^*. In the efficient equilibrium leading to Q^* individuals conjecture — as in the proof of Theorem 9.1 — that prices will be $\varrho(a,a) = p^* = (0, .75)$ if no one deviates, buyers will obtain their marginal products if there is any seller deviation, and sellers will obtain their marginal products if there is any buyer deviation. Since ν is not perfectly competitive, it is not ex post competition that solves the holdup problem in this equilibrium, but rather a certain configuration of ex post bargaining abilities encapsulated in ϱ. $\diamond$

10. The supermodular model: Some special results

In this section we will develop a pair of conditions that is both necessary and sufficient for perfect competition in the supermodular model, we then apply the conditions to examine the efficiency of perfectly competitive investment equilibria in this model. Since the supermodular model is a simple one-dimensional special case, the results we can obtain are much more concrete.

10.1. **Characterizing perfect competition in the supermodular model.** The supermodularity of v implies that if x is an efficient matching for μ, then x is *positively assortative*, that is, $(b,s), (b', s') \in \text{supp } x$ and $b' > b$ implies $s' \geq s$. We will use this fact to develop two necessary and sufficient conditions for perfect competition in the supermodular model.

Let $\underline{s} = \min\{s \in S\}$ and $\overline{s} = \max\{s \in S\}$; define $\underline{b}$ and $\overline{b}$ analogously. Since S may not be a connected set, for any $s \in (\underline{s}, \overline{s})$ define

$$s- = \sup\{s' \in S : s' < s\}$$

and

$$s+ = \inf\{s' \in S : s' > s\}.$$

Let's call $s \in (\underline{s}, \overline{s})$ a *continuity point* of S if $s- = s = s+$. On the other hand if $s \neq s+$ we will say that there is a *seller jump* in S from s to $s+$. For any $b \in (\underline{b}, \overline{b})$, define $b+$ and $b-$ analogously; also define continuity points of B, and points where there is a buyer jump in B, analogously. For the left-hand tail $s = \underline{s}$, it will be convenient to set $\underline{s}- \equiv \mathbf{0}$; similarly set $\underline{b}- \equiv \mathbf{0}$.

The first necessary condition for μ to be perfectly competitive has already been developed: there must be a marginal trader in μ (Theorem 5.2). Because the supermodular model is one dimensional, we can be more specific about the identity of the marginal trader. The set of productive matches in x is

$$x_+ = \{(b,s) \in \text{supp } x : v(b,s) - r(b) - r(s) > 0\}.$$

In the supermodular model $r(\cdot) \equiv 0$, so this equals the set of matches in which $v(b,s) > 0$. Assume there are gains from trade in μ, hence $x_+ \neq \emptyset$. Denote by (b_x, s_x) the match in the closure of x_+ that satisfies

$$v(b_x, s_x) = \inf\{v(b,s) : (b,s) \in x_+\}.$$

We will call (b_x, s_x) the marginal productive couple in x. (Such a couple exists because supp x is closed; in addition, $(b_x, s_x) \in B \times S$ because $v(b,s) > 0$ implies $(b,s) \in B \times S$; finally, there is only one (unique) marginal productive couple because v is strictly increasing for $(b,s) \in x_+$.) We'll call b_x a *marginal buyer* if at least one of the following three conditions holds:

$$v(b_x, s_x) = 0, \quad b_x - = b_x, \quad \text{or} \quad (b_x, \mathbf{0}) \in \text{supp } x.$$

Similarly, s_x is a marginal seller if

$$v(b_x, s_x) = 0, \quad s_x - = s_x, \quad \text{or} \quad (\mathbf{0}, s_x) \in \text{supp } x.$$

The following lemma shows that this terminology is appropriate.

Lemma 10.1 (identifying the marginal trader)**.** *Suppose μ is not trivial and (x,p) is Walrasian for μ. Then if either b_x is a marginal buyer or s_x is a marginal seller, there is a marginal trader in (x,p).*

Proof. If either s_x is a marginal seller or b_x is a marginal buyer, the closure of $A(x_+)$ intersects $A(x_0)$, hence there is a nmarginal trader in (x,p). □

Henceforth, if s_x is a marginal seller, we'll also refer to him as a marginal trader; similarly, if b_x is a marginal buyer, we'll also refer to b_x as a marginal trader. Lemma 10.1 and its proof shows this terminology is appropriate.

The next result will show that the converse of Lemma 10.1 also holds: if neither b_x nor s_x is a marginal trader, there is no marginal trader in μ, hence μ is not perfectly competitive. If neither b_x nor s_x is a marginal trader, the couple (b_x, s_x) is in the set of productive couples x_+, and there is an $\epsilon > 0$ such that $b, s \in A(x_0)$ implies $b_x - b > \epsilon$ and $s_x - s > \epsilon$. We will see that this leads to a bilateral monopoly surplus that b_x and s_x must bargain over—reflecting the absence of perfect competition. On the other hand, if either b_x or s_x is a marginal trader, competition for a productive match from the marginal trader and traders with nearby attributes pins down the prices of all commodities at the lower tail of S, eliminating any bilateral monopoly surpluses at the lower tail.

Lemma 10.2 (Pricing commodities at the lower tail)**.** *Suppose supermodularity and that μ is nontrivial. Also suppose (x,p) is Walrasian* *for μ with $p \in \hat{P}(\mu)$. Then*

$$p(s) = 0, \quad \forall s \in S \text{ s.t. } s < s_x.$$

If in addition s_x is a marginal seller, then $p(s_x) = 0$, while if b_x is a marginal buyer, then $p(s_x) = v(b_x, s_x)$. On the other hand, if neither b_x nor s_x is a marginal trader, then μ is not perfectly competitive because there is a range of possible Walrasian equilibrium prices for s_x.

Proof. The definition of the marginal productive couple implies $v(b, s) = 0$ for all $(b, s) \in \text{supp } x$ such that $s < s_x$. It follows that if such a b is in B (i.e., not the null attribute), $p(s) = 0$, otherwise b would not want to match with s. On the other hand if $b = \mathbf{0}$, $p(s) = 0$, otherwise seller s would want to match with a non-null buyer. This establishes the first statement in the lemma.

To establish the second statement, set $b' = b_x-$ and $s' = s_x-$. Since seller s_x could have matched with buyer b',

$$p(s_x) \geq v(b', s_x) \equiv \underline{p}(s_x).$$

Similarly, since buyer b_x could have matched with seller s',

$$v(b_x, s_x) - p(s_x) \geq v(b_x, s') - p(s'),$$

or alternatively expressed, since $p(s') = 0$ if $s' \in S$ while $p(s') \equiv 0$ if $s' = \mathbf{0}$:

$$p(s_x) \leq v(b_x, s_x) - v(b_x, s') \equiv \overline{p}(s_x).$$

The gap $\overline{p}(s_x) - \underline{p}(s_x)$ measures the *bilateral monopoly surplus* the couple (b_x, s_x) must bargain over since s_x has the outside option of matching with s' (which establishes the lower bound) and b_x has the outside option of matching with b' (which establishes the upper bound).

There is no gap, that is,

$$\overline{p}(s_x) - \underline{p}(s_x) \equiv v(b_x, s_x) - [v(b_x, s') - v(s_x, b')] = 0,$$

iff either b_x or s_x is a marginal trader. If $v(b_x, s_x) = 0$, then optimization by b_x implies $p(s_x) = \overline{p}(s_x) = \underline{p}(s_x) = 0$. Suppose instead that $v(b_x, s_x) > 0$. Then if b_x is a marginal trader, so either $b_x- = b_x$ or $(b_x, \mathbf{0}) \in \text{supp } x$, optimization by all buyers implies b_x will get none of the surplus in his match because

$$p(s_x) = \overline{p}(s_x) = \underline{p}(s_x) = v(b_x, s_x).$$

On the other hand, if s_x is a marginal trader, so either $s_x- = s_x$ or $(\mathbf{0}, s_x) \in \text{supp } x$, then optimization by all sellers implies s_x will get none of the surplus in his match because

$$p(s_x) = \overline{p}(s_x) = \underline{p}(s_x) = 0.$$

But if neither b_x nor s_x is a marginal trader, the bilateral monopoly gap will be positive. Let $\underline{\gamma} = \underline{p}(s_x)$, $\overline{\gamma} = \overline{p}(s_x)$, and let Γ equal the interval $[\underline{\gamma}, \overline{\gamma}]$. For any $\gamma \in \Gamma$, let

$$p_\gamma = p + (\gamma - p(s_x)).$$

It is a straightforward exercise to check that

$$\{p_\gamma : \gamma \in \Gamma\} \subset \hat{P}(\mu).$$

□

So far we have seen how competition pins down the prices of commodities $s \leq s_x$. How does competition help to price commodities $s > s_x$? We first show how it pins down the price $p(s+)$ after a seller jump. An assignment x has the *overlap property* if, whenever there is a seller jump from s to $s+$ and $s \geq s_x$, there is a $b \in B$ such that both $(b, s), (b, s+) \in \text{supp } x$. Roughly speaking x has the overlap property if, whenever there is a seller jump from s to $s+$, there is not a corresponding buyer jump. Competition among the buyers with attribute b that are matched with both s and $s+$ eliminates any bilateral monopoly surplus between any seller $s+$ and the buyer he is matched with. Such competition is necessary for perfect competition. It pins down $p(s+)$, given $p(s)$.

Lemma 10.3 (overlap is necessary)**.** *In the supermodular model, if μ is perfectly competitive, any efficient assignment for μ has the overlap property.*

Proof. Let x be any efficient assignment for μ. Suppose $s \geq s_x$ and there is a seller jump in S from s to $s' \equiv s+$. Let $b = \max\{b_0 : (b_0, s) \in \text{supp } x\}$, and $b' = \min\{b_0 : (b_0, s') \in \text{supp } x\}$. Since $s \geq s_x$, both $b, b' \in B$ (i.e., neither equals the null attribute). We will show that $b' = b$.

Consider any price function $p \in \hat{P}(\mu)$. Interchangeability implies (x, p) is Walrasian for μ. Hence, since seller s' could have matched with buyer b,

$$p(s') \geq v(b, s') - [v(b, s) - p(s)] \equiv \underline{p}(s').$$

Similarly, since buyer b' could have matched with seller s,

$$v(b', s') - p(s') \geq v(b', s) - p(s),$$

or in other words,

$$p(s') \leq v(b', s') - [v(b', s) - p(s)] \equiv \overline{p}(s').$$

The gap $\overline{p}(s') - \underline{p}(s')$ measures the bilateral monopoly surplus the couple (b', s') must bargain over given $p(s)$ since s' has the outside option of matching with b (which establishes the lower bound) and b' has the outside option of matching with s (which establishes the upper bound). The supermodularity of v implies $\overline{p}(s') - \underline{p}(s') = 0$ (there is no gap) iff x has the overlap property, i.e., iff $b' = \overline{b}$, so seller s' has a perfect substitute outside option to matching with buyer b', namely matching with buyer b.

Let $\underline{\gamma} = \underline{p}(s')$, let $\overline{\gamma} = \overline{p}(s')$, and let Γ equal the interval $[\underline{\gamma}, \overline{\gamma}]$. For any $\gamma \in \Gamma$, let $p_\gamma(s^0) = p(s^0)$ for all $s^0 \in \mathbf{S}$ s.t. $s^0 < s'$, while $p_\gamma(s^0) = p(s^0) + (\gamma - p(s'))$ for all $s^0 \in \mathbf{S}$ s.t. $s^0 \geq s'$. It is a straightforward exercise to check that

$$\{p_\gamma : \gamma \in \Gamma\} \subset \hat{P}(\mu).$$

Hence, $\hat{P}(\mu)$ is a singleton implies Γ is a singleton, which in turn implies x satisfies the overlap property. □

The previous lemma showed that the overlap property is necessary to uniquely price a match with seller $s+$ after a seller jump. It is also sufficient. In particular, if there is a seller jump from s to $s+$, then seller $s+$ gets the whole increase in surplus.

Lemma 10.4 (pricing jumps)**.** *Let x be any efficient assignment for μ. Suppose there is a seller jump in S from s to $s' \equiv s+$, where $s \geq s_x$. If x has the overlap property, so $(b, s), (b, s') \in \operatorname{supp} x$ for some $b \in B$, then*

$$p(s') - p(s) = v(b', s') - v(b', s), \quad \forall p \in P(\mu). \tag{10.1}$$

Proof. Let $p \in P(\mu)$. Interchangeability implies (x, p) is Walrasian for μ. Hence, since both $(b, s), (b, s') \in \operatorname{supp} x$, utility maximization by b implies $v(b, s') - p(s') = v(b, s) - p(s)$. □

Under the extra mild assumption that $v \in \mathcal{C}^2$, the above two necessary conditions for perfect competition are also sufficient. We first show that, given this extra assumption, incentive compatibility is sufficient to price matches at all continuity points — whether or not there is perfect competition.

Lemma 10.5 (pricing at continuity points)**.** *Assume a supermodular model with $v \in \mathcal{C}^2$. Consider any efficient assignment x for μ and any match $(b, s) \in \operatorname{supp} x$ satisfying $b \in [b_x, \overline{b})$ and $s \in [s_x, \overline{s})$. Assume b and s are right-continuous at B and S respectively, that is, $b = b+$ and $s = s+$. Then for any $p \in P(\mu)$, p is differentiable at s with*

$$p'(s) = \frac{\partial v(b, s)}{\partial s}. \tag{10.2}$$

Proof. Let $p \in P(\mu)$, hence (x,p) is Walrasian for μ.

Since $s < \overline{s}$, for all t in some interval $(0,m]$ there is a $b_t \in B$ such that $(s+t, b_t) \in \text{supp } x$. Since B is a right-continuous at b, we can take for granted that $\lim_{t \downarrow 0} b_t = b$. Notice for every $t \in (0,m]$:

$$p(s+t) - p(s) \geq v(b, s+t) - v(b,s),$$

otherwise b would want to match with $s+t$, and

$$p(s+t) - p(s) \leq v(b_t, s+t) - v(b_t), s),$$

otherwise b_t would want to match with s. Combining leads to to the conclusion that for all $t \in [0,m]$ and all $n \in [t,m]$:

$$\begin{aligned} v(b,s+t) - v(b,s) \leq p(s+t) - p(s) \leq v(b_t, s+t) - v(b_t, s) \\ \leq v(b_n, s+t) - v(b_n, s), \end{aligned}$$

where the last inequality follows from the supermodularity of v. Dividing by t, for all $t \in [0,m]$ and all $n \in [t,m]$:

$$\frac{v(b,s+t) - v(b,s)}{t} \leq \frac{p(s+t) - p(s)}{t} \leq \frac{v(b_m, s+t) - v(b_n, s)}{t}.$$

As $t \to 0$, the left fraction converges to $\dfrac{\partial v(b,s)}{\partial s}$ while the right fraction converges to $\dfrac{\partial v(b_n,s)}{\partial s}$. Since $b_n \to b$, the partial derivative $\dfrac{\partial v(b_n,s)}{\partial s} \to \dfrac{\partial v(b,s)}{\partial s}$. We conclude that p is differentiable at s with

$$p'(s) = \frac{\partial v(b,s)}{\partial s}.$$

□

Theorem 10.1 (characterizing perfect competition in the supermodular model)**.** *Consider the supermodular model with $v \in \mathcal{C}^2$. Suppose μ is nontrivial. Then μ is perfectly competitive iff it has the overlap property and there is a marginal trader in μ.*

Proof. Necessity has already been proved, we need only prove sufficiency.

The overlap property says that whenever there is a seller jump, there is not a corresponding buyer jump. Hence, looking at the contrapositive, the overlap property implies that whenever there is a buyer jump, there is not a corresponding seller jump. More precisely, whenever there is a buyer jump in B from b to $b' \equiv b+$, where $b \geq b_x$, there is an s such that both $(b,s), (b',s) \in \text{supp } x$. [Proof: Let $s = \max\{s_0 \in S : (b, s_0) \in \text{supp } x\}$, and let $s' = \min\{s_0 \in S : (b', s_0) \in \text{supp } x\}$. If

$s' > s$, there would be a seller jump from s' to s and no b such that both $(b, s), (b, s') \in \text{supp } x$, contradicting overlap.]

Now suppose x has the overlap property. Since S and B are compact (hence bounded), there are at most a countable number of jump points in S and B. Hence there is a partition of $S \cap [s_x, \infty)$ into a set of closed intervals

$$[s'_1, s_2], [s'_2, s_3], \ldots, [s'_{m-1}, s_m], [s'_m, s_{m+1}], \ldots$$

satisfying $s_x \equiv s'_1 \leq s_2 \leq s'_2 \leq \ldots$, and for each m ($m = 2, 3, \ldots$), either S jumps from s_m to $s'_m \equiv s_m+$ (in which case $s'_m > s_m$) or $s_m = s'_m$ and B jumps from some $b_m \in B$ to b_m+ where both $(b_m, s_m), (b'_m, s'_m) \in$ supp x.

Lemma 10.4 and Lemma 10.5 imply that for any given price for a match with s_x, say γ, there is a unique price function $p_\gamma \in C(S \cap [s_x, \infty))$ satisfying Equation (10.1) and Equation (10.2) along with the boundary condition that $p_\gamma(s_x) = \gamma$. Hence $p \in P(\mu)$ implies there is a unique γ such that $p(s) = p_\gamma(s)$ for all $s \in S$ s.t. $s \geq s_x$. Further, Lemma 10.2 pins down γ: for all $p \in P(\mu)$, $p(s_x) = 0$ if s_x is a marginal trader, while $p(s_x) = v(b_x, s_x)$ if b_x is a marginal trader. We can conclude that for any $p, p^0 \in P(\mu)$, the price vectors p and p^0 must agree for all $s \in S$ satisfying $s \geq s_x$. In view of Lemma 10.2, p and p^0 also must agree for all $s \in S$ satisfying $s < s_x$. Hence $\hat{P}(\mu)$ is a singleton. □

Remark 10.1. Readers familiar with CMP1 and CMP2 will see that many of the ideas and results leading up to Theorem 10.1 have been developed there. In particular, the importance of overlap comes from CMP1, the differential pricing equation in Lemma 10.5 is also derived in CMP2. The importance of a marginal trader, however, is not emphasized in CMP1 or CMP2 since their goal is not to characterize perfect competition in the strong sense used here. The authors downplay the potential presence of bilateral monopoly surpluses at the lower tail.

10.2. **The efficiency of equilibria in the supermodular model.** Even in the general investment model, we found that perfect competition eliminates Felli-Roberts-type investor mismatch equilibria. We will now show that, in the supermodular model, perfect competition also eliminates the possibility of overinvestment equilibria.

Theorem 10.2. *Assume supermodularity and let $\overline{S} = [0, \overline{s}]$. If (ν, ϱ) is a perfectly competitive investment equilibrium, and if (x, p) is ex post Walrasian for $\mu \equiv \nu_{\mathbf{A}}$ with $p \in \hat{P}(\mu)$, then (ν, x, p) is ex ante Walrasian relative to $\overline{S}$. Hence ν is efficient relative to $\overline{S}$.*

Proof. Let $\mu = \nu_{\mathbf{A}}$, let (x, p) be ex post Walrasian for μ with $p \in \hat{P}(\mu)$. We will show (ν, x, p) is ex ante Walrasian relative to $\overline{S}$. Since μ is perfectly competitive, $\varrho(s, s')(s') = p(s')$ for all $s \in S$ and $s' \in \mathbf{S}$. Hence, since (ν, ϱ) is an investment equilibrium, for each seller $(s, j) \in$ supp ν,

$$p(s) - c(s, j) \geq p(s') - c(s', j), \quad \forall s' \in \mathbf{S},$$

a fortiori for all $s' \in \overline{S}$. We need only verify that for each buyer $(b, i) \in$ supp ν and $(b, s) \in$ supp x:

$$v(b, s) - p(s) - c(b, i) \geq v(b', s') - p(s') - c(b', i), \quad \forall b' \in \mathbf{B},\ \forall s' \in \overline{S}.$$

Since (ν, ϱ) is an investment equilibrium, the above holds for all $s' \in S$. We need only check that it also holds for the extra commodities in $\overline{S} - S$. Assume the contrary, that there is a match $(b', s') \in \mathbf{B} \times \overline{S}$ such that

$$v(b', s') - p(s') - c(b', i) > v(b, s) - p(s) - c(b, i).$$

We will show a contradiction.

We first show that $s' > s_x$. Recall from Lemma 10.2 that $p(s_x)$ equals either 0 or $v(b_x, s_x)$. If $p(s_x) = 0$, b' could earn at least as much profit matching with s_x as with any $s' < s_x$. That is, if (b', s') is a profitable deviation for i, so is (b', s_x). But $s_x \in S$, contradicting (ν, ϱ) is an investment equilibrium. On the other hand if $p(s_x) = v(b_x, s_x) > 0$, then $p \in \hat{P}(\mu)$ implies

$$p(s) = v(b_x, s), \quad \forall s \in [0, s_x].$$

Thus if $b' \leq b_x$ and $s' < s_x$, the fact that v is increasing implies

$$v(b', s') - p(s') - c(b', i) \leq v(b_x, s') - p(s') - c(b', i) = -c(b', i) \leq 0,$$

contradicting b' is a profitable deviation. On the other hand, if $b' > b_x$, supermodularity implies

$$v(b', s_x) - v(b_x, s_x) > v(b', s') - v(b_x, s'),$$

hence

$$v(b', s_x) - p(s_x) > v(b', s') - p(s').$$

That is, if (b', s') is a profitable deviation for i, then (b', s_x) is even more profitable, again contradicting (ν, ϱ) is an investment equilibrium.

Since $s' > s_x$ and $s' \notin S$, there must be a seller jump from some l to h in S such that $l < s' < h$; further, since μ is perfectly competitive, overlap implies there must a buyer $k \in B$ such that both $(k, l), (k, h) \in$ supp x. There are two cases to consider. Suppose first that $b' \geq k$. Since in equilibrium i chose (b, s):

$$v(b', h) - p(h) - c(b', i) \leq v(b, s) - p(s) - c(b, i).$$

Hence it will suffice to show $v(b', s') - p(s') - c(b', i) \leq v(b', h) - p(h) - c(b', i)$ or $v(b', s') - p(s') \leq v(b', h) - p(h)$. Since competition implies $p(h) - p(s') = v(k, h) - v(k, s')$, it will suffice to show $v(k, h) - v(k, s') \leq v(b', h) - v(b', s')$. But this follows immediately from the supermodularity of v.

Suppose instead that $b' < k$. Then since i chose (b, s) in equilibrium:

$$v(b', l) - p(l) - c(b', l) \leq v(b, s) - p(s) - c(b, i).$$

It will suffice to show $v(b', s') - p(s') - c(b', i) \leq v(b', l) - p(l) - c(b', i)$ or $v(b', s') - v(b', l) \leq p(s') - p(l)$. Since competition implies $p(s') - p(l) = v(k, s') - v(k, l)$, it suffices to show $v(b', s') - v(b', l) \leq v(k, s') - v(k, l)$. But this also follows immediately from the supermodularity of v. □

Since the supermodular model is one dimensional, "overinvestment" and "underinvestment" are meaningful. In this model, the following stronger version of Corollary 6.2 holds. Example 6.1 already illustrated the possibility of two-sided underinvestment in a perfectly competitive equilibrium.

Corollary 10.1 (no overinvestment equilibria)**.** *Suppose supermodularity and that (ν, ϱ) is a perfectly competitive investment equilibrium. If ν' is feasible and*

$$G(\nu') > G(\nu),$$

then there must be underinvestment by sellers in μ:

$$\max\{s \in \text{supp } \mu'\} > \max\{s \in \text{supp } \mu\} \equiv \overline{s}$$

and also underinvestment by buyers in μ:

$$\max\{b \in \text{supp } \mu'\} > \max\{b \in \text{supp } \mu\} \equiv \overline{b},$$

where $\mu = \nu_{\mathbf{A}}$ and $\mu' = \nu'_{\mathbf{A}}$.

Proof. Since μ is efficient relative to $\overline{S}$, the first assertion of the corollary is immediate. Given Theorem 10.2, the proof of the second assertion is analogous to that of Corollary 6.2. □

10.3. **Contrast with CMP2.** In contrast to Corollary 10.1, the CMP2 approach to perfect competition leads to the conclusion that underinvestment *and* overinvestment equilibria are possible even under perfect competition. CMP2's overinvestment example is presented next, along with an explanation of how to reconcile their conclusion with ours.

Example 10.1 (CMP2's overinvestment example)**.** Assume $\mathbf{B} = \mathbf{S} = [0, \infty)$. (The lack of compactness will not matter.) Consider the supermodular model in which the value function is

$$v(b, s) = \begin{cases} bs, & \text{if } bs \le 1/2 \\ 2(bs)^2 & \text{if } bs > 1/2. \end{cases}$$

In the ex ante population $\mathcal{E}$, both buyers' and sellers' types are uniformly distributed on the interval $[\underline{i}, \overline{i}] = [.2, .3]$. Take $\mathcal{E}(I) = \mathcal{E}(J) = .1$, so we can imagine there is one buyer and one seller of each type. The cost functions are $c(b, i) = b^5/(5i)$ for all b and $c(s, j) = s^5/(5j)$ for all s.

CMP2 show that, because of the absence of differentiability in v when $bs = 1/2$, total surplus is maximized when the buyer of type i and seller of type $j = i$ choose attributes $b^*(i)$ and $s^*(i)$, respectively, satisfying

$$\begin{aligned} b^*(i) &= s^*(i) = \sqrt[3]{i}, & \forall i \in [.2, i^*) \\ b^*(i) &= s^*(i) = 4i, & \forall i \in [i^*, .3], \end{aligned}$$

where $i^* = (3/2^9)^{3/10} \approx .21$. So in the efficient allocation for this economy there is a buyer jump from $b = \sqrt[3]{i^*}$ to $b' = 4i^*$, and a corresponding seller jump.

Although there exists an investment equilibrium in which everyone invests efficiently (recall Theorem 9.1), this is not the only possibility. In particular, there also exists an overinvestment equilibrium in which the buyer of type i and seller of type $j = i$ choose attributes $b(i)$ and $s(i)$ satisfying

$$b(i) = s(i) = 4i, \quad \forall i \in [.2, .3],$$

so all types of buyers and sellers in $[.2, i^*)$ overinvest. Let ν denote these inefficient attribute choices, let $\mu = \nu_{\mathbf{A}}$, and let x denote an efficient assignment for μ. Since x is positively assortative, it matches the lowest attribute buyer $\underline{b} = .8$ with the lowest attribute seller $\underline{s} = .8$, and the highest attribute buyer $\overline{b} = 1.2$ with the highest attribute seller $\overline{s} = 1.2$.

Since v is symmetric the differential equation in Lemma 10.5 implies that for any $p \in \hat{P}(\mu)$,

$$p(s) = \frac{1}{2} v(b, s) + \gamma, \quad \forall s \in S \text{ and } (b, s) \in \text{supp } x,$$

where γ (coming from the constant of integration) can be any value

$$\gamma \in [\underline{\gamma}, \overline{\gamma}) \equiv [-\frac{1}{2} v(\underline{b}, \underline{s}),\ \frac{1}{2} v(\underline{b}, \underline{s})].$$

So, modulo the constant, each matched couple evenly splits the ex post surplus they generate.

The reader can verify that there exist rational conjectures ϱ that support ν as an investment equilibrium, with each individual conjecturing prices will equal p_0 (split the surplus) if no one deviates, and each individual conjecturing that prices will move disadvantageously for him if he deviates. [In particular, analogous to the proof of Theorem 9.1, let each buyer (seller) conjecture that any deviation from ν will lead sellers (buyers) to be tough bargainers that demand their entire marginal products. In accord with the lattice property of prices in the assignment model, in this example $\overline{p} \equiv p_0 + \overline{\gamma}$ (resp. $\underline{p} \equiv p_0 + \underline{\gamma}$) is the price function in which all sellers (resp. all buyers) get their marginal products.]

In preparation for contrasting our analysis to the analysis in CMP2, we emphasize that the above overinvestment equilibrium is not perfectly competitive. Indeed there exists no perfectly competitive investment equilibrium supporting ν because the ex post distribution of attributes μ is not perfectly competitive. This follows immediately from the multiplicity of possible ex post Walrasian prices $\hat{P}(\mu)$ (recall Proposition 5.1), or equivalently from the non-uniqueness of the ex post core, or from the absence of a marginal trader in μ (Theorem 5.2). In particular, the size of the bilateral monopoly surplus that any matched couple must bargain over in the above equilibrium equals $\overline{\gamma} - \underline{\gamma} = v(\underline{b}, \underline{s})$. For (ν, ϱ) to be an equilibrium, each individual must bargain vigorously in the ex post assignment game to obtain 50% of the surplus he helps to generate in his match. For example, if sellers conjecture that buyers will be tough ex post bargainers that force equilibrium prices down to $\underline{p}$, the seller of type $\underline{i}$ would invest $s = 0$ instead of $\underline{s} = .8$, anticipating a 100% holdup.

To illustrate Lemma 10.3, we briefly point out that the efficient investment equilibrium for this economy, which requires a buyer jump from $b = \sqrt[3]{i^*}$ to $b' = 4i^*$ and a corresponding seller jump, is also not perfectly competitive. This time for 2 reasons: In the efficient ex post population, say μ^*, there will be no marginal trader (just as in μ), so the terms of trade at the lower tail will still be indeterminate. In addition, since buyer and seller attributes jump simultaneously, the overlap condition will be violated, which implies a second bilateral monopoly surplus will emerge at the simultaneous jump point.

Recalling our motivating question — When will competition solve holdup problems?, — we point out that in the investment equilibrium that supports μ^* it is not competition that solves the holdup problem, but a particular conjecture about individuals' relative bargaining skills; i.e., that everyone will be able to bargain for 50% of the surplus in his match and hence avoid holdups. By contrast, if the ex post population were perfectly competitive, relative bargaining skills would be irrelevant. Competition between buyers and between sellers would determine ex post prices uniquely, there would be no bilateral monopoly surpluses to haggle over. $\diamond$

We now contrast the above analysis of Example 10.1 with the analysis in CMP2. As in GOZ2, in this paper we view each individual in a nonatomic economy as an infinitesimal, — the limit of a sequence of smaller and smaller groups with positive mass. A consequence of this approach is that perfect competition is not automatic in the continuum, indeed it can easily fail as illustrated by both the continuum version of Edgeworth's master-servant example and by CMP2's more complex example with a continuum of different types.

By contrast, CMP2 view each individual in the continuum as a measure zero point on the real line. From this alternative perspective, it is natural to jump to the conclusion that each individual must be a perfect competitor when he is nonatomic. After all, how can a measure zero point influence equilibrium prices by deviating? To their credit, CMP2 realize that things are not quite so simple. They construct a clever definition of feasibility in the continuum to ensure that any deviating individual will not be able to influence equilibrium prices, and hence to give a foundation to their strong priors that in the continuum each individual should have competitive conjectures (by which they mean, each individual should conjecture that he will not be able to influence equilibrium prices by deviating).[6]

CMP2 show that the overinvestment allocation ν in the above example will be an equilibrium if prices are p_0 (split the difference) and everyone holds the competitive conjectures that prices will remain at p_0 after any deviation. CMP2 also show that the same competitive conjectures can lead to an underinvestment equilibrium. Thus the economy in Example 10.1 is used to illustrate the authors' assertion that both underinvestment and overinvestment equilibria can survive even under

[6]See Definition 1 on p. 344 of CMP2 and the discussion following this definition. It is interesting to find that a master-servant-type example is presented by the authors to motivate the definition, whose explicit role is to ensure even such examples will be catalogued as perfectly competitive in the continuum. By contrast the GOZ2 approach, with an eye toward asymptotic meaningfulness, insists that such examples are imperfectly competitive even in the continuum.

perfect competition, although Felli-Roberts-type investor mismatches canot survive.

As we emphasized in Section 4.1, the danger of taking perfect competition for granted in the continuum is that one's continuum analysis may not be asymptotically meaningful: it may not be representative of what will happen in large but finite economies. The CMP analysis of the above example illustrates the danger. There is a sequence of finite populations $\langle m^n \rangle$ whose normalized distribution converges to μ, and each finite population m^n is not even approximately perfectly competitive — no matter how large the population is. Indeed, the same bilateral monopoly surpluses we have highlighted in analyzing the continuum population μ in Example 10.1 will re-appear in analyzing the large finite populations m^n. (See the proof of Theorem 5 in GOZ2 for the construction of large-but-finite approximations to imperfectly competitive continuum populations μ.)

Let us return to Example 10.1 one more time to illustrate the difference between the two approaches. This time we will focus on the rationality of competitive conjectures, in the sense of CMP2. (Keep in mind we have defined an investment equilibrium as one in which individuals' conjectures are rational.) We observe that no matter which $p \in \hat{P}(\mu)$ one picks — including $p = p_0$, — an arbitrarily small group of similar deviating individuals with positive mass can affect prices. In particular, if an arbitrarily small group of buyers deviates from any $b \in [.8, 1.2]$ to $b' = 0$, there would be some sellers who would have to match with these unproductive buyers or remain unmatched since $\mathcal{E}(i) = \mathcal{E}(j)$. Competition among sellers for productive matches — even for matches with buyer $\underline{b}$ or buyers close to $\underline{b}$, — would force $p(\underline{s})$ to drop all the way down to 0, and hence force the equilibrium price function to drop to $\underline{p}$. Thus the only rational conjecture for any deviating buyer $b \in \text{supp}\ \mu$ is

$$\varrho(b, b') = \underline{p}, \qquad \text{when } b' = 0.$$

Similarly, if an arbitrarily small group of sellers deviates from any $s \in [.8, 1.2]$ to $s' = 0$, buyers competing for productive matches would force equilibrium prices all the way up to $\overline{p}$, so the only rational conjecture for any seller $s \in \text{supp}\ \mu$ is

$$\varrho(s, s') = \overline{p}, \qquad \text{when } s' = 0.$$

We concluded that the competitive conjectures

$$\varrho(b, b') = \varrho(s, s') = p_0, \quad \forall b, s \in A, \quad \forall b', s' \in \mathbf{A}$$

are not rational in Example 10.1 when we view each individual as infinitesimal, the limit of small groups having positive mass. The analogy to the master-servant example should be clear.

To pay one last visit to asymptotics, we emphasize that whether or not one views Example 10.1 as perfectly competitive is not just a matter of "definitions." If one accepts that continuum analysis should be representative (in an idealized way) of what will happen in large but finite economies, one cannot accept the view that the continuum economy in this example is perfectly competitive. Return to the sequence of finite economies $\langle m^n \rangle$ approaching μ. No matter how large n is, many (discrete) individuals in m^n will rationally observe that they can significantly affect equilibrium prices by their deviations. Thus each large economy m^n approaching μ is not even approximately perfectly competitive.

Appendix A. Application: The GOZ2 revelation game

In the GOZ2 model, attributes are endowed, not produced. That is, the ex post population is exogenously given. They focus on a housing model, so any seller's attribute $s = (h, r) \in \mathbf{S}$ is interpreted as a particular type of house initially owned by a seller with reservation value r. A buyer's attribute $b \in \mathbf{B}$ is interpreted as a taste parameter: any buyer with attribute b would obtain utility $v^H(b, h)$ from owning any house of type $h \in H$.

GOZ2 assumes asymmetric information: Nature picks the actual ex post population $\mu \in M_+(\mathbf{A})$, but each individual only knows his own attribute a, not others' attributes — hence no one knows μ. In the spirit of mechanism design, GOZ2 is interest in when individuals will truthfully reveal their types, so the mechanism can assign matches efficiently. The authors show that — for finite assignment models — perfect competition leads to truthful revelation. They also establish upper bounds on the incentive to misrepresent in finite models that are nearly perfectly competitive. We can use our machinery to extend the GOZ2 finding to continuum assignment models: In such models perfect competition also leads to truthful revelation. Here is a sketch of the details.

As in the text, let ϱ be rational conjectures about what will happen to Walrasian prices (hence the core allocation) if an infinitesimal individual—the limit of small groups of similar individuals with positive mass—deviates starting from μ. Now an $a \in A$ deviating to a $d \in \mathbf{A}$ is interpreted as an individual with attribute a claiming his true attribute is d. As in GOZ2, we permit any buyer b to claim he is any

$d \in \mathbf{B}$, and we permit any seller $s = (h, r)$ to claim he is any $d = (h, r')$, where $r' \in [0, \overline{V}]$. So buyers can misrepresent their willingness to pay, while sellers can misrepresent their opportunity cost of selling.

Again as in GOZ2 we consider the revelation game in which each individual announces a permissible attribute d, the mechanism aggregates these announcements into a putative population μ', and then assigns a Walrasian allocation for μ'.

Assume it is common knowledge among individuals that they are in *some* perfectly competitive population. We will show that truthtelling is a Bayesian Nash equilibrium of the above revelation game. The argument is simple. The key is that everyone rationally believes that he cannot affect the final equilibrium price function p by misrepresenting his attribute, assuming no one else misrepresents — even though no one knows μ, and hence p, when announcing his type. Here are the details.

Assume μ is perfectly competitive with p the unique price function in $\hat{P}(\mu)$. Let $S^*(b) = \{s \in S^0 : v(b, s) - p(s) = v_b{}^*(p, S)\}$. If b deviates to $d \in \mathbf{B}$, since he cannot affect the price of any available good, his utility would (weakly) decrease from $v_b{}^*(p)$ to $v(b, s') - p(s')$, where $s' \in S^*(d)$. So, truth-telling is a best response for each buyer $b \in B$.

Similarly, if $s = (h, r) \in S$ deviates to $d = (h, r')$, his utility would (weakly) decrease from $v_s{}^*(p)$ to either $r(s)$ or $p(s)$ depending on his choice of d (e.g., if $r' < p_h$, then his utility from deviating would be p since he would be matched with a buyer; and if $r' > p_h$, then his utility from deviating would be r since he would not be matched with a buyer). So truthtelling is also a best response for each seller $s \in S$.

It is interesting to see that the above argument does not depend on any particular distribution of beliefs over the populations in $M_+(\mathbf{A})$; it only depends on each individual's probability beliefs having support in the subset of perfectly competitive populations. So the above Bayesian Nash equilibrium has a robustness feature not shared by Bayesian equilibria in general. Since in the housing model perfectly competitive populations are generic, one's confidence in the above beliefs increases.

GOZ2's proof is different, although also simple. They show that in finite models, $mp^-(a)$ is the most any individual with attribute a can hope to gain by misrepresenting (i.e., by deviating). Hence perfect competition in a finite model immediately implies no incentive to misrepresent. We point out that a similar proof is possible in the continuum: $MP^-(a)$ is an upper bound for the incentive to misrepresent of any infinitesimal individual with attribute a. This alternative line of proof will not be amplified on here.

Based on their continuum analysis, GOZ2's Theorem 5 gives bounds on the incentive to misrepresent in nearly perfectly competitive finite populations. Theorem 8.2 in the current paper significantly improves on these bounds for deviations outside the support of the population (see Remark 8.1 in the current text). Going to the other way, starting from the continuum, one can ask how representative is our conclusion — that there is no incentive to misrepresent in perfectly competitive continuum economies, — representative of what will happen in large but finite economies? The asymptotic results in Section 8 of the current paper can be used to answer this question.

Remark A.1. One goal of the current paper is to correct an important error discovered in GOZ2. Here we will give a brief description of the error.

GOZ2 define PEDS at μ as the price correspondence P being continuous at μ. Their Theorem 3 claims this strong version of PEDS is equivalent to a unique core, a unique normalized Walrasian price vector, etc. Their Theorem 3 is true if μ has full support, but false in general. [The slip in their proof of Theorem 3 may be hard to spot. They consider a sequence of prices $\langle p_n \rangle$ converging to a price $p \in P(\mu)$. The slip is their inference that p must be in the set of *normalized* prices.] Our fix of their Theorem 3 involves two steps. First our definition of PEDS is more qualified, limiting attention to deviations from μ inside the support of μ. This leads to a more qualified version of their Theorem 3, Proposition 5.1 in the text, which focuses on deviations inside the support of μ. Then the idea of strong PEDS is introduced. Notice even strong PEDS is weaker than the GOZ2 definition of PEDS: strong PEDS recognizes that even infinitesimal buyers may be able to influence some reservation prices of unavailable commodities. Example 5.1 illustrates that strong PEDS is as good as one can hope for if one wants perfect competition to be equivalent to a unique core, a unique normalized Walrasian price vector, etc.

There is a related slip in GOZ2's normalization of subgradients. Given any $q \in \partial g(\mu)$, they define

$$\overline{q}(b) = \inf_{s \in S(\mu)} v(b,s) - q(s)$$
$$\overline{q}(s) = \inf_{b \in B(\mu)} v(b,s) - q(b).$$

They take this family of $\overline{q}$'s as their family of normalized subgradients. The slip is their assertion that $\overline{q}$ is in $\partial g(\mu)$. Typically it is not. Our fix is to use the shrink-fit method of normalization in GOZ1. Roughly speaking, the order of shrink fitting leads to one or the other of the above two equations holding, but not both. A consequence of this slip is that GOZ2's proof that the family of normalized subgradients is equicontinuous (GOZ2, Proposition 4) is flawed. But GOZ1's proof of equicontinuity, which uses the shrink-fitted family of normalized subgradients, can be substituted for this part of Proposition 4 in GOZ2. So the problem with Proposition 4 has a simple fix.

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