A Behavioral Characterization of Common Priors

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January 2003

Abstract
A state-contingent claim has positive expectation with respect to the common prior if and only if it can be decomposed into $n$ agent-specific claims such that it is commonly known that each agent’s expectation of his claim is strictly positive.

1. INTRODUCTION

The Common Prior Assumption (CPA) is a fundamental and pervasive assumption in much of game theory and the economics of information. It is often motivated as expressing the principle “that differences in probability estimates of distinct individuals should be explained by differences in information” (Aumann (1987, p.7). It can be justified either normatively as a requirement of intersubjective rationality, or methodologically, as enabling one to “zero in on purely informational issues in analyzing economic models with uncertainty” (ibid., p. 13). While under asymmetric information, i.e. when there is a real prior stage with commonly known beliefs, the CPA is controversial normatively and descriptively already, it is conceptually unproblematic in this case. By contrast, under genuinely incomplete information without a prior stage, the CPA is not a transparent assumption on the primitives of the model, which are the agents’ belief hierarchies (see especially Gul (1998) and Lipman (1997)). This issue has motivated a recent literature that establishes the CPA’s
meaningfulness via representation theorems. Most of these contributions characterize the CPA as equivalent to the absence of mutually profitable bets or variants thereof; see Feinberg (2000) and Bonanno-Nehring (1999)^1, as well as the subsequent contributions by Samet (1998b) and Halpern (1998)^2. The underlying unity of these two approaches has been shown in Nehring (2001).

A separate important issue is the characterization of the common prior itself. One such characterization has been provided in Samet’s (1998a) work using infinitely deep expectations about expectations ... about expectations of random variables. This is an important achievement, since modelling specific economic situations typically involves assumptions on the “artificial” common prior; Samet’s characterization shows how these can be interpreted in terms of agents’ belief hierarchies.

In this note, we complement Samet’s epistemic characterization by a behavioral one as follows. Assume that all agents are risk-neutral. An outsider offers a bet (state-contingent payoff) to the group which will accept the bet if it can be split up into individual bets in such a way that each member of the group is commonly known to be made strictly better off by accepting his part of the bet. We show that the group will accept the bet if and only if its expectation under the common prior is positive; in particular, the group may accept the bet even if every knows the bet to have a negative payoff (as long as this is not common knowledge). Our result explains how the common prior can have behavioral implications even though it is located at the limit of infinitely iterated higher-order expectations. In a follow-up paper (Nehring 2003), this result is used to axiomatize a utilitarian group decision criterion under incomplete information.

2. FRAMEWORK AND NOTATION

**Definition 1** A rooted type space is a tuple \( \langle I, \Omega, \{ p_i \}_{i \in I}, \tau \rangle \), where

- \( I \) is a finite set of agents.
- \( \Omega \) is a finite set of states; the subsets of \( \Omega \) are called events.
- for every agent \( i \in I \), \( p_i \) is a function that specifies, for each state \( \alpha \in \Omega \), his probabilistic beliefs \( p^\alpha_i : 2^\Omega \rightarrow \mathbb{R} \) at \( \alpha \).

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^1 Bonanno-Nehring (1999) emphasize in addition the role of truth conditions on probability-one beliefs.

^2 These were preceded by Morris’s (1994) work under asymmetric information.
• $\tau$ is the true state.

A type space is simply a state space in which at any state $\alpha$, the agents’ beliefs at that state $p^\alpha_i$ are specified. As a result, an agents’ belief at a state describes not only his beliefs about facts of nature, but also his beliefs about other agents’ (first-order) beliefs about states of nature, hence also his beliefs about agents’ higher-order beliefs about states of nature, thus in effect: an entire belief hierarchy. For example, $p^\alpha_i(\{\omega | p^\omega_j(rain) \geq 0.7\})$ denotes agent $i$’s probability at state $\alpha$ that agent $j$ believes that it will rain with at least 70% probability. A state in a type space can be thus be thought of as a notational device for describing the belief hierarchies of each agent. 3 Fixing a particular state $\tau$ as the “root” fixes a particular profile of belief hierarchies. 4

We will maintain the following two assumptions.

**Assumption 1** (Introspection) For all $\alpha \in \Omega$ and all $i \in I$: $p^\alpha_i(\{\omega \in \Omega | p^\omega_i = p^\alpha_i\}) = 1$.

**Assumption 2** (Truth) For all $\alpha \in \Omega$ and all $i \in I$: $p^\alpha_i(\{\alpha\}) > 0$.

Introspection says that agents are always (at any state $\alpha$) certain of own belief $p^\alpha_i$. Truth states that, for any state that may occur, agents will have put positive probability on that state if it occurs; thus Truth assumes that agents are never wrong in their probability-one beliefs. While standard, this assumption is not unrestrictive.5

For any $\alpha \in \Omega$, let $\Pi_i(\alpha) := \text{supp} p^\alpha_i$. By Introspection and Truth, the family $\Pi_i := \{\Pi_i(\omega) | \omega \in \Omega\}$ is a partition of $\Omega$, $i$’s *type partition*. Note that the type-partition is thus a derived entity rather than a primitive; this reflects the fact that under a general incomplete information interpretation types cannot be interpreted as “signals”.

An agent “knows” an event $E$ at $\alpha$ if he is certain of it, i.e. if $E \supseteq \Pi_i(\alpha)$. Let $\Pi$ denote the meet (finest common coarsening) of the partitions $\{\Pi_i\}_{i \in I}$, with $\Pi(\alpha)$ denoting the cell of the meet containing state $\alpha$. $E$ is *common knowledge* if everybody knows that $E$, and if everybody knows that everybody knows that $E$, and so forth. Formally, $E$ is “common knowledge” at $\alpha$ if $E \supseteq \Pi(\alpha)$. Since type spaces serve as a notational vehicle to represent hierarchies of (interim) beliefs, we will assume throughout that the state space includes only states that are relevant to their description.

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3By results due to Armbruster-Boege (1979) and Mertens-Zamir (1985), any profile of probabilistic belief hierarchies has a type-space representation; the assumption that the state space $\Omega$ is finite is restrictive but entirely standard. Infinite state-spaces are considered in Halpern (1998) and Feinberg (2000).

4Rooted type spaces have been introduced in Bonanno-Nehring (1999).

5See Bonanno-Nehring (1999) for a detailed study of its relaxation.
i.e. that $\Pi = \{\Omega\}$. It is then unambiguous to speak of “common knowledge of an event”, without reference to the state.

A probability measure $\mu : 2^\Omega \to \mathbb{R}$ is a common prior if, for all $i \in I$, $\omega \in \Omega$ and $A \subseteq \Omega$, $p_i^\alpha(A) = \mu^\alpha(A/\Pi_i(\omega))$ whenever $\mu(\omega) > 0$. In view of the partitional structure of the $\Pi_i$ and the assumption that $\Omega$ is the only common knowledge event, it is easily verified that if a common prior exists, it is unique, hence commonly known, and assigns positive probability to every state.

A random variable $f$ is a real-valued function on $\Omega$. For any random variable $f$, agent $i$’s expectation of that random variable, when viewed as a function of the state, is again a random variable $E_i f$ given by $E_i f = \sum_{\omega \in \Omega} p_i^\alpha(\omega)f(\omega)$. For a probability measure $\mu$ on $\Omega$, let $E_\mu f$ denote the expectation of $f$ with respect to $\mu$, $E_\mu f = \sum_{\omega \in \Omega} \mu(\omega)f(\omega)$. Finally, the indicator function associated with the event $A$ is denoted by $1_A$; constant random-variables are denoted in boldface.

### 3. CHARACTERIZING COMMON PRIORS

Samet (1998) has obtained the following fundamental result about the structure of finite type spaces. It relies on agents’ higher-order expectations and taking their limits. A sequence $s = (i_1, i_2, \ldots)$ of elements in $I$ is an $I$–sequence if, for every $i \in I$, $i_k = i$ for infinitely many $k$’s.

**Proposition 1 (Samet)** For any random-variable $f$ on $\Omega$ and $I$–sequence $s$, the limit of the iterated expectations $\lim_{k \to \infty}(E_{i_k} \ldots E_{i_1} f)$ exists and its value is common knowledge at each state.

We will write $E_s f$ for $\lim_{k \to \infty}(E_{i_k} \ldots E_{i_1} f)$ and refer to it as the asymptotic iterated expectation of $f$ with respect to $s$. Proposition 1 leads directly to the following characterization of common priors and their existence due to Samet (1998).

**Theorem 1** A common prior exists, if and only if, for any random-variable $f$ on $\Omega$ and any two $I$–sequences $s$ and $s'$, it is common knowledge that $E_s f = E_{s'} f$.

The common prior $\mu$ is given by $\mu(A) = E_s^\alpha 1_A$ for all $A \subseteq \Omega$, and any $\alpha \in \Omega$ and $I$–sequence $s$.

Theorem 1 describes precisely where in the belief hierarchy the common prior resides. Localized at a limit of infinitely deep expectations, the common prior appears far removed from any behavioral implications. To fill this gap, the following result characterizes the common prior by explicitly describing its betting implications. It assumes the existence of a transferable commodity with respect to which all agents are risk-neutral. The idea is to determine those random-variables (viewed as contingent payments to the group, “bets”) which the group $I$ would be willing to bet on collectively.
using an appropriate sharing arrangement; as part of the “rules of the game”, it must be commonly known that the sharing arrangement is strictly profitable to each agent. This is formalized in the following definition.

**Definition 2** \( f \) is acceptable for \( I \) if there exist \( f_i : \Omega \to \mathbb{R} \) for \( i \in I \) such that \( f = \sum_{i \in I} f_i \) and such that it is common knowledge that \( E_\mu f_i > 0 \) for all \( i \in I \).

**Theorem 2** i) A common prior exists if and only if 0 is not acceptable for \( I \).

ii) If a common prior \( \mu \) exists, \( f \) is acceptable for \( I \) if and only if \( E_\mu f > 0 \).

While part i) is well-known (see Morris (1994)), part ii) is the novel contribution of this note. To illustrate its logic, consider the following simple example.

**Example.** Let \( I = \{1, 2\} \), \( \Omega = \{\tau, \beta, \gamma, \delta\} \), with \( \tau \) as the true state. Let the agents’ beliefs given by \( \Pi_1 = \{\{\tau, \beta\}, \{\gamma, \delta\}\} \), \( \Pi_2 = \{\{\tau, \gamma\}, \{\beta, \delta\}\} \), with \( \mu = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \) as the common prior. Let \( f = (-4, -4, -4, 20) \) with a prior expectation \( E_\mu f \) of 2. The bet \( f \) is made acceptable to both by decomposing it into bets \( f_1 = (-2, 4, -8, 10) \) and \( f_2 = (-2, -8, 4, 10) \), ensuring expected gains of \( E_1 f_1 = E_2 f_2 = (1, 1, 1, 1) \). Note that at the true state \( \tau \), both agents are willing to accept their part of the bet, even though they would reject the bet \( f \) individually. Indeed, at \( \tau \) both know that together they loose by accepting the bet (whatever the side-payments), but this is not common knowledge.

Theorem 2 can be easily generalized to allow for risk-averse agents along the following lines. Assume that all agents possess a strictly positive, common known endowment of money, and have differentiable von-Neumann Morgenstern utility functions that need not be commonly known. Say that \( f \) is “locally acceptable for \( I \)” if it is common knowledge that \( \epsilon f_i \) is strictly preferred to 0 (no trade) for all \( i \in I \). Since under the assumptions made all agents are commonly known to be locally risk-neutral, Theorem 2 remains valid if one replaces “acceptable for \( I \)” by “locally acceptable for \( I \)”.

**APPENDIX: PROOF OF THEOREM 2, II).**

To verify necessity, one has from acceptability that \( E_\mu f_i > 0 \) for all \( i \). By linearity, one obtains \( E_\mu f = \sum_{i \in I} E_\mu f_i > 0 \), as desired.
For the converse, let \( I = \{1, \ldots, n\} \) and define a sequence of \( n \)-tuples of random variables \( \{(f^{k})_{i \in I}\}_{k=1,\ldots,\infty} \) as follows. First, set \( f^{1}_i = f \) and \( f^{1}_i = 0 \) for \( i \neq 1 \), and let \( i_k = k \mod n \). Given \( \{(f^{k})_{i \in I}\} \), define \( \{(f^{k+1})_{i \in I}\} \) by setting

\[
 f^{k+1}_i = \begin{cases} 
 f^k_i - E_i f_{i_k-1} \ldots E_1 f & \text{if } i = i_k \\
 f^k_i + E_i E_{i_k-1} \ldots E_1 f & \text{if } i = i_k+1 \\
 f^k_i & \text{if } i \neq i_k, i_k+1.
\end{cases}
\]

One easily verifies inductively that \( \sum_{i \in I} f^{k}_i = 0 \) for all \( k \).

Moreover, one verifies inductively the following lemma.

**Lemma 1**

i) For all \( i, k \) such that \( i \neq i_k : E_i f^{k}_i = 0 \).

ii) For all \( i, k \) such that \( i = i_k : E_i f^{k}_i = E_i E_{i_k-1} \ldots E_1 f \).

To verify, the lemma holds clearly for \( k = 1 \). Assume it to hold for some \( k \geq 1 \). First, the claim holds for \( k + 1 \) whenever \( i \neq i_k, i_{k+1} \), since then \( f^{k+1}_i = f^k_i \) by construction.

Second, the lemma holds for \( k + 1 \) whenever \( i = i_{k+1} \), since \( f^{k+1}_{i_{k+1}} = f^k_{i_{k+1}} + E_{i_k} E_{i_{k-1}} \ldots E_1 f \), hence

\[
 E_{i_{k+1}} f^{k+1}_{i_{k+1}} = E_{i_{k+1}} f^k_{i_{k+1}} + E_{i_k} E_{i_{k-1}} \ldots E_1 f = E_{i_{k+1}} E_{i_k} \ldots E_1 f
\]

by validity of part i) for \( k \).

Finally, the lemma holds for \( k + 1 \) whenever \( i = i_k \), since then \( f^{k+1}_{i_k} = f^k_i - E_{i_k} E_{i_{k-1}} \ldots E_1 f \), hence

\[
 E_{i_k} f^{k+1}_{i_k} = E_{i_k} f^k_i - E_{i_k} E_{i_k-1} \ldots E_1 f = 0
\]

by validity of part ii) for \( k \). \( \square \)

The sequence \( \{i_k\}_{k=1,\ldots,\infty} \) is an \( I \)-sequence. Hence, by Theorem 1, using linearity,

\[
 \lim_{k \to \infty} E^{\alpha}_{i_k} E_{i_{k-1}} \ldots E_1 f = E_\mu f > 0 \quad \text{for all } \alpha \in \Omega.
\]

It follows that if \( k \) is chosen suitably large (as equal to \( "K" \)), \( E^{\alpha}_{i_k} E_{i_{k-1}} \ldots E_1 f > 0 \) for all \( \alpha \in \Omega \).

Set \( f_i = \begin{cases} 
 f^K_i - (n-1)\epsilon \mathbf{1} & \text{if } i = i_K \\
 f^K_i + \epsilon \mathbf{1} & \text{if } i \neq i_K.
\end{cases} \)

By Lemma 1, for sufficiently small but strictly positive \( \epsilon \), it is therefore common knowledge that \( E_1 f_i > 0 \) for all \( i \), demonstrating the acceptability of \( f \). \( \square \)
REFERENCES


