Is it Possible to Define Subjective Probabilities in Purely Behavioral Terms? A Comment on Epstein-Zhang (2001)

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Abstract

It is shown that well-behaved preference orderings may exhibit the Ellsberg paradox on the set of unambiguous events as defined by Epstein and Zhang (2001). Moreover, since such counterexamples can be constructed even when the set of unambiguous events is rich, EZ's main representation result does not clarify satisfactorily when the proposed definition delivers probabilistic sophistication on unambiguous events. We conclude by conjecturing that these problems indicate the existence of inherent limitations of a strictly behavioral approach to identifying probabilistic beliefs in the presence of ambiguity, rather than deficiencies in EZ's implementation of that approach.

Keywords: ambiguity, risk, subjective probability.

1. INTRODUCTION

A fundamental question in the theory of decision-making under uncertainty is the identification of a decision-maker's probabilistic beliefs from preferences. This has been addressed in the literature so far mainly under the heading of finding an appropriate definition of "unambiguous events". The earliest proposals of Ghirardato-Marinacci (2001), Nehring (1999)[1] and Zhang (2002) all imply expected utility maximization over unambiguous acts (acts measurable with respect to unambiguous events), and are therefore inapplicable to decision-makers who depart from SEU not just for reasons of ambiguity, but also for reasons of probabilistic risk-attitudes as exemplified by the Allais paradox. Epstein-Zhang's (2001, henceforth EZ) contribution is the first and hitherto only contribution to attempt a "behaviorally general" ("model-free") definition. This appears to be a big step forward beyond the existing definitions in that it promises to deliver a generally applicable distinction between risk and ambiguity. In this note, we ask whether their definition works as intended.

EZ's principal criterion of the success of a definition of unambiguous events is that it delivers probabilistic sophistication on unambiguous acts. To demonstrate the success of their definition, the main representation result in EZ provides sufficient conditions that entail this property. While these look fairly weak, we show by example that even when preferences are extremely well-behaved (e.g. of the MEU variety) and the set of unambiguous events is "rich", they may exhibit the Ellsberg paradox *on the set of EZ-unambiguous acts;* a fortiori, such preferences cannot be probabilistically sophisticated on unambiguous acts. EZ thus fail to achieve the intended separation of probabilistic risk from ambiguity.[2]

[1]Building in part on Ghirardato et al. (2003), an equivalent definition is also proposed in Ghirardato et al. (2004).

[2]For a technically deep analysis of EZ unambiguous events in the context of the MEU model that in part builds on and develops the observations of this note, see Amarante-Filiz (2004).

Our counterexamples do not contradict EZ's representation result at the formal level. Instead, they reveal that the content of one of their key axioms on preferences is not what it seems to be; in particular we shall argue that this axiom ("Small Unambiguous Event Continuity") cannot interpreted as merely imposing "richness" on the set of unambiguous events, contrary to what is suggested by EZ. It thus remains an open question when (that is: under what conditions formulated directly in terms of preferences) the EZ definition "works" in the sense of delivering probabilistic sophistication on unambiguous events.[3]

There are two possible types of responses to our observations. On the one hand, one may conjecture that EZ's choice of a particular purely behavioral definition of unambiguous events was not right one, and try to come up with a better one by either refining EZ's or starting from scratch. Alternatively, one may conclude that the problems identified here are likely to resurface for alternative purely behavioral definitions, and that some non-behavioral element such as the exogenous identification of a subfamily of unambiguous events must be assumed to begin with. We have written this note out of a belief in the second type of response as the more promising, that is: out of a belief that the limitations of EZ's proposal are not accidental, but indicative of a fundamental, deep-seated difficulty in conceptualizing decision making under ambiguity itself.

2. THE EPSTEIN-ZHANG DEFINITION FOR BETTING PREFERENCES

Let (S, Σ) be a measurable space where S is the set of states and the universe of events Σ is a σ-algebra. Throughout, we will focus on domains with two possible

[3]Kopylov (2003) has provided an elegant modification of EZ's representation result. However, as explained below in section 2, Kopylov's result does not help address the issues raised in this note.

outcomes only, *win* and *lose*. This case not only allows to simplify the exposition significantly, it is also central to the intuitive motivation of EZ's definition of unambiguous events. With only two outcomes, one can denote the act [*win* on A, *lose* on A] ("betting on A") simply by the event $A \in \Sigma$. Thus, the decision maker is characterized by a preference ordering over events $\succsim$; the preference ordering $\succsim$ is assumed to be monotone and non-degenerate.

Axiom 1 (Monotonicity) *$A \succsim B$ whenever $A \supseteq B$, and $A \succ B$ whenever $A \backslash B$ is non-null, i.e. whenever $(A \backslash B) \cup C \succ \emptyset$ for some $C \in \Sigma$.*

Axiom 2 (Non-Degeneracy) $S \succ \emptyset$.

In the two outcome context, EZ's definition of an unambiguous event amounts to the following.

Definition 1 *An event T is* **unambiguous** *if, for all A, B disjoint from T, $A \succsim B$ if and only $A \cup T \succsim B \cup T$, and if the same holds for T^c instead of T.*

The family of all unambiguous events is denoted by $\mathcal{A}$. Intuitively, an event is unambiguous if it is evaluated separably by the decision maker. As pointed out by EZ, in the context of the typical ambiguity-averse preferences in the Ellsberg experiment with one urn with three colors the definition successfully identifyies the color with known frequency as unambiguous, and the other two as ambiguous.

The question at the center of EZ and of this note is the extent to which this definition yields well-defined subjective probabilities over the family of unambiguous events $\mathcal{A}$ in general. There are several criteria to determine whether, given a preference ordering $\succsim$, the decision-maker has "well-defined subjective probabilities" over a given family of events $\mathcal{B} \supseteq \{\emptyset, S\}$. The central criterion of EZ is the requirement that preferences over acts measurable with respect to $\mathcal{B}$ be "probabilistically sophisticated".[4] In a two-outcome context, this boils down to the following requirement.

(Probabilistic Sophistication on $\mathcal{B}$) There exists a finitely additive set-function $p : \mathcal{B} \to [0, 1]$ with $p(S) = 1$ such that, for all $A, B \in \mathcal{B}$,

$$A \succsim B \text{ if and only if } p(A) \geq p(B). \tag{1}$$

If it is indeed appropriate to attribute well-defined subjective probabilities to all events in $\mathcal{B}$, then one should be able to attribute such probabilities to all events whose probability can be deduced from those in $\mathcal{B}$. This leads to the requirement that $\mathcal{B}$ be a λ-system; a family $\mathcal{B}$ is a (finite) λ-system if it is closed under complementation and disjoint union, i.e. if i) $S \in \mathcal{B}$, ii) $A \in \mathcal{B}$ implies $A^c \in \mathcal{B}$, and iii) $A, B \in \mathcal{B}$ and $A \cap B = \emptyset$ implies $A \cup B \in \mathcal{B}$; $\mathcal{B}$ is a countable λ-system if it is closed under countable disjoint unions, i.e. if $\cup_{i=1}^{\infty} A_i \in \mathcal{B}$ whenever the $A_i \in \mathcal{B}$ are mutually disjoint.

When does EZ's definition of unambiguous events $\mathcal{A}$ deliver probabilistic sophistication on unambiguous events $\mathcal{A}$, preferably with $\mathcal{A}$ a λ-system? It is clear from the outset that one cannot expect this to happen in full generality, even when all events are unambiguous ($\mathcal{A} = \Sigma$); for in this case, $\succsim$ is what is called a "qualitative probability" in the literature, and Kraft et al. (1959) have shown that qualitative probabilities on finite state spaces need not be representable in terms of numeric probabilities.

To support their definition and overcome this difficulty, the main result in EZ provides assumptions on preferences which ensure that preferences on unambiguous acts are probabilistically sophisticated. Applied to betting preferences, EZ make the

[4] Arguably, having beliefs p entails substantially more than condition (1), for example the extendability of p to an additive set-function on all of Σ; however, since the main point of this note concerns the difficulty of guaranteeing Probabilistic Sophistication on $\mathcal{A}$, the family of EZ-unambiguous events, these additional desiderata are not be discussed here.

following assumptions. The first one is central to their result, while the technical is merely auxiliary.

Axiom 3 (Small Unambiguous Event Continuity) *For any $A, B \in \mathcal{A}$ such that $A \succ B$, there exist partitions $\{C_i\}_{i\in N}$ and $\{D_j\}_{j\in M}$ in $\mathcal{A}$ that refine the partitions $\{A, A^c\}$ respectively $\{B, B^c\}$ such that $A\backslash C_i \succ B$ for all $i \in N$ and $A \succ B \cup D_j$ for all $j \in M$.*

Axiom 4 (Monotone Continuity) *Consider any decreasing sequence $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{A}$, B, C in $\mathcal{A}$ with B disjoint from A_1. Then $A_i \cup B \succsim C$ for all i implies $(\cap_{i=1}^{\infty} A_i) \cup B \succsim C$.*

Restricted to bets, a corrected version of EZ's main result is the following:

Theorem 1 *Let $\succsim$ a monotone preference order over bets and assume that the corresponding set of unambiguous events $\mathcal{A}$ is a countable λ-system. Then the following two statements are equivalent:*

1. *$\succsim$ satisfies Non-Degeneracy, Small Unambiguous Event Continuity, and Monotone Continuity.*
2. *There exists a (unique) convex-ranged and countably additive probability measure p on $\mathcal{A}$ such that, for all $A, B \in \mathcal{A}$,*

$$A \succsim B \text{ if and only if } p(A) \geq p(B).$$

In contrast to Theorem 1, in EZ's original statement (EZ, Theorem 5.2), the λ-system property of $\mathcal{A}$ is derived rather than assumed; however, according to Kopylov (2003, p. 31), this claim is false.[5] Kopylov (2003) shows instead that, in general,

[5] It is an open question to what extent this property can be derived from intuitively more primitive assumptions. Amarante-Filiz (2004) show that A is a λ-system whenever preferences have an MEU representation.

$\mathcal{A}$ is closed under disjoint countable unions of a particular kind, calling the resulting generalization of λ-systems "mosaics". In an elegant extension of EZ's work, he also provides a derivation of probabilistic sophistication on $\mathcal{A}$ for the general case of mosaics assuming a strengthened version of Axiom 3 (ibid., Corollary 4.2) while dispensing with Monotone Continuity. Since we will argue that Axiom 3 is too restrictive already, the following analysis applies to Kopylov's version of EZ's result as well.

3. DO EZ UNAMBIGUOUS EVENTS SEPARATE RISK FROM AMBIGUITY?

Crucial to the assessment of the import of Theorem 1 is an assessment of the domain of its applicability. The following examples will show that it is substantially smaller than apparent and, more importantly, not transparently defined.

Example 1. Fix any event $T \in \Sigma$. Let Π_1 and Π_2 denote two (weak$^*-$)closed and convex sets of finitely additive probability measures such that $\pi(T) = 1$ for all $\pi \in \Pi_1$ and $\pi(T) = 0$ for all $\pi \in \Pi_2$, and fix α, β such that $0 < \alpha < \beta < 1$. Define the (weak$^*-$)closed and convex set Π as follows,

$$\Pi := \{\gamma\pi_1 + (1-\gamma)\pi_2 \mid \alpha \leq \gamma \leq \beta,\ \pi_1 \in \Pi_1,\ \pi_2 \in \Pi_2\}, \tag{2}$$

and let preferences over events determined by their lower probability given Π,

$$A \succsim B \text{ if and only if } \min_{\pi\in\Pi} \pi(A) \geq \min_{\pi\in\Pi} \pi(B). \tag{3}$$

It is easily seen that for *any* such preference relation, both T and T^c are EZ-unambiguous!

Observation 1 *For any preference relation defined by (2) and (3),* $\{T, T^c\} \subseteq \mathcal{A}$

Indeed, for any $A \subseteq T$,

$$\min_{\pi \in \Pi} \pi(A) = \alpha \min_{\pi \in \Pi_1} \pi(A),$$

as well as

$$\min_{\pi \in \Pi} \pi(A \cup T^c) = \beta \min_{\pi \in \Pi_1} \pi(A) + (1 - \beta).$$

Hence evidently, for all $A, B \subseteq T$,

$$A \succsim B \text{ if and only if } \min_{\pi \in \Pi_1} \pi(A) \geq \min_{\pi \in \Pi_1} \pi(B) \text{ if and only if } A \cup T^c \succsim B \cup T^c.$$

Similarly, for all $A, B \subseteq T^c$

$$A \succsim B \text{ if and only if } \min_{\pi \in \Pi_2} \pi(A) \geq \min_{\pi \in \Pi_2} \pi(B) \text{ if and only if } A \cup T \succsim B \cup T,$$

establishing that both T and T^c are unambiguous.

Since $\alpha < \beta$, this classification is clearly counterintuitive. In this example, the fact that preferences are separable in $\{T, T^c\}$ (that is: $\{T, T^c\} \subseteq \mathcal{A}$) picks up the separability of Π implied by (2), but has nothing to do with the existence of probabilistic beliefs with respect to the events T and T^c.[6]

In general, it might be the case that T and T^c are the only non-trivial unambiguous events, in which case preferences are "probabilistically sophisticated on $\mathcal{A}$" in a trivial way. Consider, however, cases in which the set of events over which the decision-maker has probabilistic beliefs is rich in an intuitive sense.

Example 2. Specifically, assume that the state space is the product of a space with two "subjective" states and a continuous state space representing a continuous random device, $S = \{b, r\} \times [0, 1]$, with $\Sigma = 2^{\{b,r\}} \times \Sigma_2$ where Σ_2 denotes the Borel-σ-algebra on $[0, 1]$. One may think of $\{b, r\}$ as representing the outcome of a draw

[6]The example easily extends to multi-outcome domains by adopting, for example, the MEU model; separability conditions as in (2) arise naturally in dynamic versions of this model, see Epstein/Schneider (2003).

from an unknown urn with *b*lack and *r*ed balls. Let μ denote some convex-ranged probability measure on Σ_2, and let $T = \{b\} \times [0,1]$. Say that betting preferences are *compatible with independent randomization* if and only if, for all $A, B \in \Sigma$, $A \succsim B$ whenever $\mu(A \cap T) \geq \mu(B \cap T)$ as well as $\mu(A \cap T^c) \geq \mu(B \cap T^c)$, that is: whenever A is at least as likely than B, conditional on either T or T^c. Independent randomization gives rise to a class of counterexamples that do not make use of the MEU functional form.

Observation 2 *Whenever the preference ordering $\succsim$ is compatible with independent randomization, $\{T, T^c\} \subseteq \mathcal{A}$.*

Again, as in Example 1, the unambiguity of the events T and T^c reflects a separability structure that has nothing to do with the existence of probabilistic beliefs over these events. It is thus not very surprising that combining these two examples, one obtains the 2-color version of the Ellsberg paradox within the family of unambiguous events.

Example 3. Indeed, let Π_1 be the singleton $\{\pi_1\}$, with $\pi_1(A) := \mu(A \cap T)$; likewise, let Π_2 be the singleton $\{\pi_2\}$, with $\pi_2(A) := \mu(A \cap T^c)$. The preference relation $\succsim_3$ defined by (2) and (3) can be seen as a Gilboa-Schmeidler (1989) MEU preference relation in a two-state Anscombe-Aumann framework translated into a Savage setting. Let $\mathcal{A}_3$ denote the associated family of unambiguous events. The following Fact is easily verified.

Fact 1 $A \in \mathcal{A}_3$ *iff*

i) $\mu(A \cap T) = \mu(A \cap T^c)$ *or*

ii) $\mu(A \cap T) = 1$ *and* $\mu(A \cap T^c) = 0$, *or* $\mu(A \cap T) = 0$ *and* $\mu(A \cap T^c) = 1$.

The family of events satisfying i) (denoted by $\mathcal{R}$) reflects the assumed compatibility with independent randomization, while those satisfying ii) correspond to Observation 2. Fact 1 immediately entails the following Observation.

Observation 3 $\succsim_3$ *displays the Ellsberg paradox on* $\mathcal{A}_3$*; that is, there exist events* $A, B \in \mathcal{A}_3$ *such that* $A \succ B$ *and* $A^c \succ B^c$.

To see this, take any $E \in \Sigma_2$ such that $\alpha < \mu(E) < \beta$; then by construction

$$\{b\} \times [0,1] \prec_3 \{b,r\} \times E \text{ and } \{r\} \times [0,1] \prec_3 \{b,r\} \times E^c. \tag{4}$$

Since in view of Fact 1 all four events are EZ-unambiguous, preferences display the Ellsberg paradox on $\mathcal{A}_3$.

4. IMPLICATIONS FOR THE INTERPRETATION OF EZ'S REPRESENTATION THEOREM

What is going wrong here? Clearly, Theorem 1 cannot apply since (4) is inconsistent with Probabilistic Sophistication on $\mathcal{A}_3$. As $\mathcal{A}_3$ is evidently a λ-system in view of Fact 1, the culprit must be Axiom 3. Indeed, Axiom 3 fails to hold since the events $\{b\} \times [0,1]$ and $\{r\} \times [0,1]$ cannot be partitioned into strictly smaller unambiguous subsets. On the other hand, $\mathcal{A}_3$ does contain a "rich" subset of unambiguous events, namely the set of "random" events $\mathcal{R}$. Note in particular that Axiom 3, *when restricted to random events* $\mathcal{R}$, is satisfied by $\succsim_3$. In view of this, the richness motivation of Axiom 3 would only justify requiring Small Event Continuity *with respect to some subset* $\mathcal{B}$ of $\mathcal{A}$, a condition much weaker than Axiom 3. [7]

[7] The surprising strength on Small Event Continuity on λ-systems can be viewed as mirroring a disanalogy of convex-rangedness of probability measures defined on $\sigma-$algebras and those on countable λ-systems. Specifically, suppose that p is a probability measure on a countable λ-system $\mathcal{D}$ that is convex-ranged on some family $\mathcal{B} \subseteq \mathcal{D}$. If $\mathcal{D}$ is a closed under intersections (hence a $\sigma-$algebra), p is must be convex-ranged on the larger family $\mathcal{D}$ as well; yet as shown by an appropriate specialization of Example 2, this conclusion does not hold if $\mathcal{D}$ is merely a λ-system.

If one interprets $\mathcal{B}$ as an exogeneously given set of unambiguous events, and $\mathcal{D}$ as the "true" set of unambiguous events (however defined) this observation suggests that in general, one cannot expect convex-rangedness of p to extend to the endogeneously defined "true" set of unambiguous events.

Requiring Small Event Continuity with respect to $\mathcal{A}$ itself yields an assumption that lacks transparent intuitive content and is more and intransparently restrictive strong. The lack of transparency of the Small Event Continuity axiom is attributable in part to its reliance on the endogeneously defined family of unambiguous events $\mathcal{A}$, a notion whose content EZ's main result, Theorem 1, was meant to clarify with the crucial help of this axiom itself.

It thus remains an open question whether there are sufficient conditions of reasonable generality that are formulated directly in terms of preferences and ensure Probabilistic Sophistication over unambiguous acts. Indeed, it is not obvious that such conditions exist. As Example 2 shows, even the presence of a continuous independent random device is not enough.

Examples 1 through 3 also raise questions concerning the conceptual interpretation of the EZ definition. EZ view their definition of unambiguous events as meaningful whenever preferences are monotone (satisfy P3), including, for example, situations in which the state space is finite. This conceptual assumption appears to be necessary if the definition is used in some of the preference axioms (such as Axiom 3); furtm hermore, there is nothing in the behavioral pattern identified by the EZ definition that would warrant a restriction to particular kinds of monotone preference relations. The existence of even one preference relation within its domain of legislation displaying the Ellsberg paradox on unambiguous events suggests that the EZ definition does not capture absence of ambiguity in a consistent, conceptually primitive manner.[8]

[8] Thus, the upshot of our examples is that the EZ definition is "too weak". Conversely, one may ask whether it always finds all "truly" unambiguous events. In this regard, it has been argued before by Klibanoff et al. (2003) that the EZ definition may sometimes be "too strong" by classifying genuinely unambiguous events as ambiguous; put differently, their criticism is to point out that the EZ definition builds in assumptions on preferences over unambiguous acts that do not follow from the existence of probabilistic beliefs over unambiguous events per se.

5. CONCLUSION

We conclude from the above observations, reinforced by those of Kopylov (2003) and Klibanoff et al. (2003), that EZ's definition of "unambiguous" events fails to deliver a satisfactory separation between risk and ambiguity,

There are two basic responses to this state of affairs. On the one hand, it may be the case that the EZ definition is basically on the right track, but needs to be "fixed" somehow. For example, in the context of Example 2, one feels that the event $\{b\} \times [0,1]$ ("black") cannot really be unambiguous unless events of the from $\{b\} \times E$ representing conjunctions of the original event $\{b\} \times [0,1]$ with independent random events $[0,1] \times E$ are unambiguous as well. A natural approach to fixing the EZ definition would therefore be to try to refine it by building in closure with respect to such conjunctions. While such a move may have some appeal at a formal level, it is not clear whether it can made without losing the intuitive behavioral motivation that makes the EZ definition attractive in the first place; furthermore, it raises the question of how to identify the existence of a rich set of independent random events in purely behavioral terms (as opposed to fixing it exogeneously as done here in the definition of "compatible with independent randomization"), a question that may not be more easily solvable than that of identifying unambiguous events in the first place.

Alternatively, one may conclude that a non-epistemic definition of "unambiguous" events is unlikely to yield probabilistic sophistication on unambiguous acts, and, even more so, epistemically motivated properties such as the closure under disjoint unions characteristic of λ-system. In other words, one needs to put in more epistemic content into the definition of "unambiguous" from the very beginning.

More drastically, it may be necessary to exogeneously specify some events as unambiguous, in order to infer the unamiguity of others behaviorally. This could be justified by imputing certain probabilistic beliefs to the agent on the basis of verbal testimony or a hypothesized and behaviorally falsifiable shared understanding of certain aspects of the decision situation, an approach developed in detail in Nehring (2001,2006). Beliefs of this kind are, in fact, already imputed implicitly in applications of the Anscombe-Aumann (1963) framework to the modeling of preferences under ambiguity.

While such a move has a lot going for it, it represents a break with the strictly behaviorist revealed preference approach to decision theory pioneered by Ramsey and Savage, an approach that continues to dominate much of decision theory and that centrally inspired the Epstein and Zhang's contribution. We conjecture that its limitations do not reflect limitations of the authors in implementing their behaviorist approach but instead reflect deep-seated limitations of that approach itself.

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