Modelling Economies of Scope in Terms of Joint Production*

Klaus Nehring

Department of Economics, University of California at Davis
Davis, CA 95616, U.S.A.
kdnehring@ucdavis.edu

and

Clemens Puppe

Department of Economics, University of Bonn
Adenauerallee 24-42, D – 53113 Bonn, Germany
clemens.puppe@wiwi.uni-bonn.de

September 17, 2002

*This paper is based on a former working paper entitled "Modelling Economies of Scope" (Nehring and Puppe (1999)). For valuable comments we are indebted to Louis Makowski. Versions of this paper have been presented at the 2nd Southern California Economic Theory Conference 2000, at ESEM 1999 in Santiago de Compostela, and in seminars at the Universities of Alicante, Berlin (Humboldt), Bonn, Davis, Mainz and St. Gallen. We are grateful for comments of the participants. Financial support from SFB 303 at Bonn University and DFG grant Pu 105/3-1 is also gratefully acknowledged.

Abstract. *Economies of scope, conceptualized here as cost complementarities, arise from synergies in the production of heterogeneous goods. It is shown that synergies can be accounted for in terms of shared public inputs (roughly) if and only if synergies decrease as the scope of production increases; this case of "substitutive" synergies is argued to be typical. As an application, we consider a model of "learning in steps," in which the public inputs represent different knowledge items.*

JEL Classifications: D24, L23
Keywords: Cost functions, economies of scope, submodularity, conjugate Moebius inversion, knowledge.

1 Introduction

Economies of scope arise from synergies in the production of heterogeneous goods. The classical Marshallian notion of joint production (Marshall (1890/1920)) explains instances of such synergies by the fact that some factors of production are pure public inputs. That is, once the inputs have been used for producing one good, they become costlessly available for use in the production of others. We refer to this as the *joint public input* interpretation of economies of scope. In this paper, we demarcate the domain of applicability of the joint public input explanation to economies of scope by studying the entailed structure of synergies, i.e. the pattern of marginal cost reductions due to the production of related goods. Our main conclusion is that synergies can be accounted for in terms of pure public inputs essentially if and only if synergies *decrease* as the scope of production increases (cf. Theorem 2.2). We also show that the opposite case of increasing synergies is very restrictive (see Theorem 2.3).

A prime example motivating our work is *knowledge* as a factor of production. Our analysis is applicable to a notion of *explicit* knowledge as the possession of specific pieces of information or understanding, as exemplified by the idea that knowledge items can be patented, or listed in a curriculum. This contrasts with a notion of *implicit* knowledge as skill acquired through practice, such as the diagnostic competence of a medical doctor honed by long experience. In the economics literature, such implicit knowledge is often addressed under the label of "learning by doing" and frequently viewed as a major source of increasing returns to scale (Arrow (1962)). While learning by doing, as increasing returns to scale more generally, raises difficult issues about the nature of competition and equilibrium analysis, the phenomenon itself can be adequately captured using conventional tools; for many purposes, even a one-input/one-output cost function with decreasing marginal costs will do. Like implicit knowledge, explicit knowledge is an important source of increasing returns due to its nonrivalrous nature as an input: the use of specific know-how for producing one good does not preclude its simultaneous use in the production of other goods. The increasing returns due to explicit knowledge naturally take the form of economies of *scope*. By definition, economies of scope involve multiple outputs. Moreover, explicit knowledge as an input is inherently heterogeneous, since by its very nature something known cannot be learnt again. In other words, a particular knowledge item can be used as an input only in quantities 0 or 1. Thus, the natural framework to model production based on explicit knowledge involves multiple inputs as well as multiple outputs. To zero-in on the combinatorial questions posed by the multi-dimensionality of the good space, we work in a discrete setting in which both inputs and outputs are formalized as *sets* of goods rather than vectors of continuous quantities.[1]

A cost function is said to be characterized by economies of scope if marginal costs decrease as more goods are produced. Technically, economies of scope will thus be identified with *submodularity* of the cost function. In the literature, this strong notion of economies of scope is also known under the name of "cost complementarity," in contrast to the weaker requirement of subadditivity (see Panzar and Willig (1981) and

[1] In this respect, our approach follows Scarf's (1981) analysis of production with indivisibilities (see also Scarf (1994)). A discrete model seems particularly appropriate in the context of economis of scope, since due to the associated non-convexities of the technology, production decisions (whether profit-maximizing or socially optimal) have a significant discrete combinatorial component even in a continuous setting in the subproblem of which goods to produce in positive quantities.

Panzar (1989)). We adopt the stronger notion in this paper for its richer structure, without disputing that the weaker one may be the more directly relevant for equilibrium analysis. The local extent of economies of scope, i.e. the reduction in the marginal cost of producing a particular good x brought about by the additional production of another good y, will be referred to as the *synergy* between x and y. Note that the synergy between x and y typically depends on the set of goods otherwise produced; submodularity can be paraphrased as the requirement that the synergy between two goods is always non-negative, no matter what other goods are being produced.

Cost functions derived from joint public inputs are always submodular; conversely, we show that any cost function satisfying a stronger condition of "*total* submodularity" can be represented mathematically as arising from the use of joint public inputs. The economic content of total submodularity is that the synergy between any pair of goods is decreasing as the scope of production increases; such decreasing (self-dampening) synergies are referred to as "substitutive" economies of scope. It turns out that the case of substitutive economies of scope is more fundamental than and presumably far more prevalent than the formally symmetric case of increasing (self-reinforcing) synergies, which is shown to be remarkably restrictive. Thus, we arrive at the main conclusion of the paper that *the step from the general notion of economies of scope as cost complementarities to the more structured joint public input interpretation is small, being largely a matter of assuming sufficient regularity.*

The public input interpretation is fruitful especially because it opens the possibility to model economies of scope in a flexible way through appropriate assumptions on the pattern of inputs. We illustrate this point with a variety of basic examples in Section 2; in Section 3, we develop in greater detail a model of economies of scope based on the use of explicit knowledge.

As a simple example illustrating the basic concepts of substitutive synergies and public inputs, consider the production of economics research papers, say a note on equilibria in extensive form games x, a theoretical analysis of entry deterrence y, and an applied IO-paper z. A researcher estimates the cost (= effort) of writing the three papers as half a year of work; she also deliberates on the cost of writing any subset of the projected papers. Her estimate of the costs, measured in weeks of concentrated work, is as follows: $c(\{x\}) = 8$, $c(\{y\}) = 15$, $c(\{z\}) = 13$, $c(\{x,y\}) = 17$, $c(\{x,z\}) = 20$, $c(\{y,z\}) = 24$, and $c(\{x,y,z\}) = 26$. As is easily checked, the example exhibits economies of scope, i.e. the marginal cost of any given paper decreases as more other papers are also written. Moreover, the economies in the example are substitutive, i.e. the synergy between two papers decreases when a third paper is also written. For instance, the synergy between x and z, i.e. the reduction of the marginal cost of writing x by also writing z is given by $c(\{x\}) - [c(\{x,z\}) - c(\{z\})] = 1$ if y is not written, and by $c(\{x,y\}) - c(\{y\}) - [c(\{x,y,z\}) - c(\{y,z\})] = 0$ if y is also written. In fact, the specified cost function is totally submodular, hence admits a representation in terms of joint public inputs which intuitively correspond to common ideas and tools shared by the different papers. For instance, the papers x and y may share the use of some game theoretic tools that the researcher has yet to learn. Similarly, there will be some overlap of concepts and ideas needed for the two IO-papers y and z.

The representation in terms of public inputs emerges from imputing specific fixed costs to the acquisition of the public inputs, i.e. the shared concepts and ideas in our example. Specifically, for each $A \subseteq \{x, y, z\}$, denote by λ_A the fixed cost of developing all concepts ("inputs") that are jointly employed by exactly the papers in A (again measured in weeks of concentrated work). There exists a *unique* assignment of fixed costs λ_A for all A such that for any set S of papers, the given cost $c(S)$ equals the total cost of all inputs that are needed for some paper in S, formally $c(S) = \sum_{A \cap S \neq \emptyset} \lambda_A$. In the example, the unique assignment is given by: $\lambda_{\{x,y,z\}} = 1$, $\lambda_{\{x,y\}} = 5$, $\lambda_{\{x,z\}} = 0$, $\lambda_{\{y,z\}} = 3$, $\lambda_{\{x\}} = 2$, $\lambda_{\{y\}} = 6$, $\lambda_{\{z\}} = 9$. In general, the fixed cost assignment is obtained from *conjugate Moebius inversion*, as explained in detail in Section 2 below. Crucial for our approach is the observation that the structure of synergies is reflected by the pattern of the associated public inputs. For instance, $\lambda_{\{x,z\}} = 0$ means that there are no concepts common to the note on extensive form equilibria x and the applied IO-paper z that are not also shared by the theoretical paper on entry deterrence y. This corresponds to the fact that there is no remaining synergy between x and z once y is being written.

Given a totally submodular cost function, conjugate Moebius inversion can of course only secure the formal possibility of a joint public input interpretation; the mathematically identified inputs need not always be economically meaningful. Indeed, even if the real production process involves impure (i.e. rivalrous) inputs, the resulting cost function may very well be totally submodular. In that case, the inputs derived from conjugate Moebius inversion, assumed to be public by construction, lack economic meaning. In Section 4, we show that it is nonetheless frequently possible to verify the absence of rivalrous inputs based on information on the cost function alone, thereby ensuring the economic meaningfulness of the joint public input interpretation.

The use of (non-conjugate) Moebius inversion is standard in the related literature on cost sharing since Shapley's seminal contribution (1953); occasionally, one also finds references to its conjugate form (see Moulin (1988) and Young (1994)). As our main mathematical contribution, we sharpen this tool by providing a novel characterization of the conjugate Moebius inverse in terms of the higher-order derivatives[2] of a cost function (Theorem 2.2). This is economically relevant since the first derivative describes marginal costs and the second derivative the local synergies; moreover, positivity of the third derivative corresponds to decreasing ("substitutive") synergies.

The body of the paper is organized as follows. The following Section 2 is divided into five subsections. Subsection 2.1 introduces the fundamental concept of a discrete cost function characterized by cost complementarities (positive synergies). Subsection 2.2 discusses the distinction between "compementary" (increasing) and "substitutive" (decreasing) synergies. It is shown that any cost function arising from joint production with public inputs exhibits substitutive synergies. Using conjugate Moebius inversion as the key technical tool, a converse statement is derived in Subsection 2.3. Specifically, it is shown that under a mild regularity condition any cost function characterized by substitutive synergies admits a joint public input representation. In Subsection 2.4, it is shown that the case of substitutive synergies is the far more relevant case. By consequence, the joint public input interpretation of economies of scope seems to be applicable under very general circumstances. A set of basic examples is presented in Subsection 2.5.

As a more detailed application, we study in Section 3 a model of "learning by insight" in which economies of scope arise from different activities requiring the same elementary skills. The proposed "stepwise learning model" is especially parsimonious in that every public input can be uniquely assigned to one activity, representing the

[2]strictly speaking: higher-order differences.

characteristic "learning step" associated with that activity. Based on a characterization in terms of the entailed structure of synergies, we argue that the "stepwise learning model" can serve as a reasonable and useful benchmark model in the context of learning based on explicit knowledge.

In the final Section 4, we discuss the problem of determining whether production is exclusively based on pure public inputs, or whether some inputs are rivalrous. All proofs are collected in an appendix.

2 Substitutive Economies of Scope

Let X be a finite set of goods that can be potentially produced by a firm. For any subset $S \subseteq X$ denote by $c(S)$ the cost of producing exactly the goods in S; these costs are summarized in a cost function $c : 2^X \to \mathbf{R}$. The elements of X are sometimes interpreted as heterogeneous *individual* objects, e.g. cars, or as *types* of goods ("product lines") distinguished by specific know-how, e.g. different car models sharing a basic common design. Under the latter interpretation, costs arise from the need to acquire the *capability* to produce goods belonging to the same product line, and increasing returns are due to shared component knowledge such as overlapping R&D benefits.

Throughout, c is assumed to be monotone in the sense that $c(W) \leq c(S)$ whenever $W \subseteq S$ ("producing more can never be cheaper"), and normalized so that $c(\emptyset) = 0$.[3]

2.1 Economies of Scope as Synergies

Economies of scope describe the reduction of marginal costs due to the production of other goods. Such *synergies* between goods can be described by second-order derivatives of the cost function. We define the (first) derivative[4] of c at S with respect to x by

$$\nabla_x c(S) := c(S \cup \{x\}) - c(S).$$

Thus, $\nabla_x c(S)$ is the marginal cost of producing x given that S is already produced. In order to emphasize this interpretation of the first derivative, we also write $m_x(S)$ for $\nabla_x c(S)$. Clearly, for each $x \in X$, $m_x(\cdot)$ is again a real-valued function on 2^X. A crucial role in the following analysis will be played by its derivative with respect y which we also refer to as the *cross-partial derivative* of c with respect to $\{x, y\}$:

$$\begin{aligned} \nabla_{\{x,y\}} c(S) &:= \nabla_y (m_x(S)) = \nabla_y (\nabla_x c(S)) \\ &= [c(S \cup \{x, y\}) - c(S \cup \{y\})] - [c(S \cup \{x\}) - c(S)] \end{aligned}$$

A cost function is said to be characterized by economies of scope, i.e. decreasing marginal costs, if the cross-partials are always non-positive. Equivalently, the *synergy* between any x and y,

$$\mathrm{syn}_{\{x,y\}}(S) := -\nabla_{\{x,y\}} c(S), \tag{2.1}$$

i.e. the reduction of the marginal cost of producing x due to the production of y, is always non-negative. It is easily verified that non-negativity of synergies is equivalent

[3] Formally, there are thus no fixed costs. Whenever we nevertheless refer to "fixed costs" in the following, strictly speaking we mean "quasi-fixed costs."

[4] strictly speaking: first-order difference.

to *submodularity* of the cost function, i.e. to the condition that, for all $S, W \subseteq X$,

$$c(S \cup W) + c(S \cap W) \leq c(S) + c(W).$$

The object under study is thus the class of all submodular cost functions on 2^X.[5]

In writing down equation (2.1) one notes at once that the synergy between x and y depends on the set S of goods already produced. This dependence is significant. Independence of S would require that

$$\nabla_{\{x,y\}} c(S) = \nabla_{\{x,y\}} c(S')$$

for all S, S' that do not contain x or y. Such independence holds if and only if all third derivatives are zero, in which case the cost function will be referred to as *quadratic*. Despite its attractiveness from a computational point of view, a quadratic model turns out to be inappropriate in most cases. Typically, synergies depend on S in a significant and economically meaningful way.

To illustrate the role of the third derivative, consider a simple example of a seller delivering to a finite number of stores in a linear city. Suppose that there are n equidistant stores, so that $X = \{1, ..., n\}$. A seller located at the edge of town (at 0) wants to serve these stores. For simplicity, assume that the cost incurred by supplying store $x \in X$ consists in the transportation cost of driving from the starting point 0 to store x plus some constant cost $a > 0$ per store for unloading. With transportation costs proportional to the distance, the cost of serving store x is simply $c(\{x\}) = a + x$.

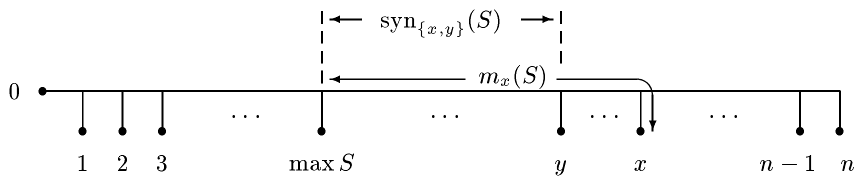

Figure 1: Serving stores in a linear city

Since in serving store x the seller has to drive beyond all stores $y < x$, the marginal cost of serving one of these is a once store x is served. The cost of supplying any set $S \subseteq X$ of stores is given by $c(S) = a \cdot (\#S) + \max S$, i.e. the costs of unloading plus the transportation cost of serving the farthest store in S. If $x \notin S$, marginal costs are thus $m_x(S) = a + [x - \max S]_+$, where $[z]_+$ is short for $\max\{z, 0\}$. Synergies are given by

$$\text{syn}_{\{x,y\}}(S) = [\min\{x, y\} - \max S]_+,$$

(see Figure 1). In particular, the synergy between x and y is always non-negative, confirming the presence of economies of scope. To illustrate, suppose that store $z = \max S$ is already being served. The marginal cost $a + (x - z)$ of supplying any farther store $x > z$ is reduced by $y - z$ whenever an intermediate store y with $z < y < x$ is also served. The dependence of $\text{syn}_{\{x,y\}}(S)$ on S is transparent: the reduction in the

[5] In identifying economies of scope with submodularity, we assume a *given* (short-term) technology. In particlular, our analysis does not take into account the effect of taking envelopes, i.e. of choosing technologies.

marginal cost $m_x(S)$ due to serving an intermediate store y with $\max S < y < x$ is smaller whenever $\max S$ is larger. In other words, $\text{syn}_{\{x,y\}}(S)$ is decreasing in S: for all x, y and all S, S',

$$S \subseteq S' \Rightarrow \text{syn}_{\{x,y\}}(S) \geq \text{syn}_{\{x,y\}}(S'). \tag{2.2}$$

The case of decreasing synergies in the sense of (2.2) will be referred to as the case of *substitutive* synergies: synergies become weaker the more synergies are already being exploited. Since $\text{syn}_{\{x,y\}}(S) = -\nabla_{\{x,y\}}c(S)$, (2.2) is equivalent to monotonicity of the cross-partials $\nabla_{\{x,y\}}c(\cdot)$, and hence to non-negativity of the third derivative of the cost function,[6]

$$\nabla_{\{x,y,z\}}c(\cdot) := \nabla_z[\nabla_{\{x,y\}}c(\cdot)].$$

This simple example demonstrates that the structure of synergies is in general complex. In particular, it shows that the qualitative behavior of synergies is closely related to the qualitative behavior of the third derivative of the cost function. A fundamental distinction concerns the *sign* of the third derivative. As outlined above, non-negativity of the third derivative everywhere corresponds to substitutive (decreasing) synergies. The polar case of *complementary* (increasing) synergies is also conceivable. However, we shall argue that these cases are not symmetric (see Section 2.4 below): substitutive synergies will figure much more prominently in our analysis.

How can we understand the presence of substitutive synergies?

2.2 Substitutive Economies of Scope due to Joint Public Inputs

The following example is meant to illustrate the notion of joint public inputs central to the further development of the theory. Consider the following very stylized description of the cost structure of producing BMWs (a well-known German car make).[7] First, to be able to produce any BMW x at all a certain amount F_{oh} of firm-wide overhead has to be incurred. Developing a specific product line, say the 5-series of BMW, requires large expenditures $F_{\text{pl}(x)}$. Similarly, designing a particular model, such as the 525td, involves additional costs $F_{\text{mo}(x)}$. Finally, the actual production of the individual car has unit costs $K_{\text{mo}(x)}$. Thus, producing a single BMW x (think of $x \in X$ as one car of a particular model) has total cost

$$c(\{x\}) = F_{\text{oh}} + F_{\text{pl}(x)} + F_{\text{mo}(x)} + K_{\text{mo}(x)}.$$

For instance, suppose one 525td is being produced. Then, the marginal cost of producing a second 525td is $K_{525\text{td}}$. By contrast, the marginal cost of, say a 528i, is $F_{528\text{i}} + K_{528\text{i}}$. More generally, the total cost of producing a set S of cars $c(S)$ is given by the sum of overhead costs, the (quasi-)fixed costs of any product category in S (line or model) plus the marginal costs of each individual car.

In this example, the presence of economies of scope is due to *joint public inputs*, i.e. inputs required by several goods (cars) that become freely available once used for one single good (car). For instance, the cost of producing a 525td and a 528i jointly

[6] Observe that the value of $\nabla_{\{x,y,z\}}c(S)$ is independent of the order of taking derivatives.

[7] While some readers may be less fond of BMWs than we are, they may nevertheless appreciate the pedagogical value of the example, especially of the crispness with which BMW's nomenclature conveys the qualitative structure of its product space.

is smaller than the sum of the cost of producing each of the two cars separately since both cars share common inputs, namely those required for developing the 5-series.

In general, the cost structure of production with joint public inputs can be described as follows. Let Ω be a set of public inputs with given prices p_ω, $\omega \in \Omega$. For any $\omega \in \Omega$ let $h(\omega) \subseteq X$ denote the set of those goods which require input ω. The total cost of producing the subset S of goods is thus given by

$$c(S) = \sum_{\omega : h(\omega) \cap S \neq \emptyset} p_\omega .$$

Note that each public input occurs only once in the sum on the right-hand side, since it becomes freely available for all outputs once it has been used by one. In the following, it will be convenient to identify inputs ω with their "extensions" $h(\omega) \subseteq X$, i.e. with the corresponding sets of goods that require these inputs. In particular, we will refer to a set A of goods as a "public input" whenever there is an $\omega \in \Omega$ such that $A = h(\omega)$, i.e. whenever there is some ω-input that is required exactly by all goods in A. Henceforth, a public input is thus simply a certain subset of goods, and the set of all public inputs is a collection of such subsets. The price of the "input" $A \subseteq X$ is then given by the aggregate cost of all ω-inputs required exactly by the goods in A:

$$\lambda_A := \sum_{\omega : h(\omega) = A} p_\omega ,$$

with $\sum_\emptyset := 0$ by convention. Expressing costs in terms of the λ_A one thus obtains for all S,

$$c(S) = \sum_{A \subseteq X : A \cap S \neq \emptyset} \lambda_A . \tag{2.3}$$

We can view λ as a measure on 2^X and write $\lambda(\mathcal{A}) := \sum_{A \in \mathcal{A}} \lambda_A$ for any family $\mathcal{A} \subseteq 2^X$. Note that, in general, there will be many subsets A for which $\lambda_A = 0$, and observe that the family of all public inputs is given by the *support* $\Lambda := \{A \subseteq X : \lambda_A \neq 0\}$ of λ.

By (2.3), one obtains

$$m_x(S) = c(S \cup \{x\}) - c(S) = \lambda(\{A : x \in A \subseteq S^c\}) \tag{2.4}$$

and

$$\text{syn}_{\{x,y\}}(S) = \lambda(\{A : \{x, y\} \subseteq A \subseteq S^c\}), \tag{2.5}$$

where S^c denotes the complement of S in X. The marginal cost of x at S is thus given by the aggregate cost of all inputs that are required by x but not already used by some good in S. Similarly, the synergy between x and y at S equals aggregate cost of all inputs common to x and y that are not required by any element of S. In particular, it is clear from (2.4) and (2.5) that any cost function of the form (2.3) is monotone and submodular, due to the non-negativity of λ. Moreover, it is also evident from the right-hand side of (2.5) that synergies are decreasing in S, i.e. that the economies of scope are substitutive. More generally, the higher-order derivatives of any cost function of the form (2.3) have alternating sign, beginning with a positive sign for the first derivative (see Theorem 2.2 below). Such cost functions will be called monotone and *totally submodular*. Equivalently, the class of totally submodular cost functions can be characterized by the property that the absolute value of any higher-order derivative is decreasing. Formally, for any $W = \{x_1, ..., x_m\} \subseteq X$, define the derivative of c with respect to W at S recursively by $\nabla_W c(S) := \nabla_{x_m}(\nabla_{W\setminus\{x_m\}} c(S))$. A monotone cost function is totally submodular if and only if, for all non-empty W, and all S, S',

$$S \subseteq S' \Rightarrow |\nabla_W c(S)| \geq |\nabla_W c(S')|. \tag{2.6}$$

Observe that submodularity corresponds to the case $\#W = 1$, and substitutivity of synergies to the case $\#W = 2$. We will refer to monotone and totally submodular cost functions as characterized by *regular* substitutive economies of scope. The term "regular" seems justified by the fact that the economic content of a particular cost function appears to reside in its first three derivatives.[8]

2.3 Implicit Joint Inputs Obtained from Conjugate Moebius Inversion

Consider now a monotone and totally submodular cost function. The above procedure for aggregating product-group specific fixed costs can be inverted, as shown by the following result.

Theorem 2.1 (Conjugate Moebius Inversion) *For any set function $c : 2^X \to \mathbf{R}$ there exists a unique measure λ on 2^X, the so-called* conjugate Moebius inverse*, such that (2.3) holds, i.e. such that for all S,*

$$c(S) = \lambda(\{A \subseteq X : A \cap S \neq \emptyset\}) = \sum_{A \subseteq X : A \cap S \neq \emptyset} \lambda_A,$$

where $\lambda_A := \lambda(\{A\})$. The measure λ has the following representation. For all A,

$$\lambda_A = \sum_{S \subseteq A} c(S^c) \cdot (-1)^{\#(A \setminus S)+1}.$$

Moreover, λ is non-negative if and only if c is totally submodular.

The first part is a standard result in combinatorics and the theory of non-additive probabilities (see Rota (1964), Chateauneuf and Jaffray (1989)). The second part, also well-known, follows from Theorem 2.2 below.

The non-negativity of the conjugate Moebius inverse (henceforth: c.m.i.) allows us to interpret it as a *cost decomposition*. In the BMW example above the cost decomposition was exogeneously given, as total costs were computed based on presupposed fixed costs specific to certain groups of products. However, even if the public inputs are not part of the physical description of the technology, the decomposition (2.3) can still admit an economic interpretation in terms of *imputed* fixed costs, with the values λ_A representing the fixed costs of the public inputs "imputed" to exactly the goods in A. Theorem 2.1 thus entails that, up to conditions on derivatives of order ≥ 4 ("regularity"), *any* cost function characterized by substitutive economies of scope admits such a cost decomposition in terms of imputed fixed costs.

[8] Note the analogy to the theory of choice under risk, in that the behavioral content of a von Neumann-Morgenstern utility function almost exclusively resides in its first three derivatives (for the economic content of the third derivative, see Kimball (1990)). Furthermore, also in this context, functions satisfying the continuous analogue of (2.6) ("mixed utility functions") have been found to play a distinguished role, see Caballé and Pomansky (1996).

The use of (non-conjugate) Moebius inversion is standard in the literature on cost sharing since Shapley (1953). Occasionally, one also finds reference to its conjugate form as defined here. For instance, Young (1994) refers to cost functions with the representation (2.3) as "decomposable." Both Young (1994, p. 93) and Moulin (1988, p. 140) note that in the totally submodular case, the Shapley value admits a particularly simple and intuitive representation. It amounts to assigning, for any implicit input, an equal cost share to each good that uses it: $Sh(x) = \sum_{A \ni x} \frac{\lambda_A}{\# A}$.

As a formal result, Theorem 2.1 can of course only secure the logical possibility of a joint public input interpretation. The mathematically identified inputs need not necessarily be economically meaningful; see Section 4.1 below for an example. Nonetheless, Theorem 2.1 establishes the remarkable generality of the joint public input *language* for talking about economies of scope *as if* originating from joint public inputs. Moreover, in Section 4.2 we provide a simple sufficient condition under which the economic meaningfulness of the imputed inputs is secured.

As a first illustration, consider again the example of serving stores in the linear city. Serving a store x requires the input "driving from $x-1$ to x." This input is in fact shared by all stores that are farther out than x. Hence, for each x, the set $\{x, ..., n\}$ is a public input with $\lambda_{\{x,...,n\}}$ as the transportation cost of driving from $x-1$ to x. In addition, each store x requires an idiosyncratic input $\{x\}$ with $\lambda_{\{x\}} = a$ representing the cost of unloading. The c.m.i. of the cost function $c(S) = a \cdot (\# S) + \max S$ is thus obtained by setting

$$\lambda_A = \begin{cases} 1 & \text{if } A = \{x, ..., n\} \text{ for } x < n, \\ a & \text{if } A = \{x\} \text{ for } x < n, \\ 1 + a & \text{if } A = \{n\}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the cost decompositon of the function $c(S) = a \cdot (\# S) + \max S$ in terms of joint public inputs is given by imputing one cost-unit to each set that consists of all stores that are farther away from the starting point than some given store x, and an unloading cost of a to each single store (see Figure 2). Note that the fixed cost $\lambda_{\{n\}}$ imputed to the farthest store subsumes both the transportation cost from $n-1$ to n and the unloading cost at n.

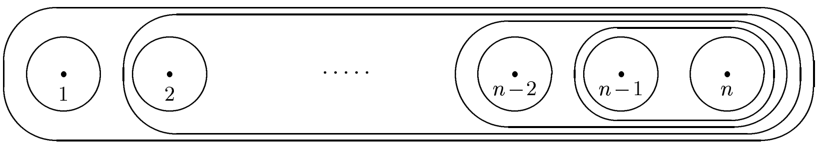

Figure 2: Extensionally identified public inputs in the linear city

The following result shows that the c.m.i. simultaneously describes the cost function and all its derivatives. Since derivatives play such a central role in the analysis of cost functions, it shows at a purely mathematical level the importance of the c.m.i.

Theorem 2.2 *Let $c : 2^X \to \mathbf{R}$ be a set function with c.m.i. λ. For all S and all non-empty $W \subseteq X$,*

$$\nabla_W c(S) = (-1)^{\# W+1} \cdot \lambda(\{A : W \subseteq A \subseteq S^c\}).$$

Taking $W = A$ and $S = A^c$ in Theorem 2.2 one obtains the following simple representation of the c.m.i.

Corollary 2.1 *Let $c : 2^X \to \mathbf{R}$ be a set function with c.m.i. λ. For all A,*

$$\lambda_A = \nabla_A c(A^c) \cdot (-1)^{\#A+1}.$$

Observe that both results apply to arbitrary set functions. In particular, the equivalence of non-negativity of λ and total submodularity of c is an immediate consequence of Theorem 2.2.[9]

Theorem 2.2 can be used to demonstrate the restrictiveness of quadratic cost functions, thereby underlining the role of third and higher-order derivatives. Suppose that for all x, y and all S with $S \cap \{x, y\} = \emptyset$, the synergy $\text{syn}_{\{x,y\}}(S)$ between x and y is strictly positive but does not depend on S. By Theorem 2.2, this implies that the support of the corresponding c.m.i. λ consists exactly of all one- and two-element subsets of X, i.e. $\Lambda = \{\{x, y\} : x, y \in X\}$. In particular, one obtains by (2.3), for all S,

$$c(S) \geq \frac{1}{2} \sum_{x \in S} c(\{x\}), \tag{2.7}$$

and more generally, $c(S \cup W) \geq \frac{1}{2} \sum_{x \in S} m_x(W)$. Note that (2.7) is easily violated when many goods in S share a common input, e.g. when there are significant overhead costs. Hence, modelling a cost function as quadratic entails a strong quantitative limitation on the extent of synergies.

Analogously, a vanishing $(k+1)$-th derivative means, by Theorem 2.2, that all public inputs are shared by at most k goods, which implies $c(S) \geq \frac{1}{k} \sum_{x \in S} c(\{x\})$ for all S.

2.4 The Privileged Status of Substitutive Synergies

As we have seen, the applicability of the joint public input interpretation, i.e. the interpretation of the c.m.i. as a cost decomposition, is limited to the case of substitutive economies of scope (decreasing synergies). This condition is not as restrictive as it may appear, since substitutivity of synergies is economically more natural than their complementarity. This intuition is confirmed by the fact that complementary (i.e. increasing) synergies impose strong restrictions on the overall extent of synergies, as expressed by the following inequality. For all x,

$$\sum_{y \in X \setminus \{x\}} \text{syn}_{\{x,y\}}(\emptyset) \leq c(\{x\}).^{10}$$

In terms of total cost, complementary synergies entail the same strong restriction as the quadratic model (cf. (2.7)):

[9] Choquet (1953, Sect. 14 and 26) introduced totally submodular set functions in terms of the alternating sign of the higher-order derivatives (thus calling them "alternating of infinite order"), and suggested that they occur more frequently and seem more useful than totally supermodular ones (belief functions). He did not state Theorem 2.2 nor Corollary 2.1; both seem to be new.

[10] For verification, let $X = \{x, y_1, ..., y_m\}$, and observe that $\sum_{i=1}^m \text{syn}_{\{x,y_i\}}(X \setminus \{x, y_1, ..., y_i\}) = m_x(\emptyset) - m_x(X \setminus \{x\}) \leq c(\{x\})$. The stated inequality thus follows from the assumption of increasing synergies. Note that, by contrast, in the substitutive case one can only deduce the inequality $\sum_{y \in X \setminus \{x\}} \text{syn}_{\{x,y\}}(\emptyset) \leq (n-1) \cdot c(\{x\})$.

Theorem 2.3 *Let $c : 2^X \to \mathbf{R}$ be monotone and submodular. Furthermore, assume that, for all x, y, $syn_{\{x,y\}}(\cdot)$ is increasing, i.e. that the third derivative of c is non-positive everywhere. Then, for all S,*

$$c(S) \geq \frac{1}{2} \sum_{x \in S} c(\{x\}).$$

By comparison, substitutive synergies entail no analogous restriction beyond monotonicity (i.e. $c(S) \geq \max_{x \in S} c(\{x\}) \geq \frac{1}{\#S} \sum_{x \in S} c(\{x\})$). The analogy to functions on the real line may be instructive. If $f : [0, \infty) \to \mathbf{R}$ is increasing and concave, then it is not possible that its third derivative is strictly negative everywhere.[11] By contrast, it is perfectly possible that its third derivative is strictly positive everywhere.

As we have already argued in the context of quadratic cost functions, the restrictions described above will be undesirable in many cases, for instance they are easily violated when overhead costs are significant. From this we conclude that the case of substitutive synergies is the by far more relevant case in applications.

2.5 Examples

The cost decomposition in terms of conjugate Moebius inversion allows one to model cost functions through appropriate *a priori* restrictions on the pattern of the public inputs. In this subsection, we present two simple and basic examples; in Section 3, we develop as a more elaborate example a model of stepwise learning.

Consider again the production of BMWs as described in Subsection 2.2 above. A distinctive feature of that example is that public inputs are *nested*: for any two product-groups corresponding to the public inputs A and B, respectively, one has $A \subseteq B$ or $B \subseteq A$ whenever $A \cap B \neq \emptyset$. In other words, public inputs do not (non-trivially) overlap. The following figure illustrates this property.

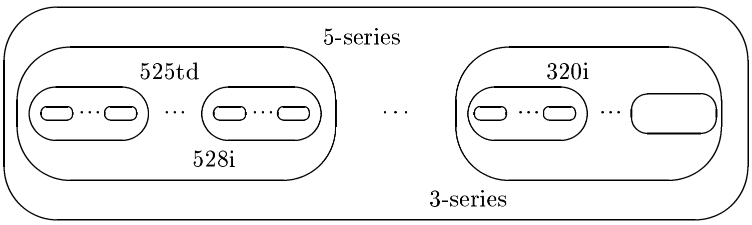

Figure 3: A hierarchy of public inputs

In general, a family $\mathcal{A} \subseteq 2^X$ will be called a *(taxonomic) hierarchy* if for all $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$ implies ($A \subseteq B$ or $B \subseteq A$). Accordingly, a cost function $c : 2^X \to \mathbf{R}$ will be called *hierarchical* if the support Λ of its c.m.i. is a hierarchy. As is evident from Fig. 2 in Section 2.3 above, the cost function $c(S) = a \cdot (\#S) + \max S$ used in the example of serving stores in the linear city is also hierarchical. As another instance of hierarchical structure, consider the introductory example of writing economics research papers. In

[11] This is easily seen by considering the first derivative f'. By assumption, f' is non-negative and decreasing everywhere. Clearly, in this case f' must have a convex part.

this context, a reasonable starting point is to deduce the public inputs from the *Journal of Economic Literature* classification system for research articles. The assumption is thus that each (sub-)category of articles, such as "D Microeconomics," "D2 Production and Organizations," or "D24 Production," represents a public input corresponding to the common concepts and ideas associated with that category. Provided that any research paper can be unambiguously assigned to exactly one subcategory at the lowest level (such as, e.g. "D24 Production"), the resulting family of public inputs exhibits a hierarchical structure. The interest in simple patterns of inputs such has a hierarchical organization lies in the fact that such patterns often correspond to specific functional forms of the cost function. For instance, by Theorem 3.1 in Nehring and Puppe (2002), a cost function is hierarchical if and only if the marginal cost of producing x given a set of goods S equals the minimal marginal cost of x given any *single* good in S, i.e. $m_x(S) = \min_{y \in S} m_x(\{y\})$.

As a different example of additional qualitative structure, suppose that goods are described by two quality dimensions. E.g., cars may be described by their body $x_1 \in X_1 = \{\text{coupe}, \text{sedan}, \text{station-wagon}\}$, and the horsepower of their engine $x_2 \in X_2 = \{150, 200, 250\}$ (see Figure 4 below). A "good" $x \in X = X_1 \times X_2$ is thus a type of product rather than an individual physical object, and $c(S)$ describes a firm's cost of being able to produce the range of goods S. We picture production of goods as involving elementary production processes. Costs arise from the capability of performing the production processes required by any good in the range. Economies of scope originate in different goods requiring the same production processes, hence sharing the same elementary competencies (= public inputs).

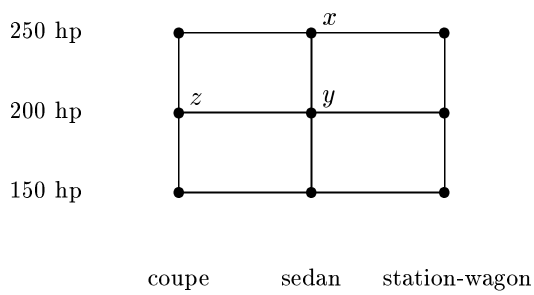

Figure 4: A two-dimensional good space

In this context, it seems natural to assume that the corresponding inputs have the form of "rectangular" subsets of $X_1 \times X_2$: if x and z share a common input, then so does any good y that is "intermediate" between x and z in both quality dimensions. In the present example, the natural orderings describing technological "betweenness" are given by 150-hp $<$ 200-hp $<$ 250-hp on X_1, and by coupe $<$ sedan $<$ station-wagon on X_2, as depicted in Figure 4. The rectangular shape of public inputs with respect to these oderings implies, for instance, that a 200-hp sedan y would share all elementary competencies that are commonly needed for producing a 250-hp sedan x and a 200-hp coupe z. As is easily verified, this entails that $\text{syn}_{\{x,z\}}(S) = 0$ for any $S \ni y$, i.e. the marginal cost of (being able to) producing x is not further reduced by providing the ability to produce z given that the production processes required by y are already available. Thus, the *a priori* assumption of the rectangular shape of public inputs yields the testable implication that there is no remaining synergy between two goods x and z once an intermediate good y is already being produced. The example is an instance of the "product of lines" and can be analyzed in more detail using the tools provided in Nehring (1999) and Nehring and Puppe (2002).

3 A Model of Learning in Steps

We now consider an application to a context in which costs arise from the need to acquire certain abilities or knowledge items. Economies of scope ("knowledge economies") are due to different abilities requiring common skills such as shared R&D know-how. As a specific example, consider a high-school math textbook containing a set X of problems. We imagine a student who tries to solve certain subsets S of problems. For this, the student has to invest a certain amount of effort $c(S)$, measured in time units, say. Economies of scope arise from different problems sharing certain "atomic" concepts. For instance, suppose that the solutions of problems x and y require solving a quadratic equation. Then, having absorbed the logic of Vieta's formula for the solutions of quadratic equations, the marginal cost of working through problem x is reduced once problem y has been solved.

Our framework assumes that mastery of problems is 0-1-valued. Either the student is able to solve a problem, or she is not. Once a problem is mastered, it is fully available (as a pure public input of sorts) for future learning. Therefore, we refer to the following model as a model of learning *in steps*, or "learning by insight." We will also assume that the total cost of absorbing a set S of knowledge items does not depend on the sequence in which the items are absorbed.[12]

Consider a set X of knowledge items and a cost function $c : 2^X \to \mathbf{R}$ with $c(S)$ quantifying the effort needed to "learn" the items in S, measured in time units. The following relation plays a central role in our analysis. Say that an item y *presupposes* an item x if the marginal effort of learning x is zero whenever y has already been absorbed; formally, define a *presupposition relation* $\geq_c$ on X from the cost function by

$$y \geq_c x :\Leftrightarrow \text{ for all } S \ni y,\ c(S \cup \{x\}) = c(S). \tag{3.1}$$

For instance, understanding how to solve a quadratic equation presupposes the concept of square root. By (3.1), this means that the marginal cost of understanding the logic of square root is zero whenever the method of solving quadratic equations has been fully absorbed. As is easily verified, for any set function c, the induced presupposition relation $\geq_c$ is reflexive and transitive, hence a preorder. To further illustrate, consider Figure 5 below which represents the presupposition relation among the basic "knowledge items" of this paper's Section 2. In Figure 5, nodes represent the knowledge items, such as the definition of a discrete cost function (item 1), the logic of conjugate Moebius inversion (item 3), the argument for the priviledged status of substitutive synergies of Section 2.4 (item 6), and so on. An arrow from item x to item y means that y presupposes x. For instance, understanding the representation of a cost function in terms of imputed fixed costs (item 7) presupposes the concept of total submodularity (item 5). As expected, the main message of Section 2 (item 8: "regular economies of scope admit a public input interpretation") presupposes every other item displayed.

[12] Certainly, this is a restrictive assumption. In an unpublished former version of this paper (Nehring and Puppe (1999)), we show how the analysis can be generalized allowing for path-dependent costs.

Figure 5 here

Denote by $x\uparrow := \{y : y \geq_c x\}$ and $x\downarrow := \{y : x \geq_c y\}$ the "up-set" of all items that presuppose x and the "down-set" of all items presupposed by x, respectively. Also, let $S\uparrow := \{y \in X : y \geq_c x \text{ for some } x \in S\}$ and $S\downarrow := \{y \in X : x \geq_c y \text{ for some } x \in S\}$.

Fact 3.1 *Let $c : 2^X \to \mathbf{R}$ be a set function with c.m.i. λ. Then, the support Λ of λ consists of sets of the form $A\uparrow$, i.e. $A = A\uparrow$ for all $A \in \Lambda$.*

Thus, a public input $A \in \Lambda$ contains with any item $x \in A$ all items y that presuppose x. Unless the presupposition relation is very rich, its knowledge imposes only loose restrictions on the family of public inputs. From a modelling perspective, greater parsimony is desirable. We will now show that there is a well-defined "most parsimonious" model consistent with the presupposition relation $\geq_c$; the model is very tractable and does not seem to be excessively restrictive.

First, observe that in the present context of learning a typical cost function will satisfy the following condition of strict monotonicity.

Strict Monotonicity For all x and S, $c(S \cup \{x\}) > c(S)$ whenever $S \cap x\uparrow = \emptyset$.

Strict monotonicity states that the marginal effort of learning x is strictly positive whenever no item that presupposes x has already been absorbed. Using Fact 3.1, strict monotonicity is easily seen to imply $x\uparrow \in \Lambda$ for all $x \in X$, i.e. Λ contains all up-sets $x\uparrow$.[13] The following *stepwise learning model* requires conversely, that *only* sets of the form $x\uparrow$ are relevant public inputs; in other words, it assumes that one can associate to any knowledge item x a unique "atomic concept" $x\uparrow$ that is specific to x and any more advanced item. As a simple example, consider the knowledge item $x = 5$ ("regular economies of scope") in Figure 5. Absorbing ("learning") that particular concept requires the notion of total submodularity which also underlies the more advanced items 7 and 8. The "atomic concept" associated with item $x = 5$ is thus represented extensionally by the public input $x\uparrow = \{5, 7, 8\}$ ("total submodularity").

The stepwise learning model is the most parsimonious model consistent with $\geq_c$, and it can be characterized by an independence condition that identifies the source of synergies. The condition says that synergies between two items x and z can occur only if they share a common presupposition that has not yet been learnt.

Synergy through Shared Presuppositions (SSP) If $\mathrm{syn}_{\{x,z\}}(S) \neq 0$, then there exists $y \notin S\downarrow$ such that $x \geq_c y$ and $z \geq_c y$.

Proposition 3.1 *A set function $c : 2^X \to \mathbf{R}$ satisfies SSP if and only if the support of its c.m.i. satisfies $\Lambda \subseteq \{x\uparrow : x \in X\}$. In this case one has, for all x,*

$$\lambda_{x\uparrow} = \nabla_x c((x\uparrow)^c) = \nabla_x c(x\downarrow \setminus \{x\}) = c(\{y : y \leq_c x\}) - c(\{y : y <_c x\}). \tag{3.2}$$

In particular, any monotone set function satisfying SSP is automatically totally submodular.

The hallmark of the stepwise learning model is thus the associating of a unique "marginal" input $x\uparrow$ with every good x. Concretely, by (3.2), the value $\lambda_{x\uparrow}$ is the marginal effort of understanding x given that all items $y \neq x$ that x presupposes have been

[13] Indeed, by strict monotonicity and Theorem 2.2, $\nabla_x c((x\uparrow)^c) = \lambda(\{A : x \in A \subseteq x\uparrow\}) > 0$, and by Fact 3.1, $\lambda(\{A : x \in A \subseteq x\uparrow\}) = \lambda_{x\uparrow}$.

learnt. Intuitively, the input $x\uparrow$ may be viewed as the characteristic *learning step* associated with knowledge item x.

To illustrate the content of Proposition 3.1 graphically, consider a cost function on the set $\mathbf{N} \times \mathbf{N}$ of pairs of natural numbers (or some large finite subset of it) such that $(y_1, y_2) \geq_c (x_1, x_2) \Leftrightarrow y_1 \geq x_1$ & $y_2 \geq x_2$. Figure 6 below depicts the shape of a typical input with and without condition SSP, respectively.

Figure 6: Typical inputs with and without SSP

The strength of condition SSP depends on the structure of the induced presupposition relation. Specifically, consider the two extreme cases of a complete presupposition relation (weak order) and an empty presupposition relation, respectively. In the former case, SSP is automatically satisfied, since by Fact 3.1 any public input is the form $x\uparrow$ for some $x \in X$; in the latter case, SSP is equivalent to the condition $\Lambda \subseteq \{\{x\} : x \in X\}$, i.e. any public input must be a singleton.

In the present context of stepwise learning, it can be argued that condition SSP is without real loss of generality by appropriate enlargement of the domain: if necessary, simply add "auxiliary knowledge items" that are shared as inputs by several items as an intermediate output in its own right. To illustrate this, consider again Figure 5 above. Does the cost function on the domain of knowledge items in that figure satisfy SSP, i.e. does Figure 5 represent a stepwise learning model? Presumably not. To see this, consider e.g. the claim of the priviledged status of substitutive synergies (item 6) and the concept of total submodularity (item 5). Suppose that every item presupposed by these two (i.e. items 1,2 and 3) has already been absorbed. Our intuition is that there still is a positive synergy between items 5 and 6: $\text{syn}_{\{5,6\}}(\{1, 2, 3\}) > 0$, in violation of SSP. Indeed, one common learning step that is needed for understanding both items 5 and 6 is the concept of substitutive synergies itself and the representation of conjugate Moebius inversion in terms of derivatives (Theorem 2.2). Adding these two items (as items 12 and 14 in Figure 7 below) accounts for the positive synergy between 5 and 6. Note that the inclusion of new items may necessitate the addition of further items if the resulting presupposition relation is to satisfy condition SSP. For instance, the positive synergy between the added items 12 and 14 must be accounted for by the inclusion of the notion of third derivative (item 10), and so on. Completing Figure 5 in this manner, we finally arrived at the representation in Figure 7 below, which presumably conforms to the stepwise learning model. In Figure 7, we also included the arguments surrounding the restrictiveness of the quadratic model, as this is probably also helpful to appreciate the basic messages of Section 2, such as items 6 and 8. The method of enlarging the domain of a cost function to achieve compatibility with the stepwise learning model exemplified here with the help of Figures 5 and 7 can be applied in full generality: for any cost function c on X, there exists an extended cost function $\tilde{c}$ on some domain $Y \supseteq X$ such that $\tilde{c}$ coincides with c on X and such that $\tilde{c}$ satisfies SSP (see Nehring and Puppe (1999) for the details).

Figure 7 here

The stepwise learning model has been presented as a useful benchmark model in the context of learning. It has been analyzed in some detail to illustrate the general modelling strategy of specifying structured families of inputs. A formal theory of step-by-step learning has been developed over the last fifteen years by Doignon and Falmagne in their theory of "knowledge spaces" (see Doignon and Falmagne (1985) for a first statement, and Doignon and Falmagne (1999) for a comprehensive treatment and references to related work by other authors). The central object of their theory is a knowledge space which can be viewed as a domain of knowledge items ordered by a generalized presupposition relation.[14] While there are some interesting mathematical connections to our work, the intent and details are quite different: focusing on knowledge assessment, Doignon and Falmagne do not postulate a cost function of learning, and do not represent the order structure of knowledge items in terms of atomic "learning steps" (public inputs) as in our present account.

4 The Interpretation of Implicit Inputs: Real versus Merely Imputed

When do the fixed cost components imputed via conjugate Moebius inversion admit an economic interpretation and when are they only mathematical constructs? This is the question addressed in this section. In Subsection 4.1, we study the production of multiple units of a homogeneous good and find that in this case the fixed cost imputation via conjugate Moebius inversion lacks an economic interpretation. Indeed, when economies of scope reduce to economies of scale, *any* combination of different units of the good shares a specific (mathematically identified) input that is not shared by all other units. Therefore, in Subsection 4.2, we derive a condition that guarantees a substantive joint public input interpretation by requiring the absence of "scale effects."

4.1 Multiple Units of Homogeneous Goods

4.1.1 One Good: Economies of Scale

Consider now the special case of a completely symmetric cost function: A cost function $c : 2^X \to \mathbf{R}$ will be called *completely symmetric* if costs depend only on the number of goods produced, i.e. for all S, $c(S) = f(\#S)$ for some function $f : \{0, 1, ..., n\} \to \mathbf{R}$, where $n = \#X$. Such a cost function describes the production of a *homogeneous* output. In this case, economies of scope specialize to *economies of scale.*

Say that a function $f : \{0, 1, ..., n\} \to \mathbf{R}$ is monotone and *totally concave* if its higher-order derivatives have alternating sign, i.e. if for all $l \geq 1$,

$$\nabla^{(l)} f(i) \cdot (-1)^{l+1} \geq 0,$$

[14] The methodology has been implemented in a computer educational system "Aleks" (for "Assessment and Learning in Knowledge Spaces") covering the high-school curriculum in basic arithmetic, algebra and geometry; it is available on the internet under www.aleks.com.

where $\nabla f(i) := f(i+1) - f(i)$, and $\nabla^{(l)} f(i) := \nabla \left[\nabla^{(l-1)} f(i) \right]$ for $l > 1$. The following result is an immediate consequence of Theorem 2.2 above.

Fact 4.1 *Let $c : 2^X \to \mathbf{R}$ be a completely symmetric cost function. Then, c is monotone and totally submodular if and only if f is monotone and totally concave.*

In the special case in which total costs are given by fixed costs plus constant marginal costs, i.e. $f(i) = b + ai$ for $i > 0$, the c.m.i. is given by $\lambda_X = b$, $\lambda_{\{x\}} = a$ for all x, and $\lambda_A = 0$ for all other A; here the set of imputed inputs Λ has clear economic meaning. In the case of decreasing marginal costs, on the other hand, the c.m.i. has typically full support, i.e. $\Lambda = 2^X$.[15] While this suggests that there is a complex interference of synergies when the production of a homogeneous good exhibits economies of scale, it also shows that the completely symmetric case does not lend itself easily to a substantive joint public input interpretation. Indeed, full support means that *any* combination of different (units of) goods corresponds to a particular public input.

4.1.2 Many Goods: Distinguishing Economies of Scope and Scale

Suppose that the good space X can be partitioned into different types of goods. In this case there are two distinct sources of costs: "fixed costs" of acquiring the capability of producing goods of a certain type, and "variable costs" of producing goods of a particular type in various quantities. For instance, fixed costs may correspond to R&D costs of developing a range of product lines as in the BMW example above.

Formally, let $X = \cup_{k \in Y} X_k$ with the X_k pairwise disjoint. The distinction between fixed and variable costs is formally captured by the following additively separable functional form of the cost function. For all S,

$$c(S) = F(\{k \in Y : S \cap X_k \neq \emptyset\}) + \sum_{k \in Y} c_k(\#(S \cap X_k)), \tag{4.1}$$

where $F : 2^Y \to \mathbf{R}$ represents fixed costs, and the $c_k : \{0, ..., \#X_k\} \to \mathbf{R}$ represent the variable cost functions for goods of type k. A cost function that admits such a separable representation will be referred to as *decomposable*.

Proposition 4.1 *A cost function $c : 2^X \to \mathbf{R}$ is decomposable in the sense of (4.1) if and only if it satisfies the following two conditions.*
(i) For all $k \in Y$, $x, y \in X_k$, and $z \notin X_k$, $m_z(S) = m_z(S \cup \{x\})$ whenever $S \ni y$, and
(ii) For all S, W, $[\#(S \cap X_k) = \#(W \cap X_k)$ for all $k] \Rightarrow c(S) = c(W)$.

Moreover, the functions c_k and F are uniquely determined up to a normalization of $c_k(1)$ for all $k \in Y$.

Condition (i) states that once a single unit of some type k is produced, increasing the output of this type entails no further reduction of the marginal cost of producing goods of any other type j. Condition (ii) states that costs only depend on the number of units produced of each type.

The function F captures potential economies of scope, while the c_k capture potential economies of scale. This clear-cut dichotomy, which is due to condition (i), seems appropriate in certain contexts such as that of R&D expenditure; it may not be

[15] Observe that the c.m.i. of a completely symmetric cost function is completely symmetric as well, i.e. for all A, λ_A only depends on the cardinality of A.

acceptable in others. For example, in a context of learning by doing in which types represent classes of (costly) activities and individual "objects" within a type instances of the same activity, the marginal cost of a repetition of some activity j will decrease only gradually as a similar activity k is performed more often.

The function c is totally submodular whenever F is totally submodular and each c_k is totally concave. Since the joint public input interpretation is less compelling within a homogeneous type, one would often assume total submodularity only of F, applying the methodology in effect only to the economies of scope aspect of c.

4.2 Economies of Scope without Scale Effects

We now provide a sufficient condition for a substantive public input interpretation; roughly, it requires that there be no "scale effects."

For the remainder of this section, we consider the class of all totally submodular cost functions $c : 2^X \to \mathbf{R}$ that can be written in the form

$$c = \sum_{\omega \in \Omega} c_\omega, \tag{4.2}$$

where each $\omega \in \Omega$ corresponds to a *real* input that may be rivalrous or not, and $c_\omega : 2^X \to \mathbf{R}$ describes the costs of acquiring it in sufficient quantity. As in Sect. 2 above, denote by $h(\omega) \subseteq X$ the set of goods that use input ω, formally, $h(\omega) = \{x : c_\omega(\{x\}) > 0\}$. For each $\omega \in \Omega$, c_ω is assumed to be monotone, submodular and of one of the following two types.

Type I: for all $x, y \in h(\omega)$: $\nabla_x c_\omega(\{y\}) = 0$.

Evidently, the type I cost functions correspond to the pure public (i.e. nonrivalrous) inputs. If c_ω is not of type I, we require $\#h(\omega) > 2$ and the following condition.[16]

Type II: for all $x, y \in h(\omega)$ and all S with $\{x, y\} \cap S = \emptyset$: $\text{syn}^\omega_{\{x,y\}}(S) > 0$,

where $\text{syn}^\omega_{\{x,y\}}$ denotes the synergy between x and y with respect to the cost function c_ω. The cost functions of type II are associated with the rivalrous inputs since the synergy between any pair of goods that use ω remains positive no matter what other goods are being produced. The following concrete functional form motivates our formulation of type II cost functions. For all S, let

$$c_\omega(S) = f_\omega\left(\sum_{x \in S} q_x\right). \tag{4.3}$$

Here, q_x denotes the quantity of input ω used by good x, and $f_\omega(\cdot)$ is the cost of acquiring the input in different quantities. A cost function of the form (4.3) is of type II whenever f_ω is strictly concave. For instance, let q_x be the metres square of the body of a car x that have to be painted, and f_ω the hours needed to do the painting job. In this case, strict concavity of f_ω occurs, e.g., due to "learning by doing."

Evidently, for a cost function c of the form (4.2), the family $\{h(\omega) : \omega \in \Omega\}$ of real inputs does, in general, not coincide with the family Λ of *imputed* inputs since the

[16]To motivate the requirement $\#h(\omega) > 2$, consider a cost function c_ω with $h(\omega) = \{x, y\}$ and $\text{syn}_{\{x,y\}}(\emptyset) > 0$. While failing to represent a nonrivalrous input, c_ω can be decomposed into three nonrivalrous cost functions with inputs $\{x\}$, $\{y\}$, and $\{x, y\}$, respectively. Therefore, the case $\#h(\omega) = 2$ can be excluded in the definition of type II cost functions.

latter are always purely public by construction. In the pure case, on the other hand, the two families do coincide.

Fact 4.2 *Let c be a totally submodular cost function of the form (4.2). If all c_ω are of type I, then $\Lambda = \{h(\omega) : \omega \in \Omega\}$.*

Given a cost function c of the form (4.2), is there a way to infer from knowledge of c alone whether all inputs are pure public inputs, or whether some are impure (i.e. rivalrous)? We will now show that one can formulate a simple sufficient condition in terms of the presence of strictly positive synergies. Then, we will argue heuristically that the condition comes close to being necessary.

Say that two goods x and y are *linked* if $\mathrm{syn}_{\{x,y\}}(S) > 0$ for all S with $\{x,y\} \cap S = \emptyset$. Furthermore, a *clique* of goods is a subset $W \subseteq X$ such that any pair in W is linked. The relation "is linked to" defines a graph, the *linkage graph* on X; a clique is a complete induced subgraph. Also, note that by Theorem 2.2, two goods x and y are linked if and only if $\{x,y\} \in \Lambda$.

Definition The *linkage index* $l(c)$ of a cost function c is defined as the maximal cardinality of a clique of the induced linkage graph.

We will now show that $l(c) = 2$ implies that all inputs must be purely public. For the converse, we shall argue that a low index makes it likely that one in fact faces the pure case.

Suppose that ω is rivalrous; then $h(\omega)$ is a clique in the linkage graph of c_ω. For any pair $x, y \in h(\omega)$ one has,

$$\mathrm{syn}_{\{x,y\}}(\{x,y\}^c) = \sum_{\omega \in \Omega} \mathrm{syn}^\omega_{\{x,y\}}(\{x,y\}^c).$$

Since, by submodularity of all c_ω, $\mathrm{syn}^\omega_{\{x,y\}} \geq 0$ for all ω, we have $\mathrm{syn}_{\{x,y\}}(\{x,y\}^c) > 0$ if and only if $\mathrm{syn}^\omega_{\{x,y\}}(\{x,y\}^c) > 0$ for at least one ω. Since synergies (with respect to c) are decreasing by the assumed total submodularity of c, we have $\mathrm{syn}_{\{x,y\}}(S) \geq \mathrm{syn}_{\{x,y\}}(\{x,y\}^c)$ for all $S \subseteq \{x,y\}^c$. Thus, any pair x, y in $h(\omega)$ is linked, whence

$$l(c) \geq \# h(\omega) > 2.$$

By contraposition, this shows that $l(c) = 2$ is sufficient for the pure case, i.e. the absence of rivalrous inputs.

To illustrate, consider the following examples. In the homogeneous case of Section 4.1.1, when c is totally symmetric and strictly submodular, the entire set X of goods forms a clique, hence $l(c) = \# X$. More generally, if c is decomposable in the sense of (4.1) with all c_k strictly concave, the cliques include all sets X_k, therefore $l(c) \geq \max_k \# X_k$. By contrast, for any hierarchical cost function c, each good is linked to at most one other good, hence $l(c) \leq 2$. By consequence, a hierarchical organization of the family Λ of imputed inputs guarantees the absence of "scale effects" by ruling out rivalrous inputs ω. The same conclusion applies if the good space has a product product structure and all inputs are rectangular sets as in the second example in Sect. 2.5 above. Indeed, if all public inputs are rectangular, the linkage graph is given by the graph depicted in Figure 4, or a subgraph of it. While a good may thus be linked to several other goods, none of these can be linked to each other. Hence, the corresponding linkage index cannot exceed 2. Finally, in the stepwise learning model again $l(c) \leq 2$ due to the specific structure of the support Λ.

Our heuristic argument for the "almost necessity" of a low value of $l(c)$ for the pure case is based on the intuition that if an input ω is used by more than one good, it is used by more than just *one* other good. In fact, it seems hard to find examples of (real) inputs that are used by exactly two different goods; goods of that kind would have to be "made for each other," such as right and left shoes, for instance. Thus, $\#h(\omega)$ will typically be large. In the pure case, this implies by Fact 4.2 that only few pairs goods are linked. By consequence, one will find even "moderately sized" cliques only under special circumstances.[17] Hence, $l(c)$ will be small if all inputs are purely public. On the other hand, in the impure case $l(c)$ is bounded below by the largest cardinality $\#h(\omega)$ of a rivalrous input, which will typically be of (at least) moderate size if the total number of goods is sufficiently high. Thus, a small linkage index is a good indicator that all inputs are nonrivalrous and that the public input interpretation is economically meaningful, while a moderate or large linkage index indicates the presence of some rivalrous input.

Appendix: Proofs

Proof of Theorem 2.2 The proof proceeds by induction over $\#W$. For $W = \{x\}$,

$$\begin{aligned}
\nabla_x c(S) &= c(S \cup \{x\}) - c(S) \\
&= \lambda(\{A : A \ni x, A \cap S = \emptyset\}) \\
&= \lambda(\{A : x \in A \subseteq S^c\}).
\end{aligned}$$

Next, let $\#W \geq 2$, and suppose the given formula applies to all derivatives of order $< \#W$. Then, for any $x \in W$,

$$\begin{aligned}
\nabla_W c(S) &= \nabla_x \left(\nabla_{W \setminus \{x\}} c(S)\right) \\
&= \nabla_{W \setminus \{x\}}[c(S \cup \{x\}) - c(S)] \\
&= (-1)^{\#W} [\lambda(\{A : (W \setminus \{x\}) \subseteq A \subseteq (S \cup \{x\})^c\}) \\
&\qquad\qquad - \lambda(\{A : (W \setminus \{x\}) \subseteq A \subseteq S^c\})] \\
&= (-1)^{\#W+1} \lambda(\{A : (W \setminus \{x\} \subseteq A \subseteq S^c, x \in A\}) \\
&= (-1)^{\#W+1} \lambda(\{A : W \subseteq A \subseteq S^c\}).
\end{aligned}$$

[17]Note, however, that even if any given good x is linked to other goods with low probability, the number of *potential* cliques of a given cardinality grows fast as the number of goods increases. Nevertheless, the intuition that few edges imply the absence of sizeable cliques can be made formally precise.

Proof of Theorem 2.3 Let $c : 2^X \to \mathbf{R}$ be monotone, submodular with non-positive third derivative. The restriction of c to any $S \subseteq X$ has these same properties; hence, for the proof of Theorem 2.3 it suffices to show that

$$c(X) \geq \frac{1}{2} \sum_{x \in X} c(\{x\}). \tag{A.1}$$

Define the average cost function $f : \{0, ..., \#X\} \to \mathbf{R}$ by $f(i) := \frac{1}{\#\mathcal{S}(i)} \sum_{S \in \mathcal{S}(i)} c(S)$, where $\mathcal{S}(i) := \{S \subseteq X : \#S = i\}$. Note that $f(0) = 0$, $n \cdot f(1) = \sum_{x \in X} c(\{x\})$ and $f(n) = c(X)$, where $n := \#X$. Consider the derivative ∇f defined by $\nabla f(i) := f(i+1) - f(i)$. By assumption, $\nabla f : \{0, ..., n-1\} \to \mathbf{R}$ is positive, decreasing and concave. By Jensen's inequality, one has for all $i = 1, ..., n-1$,

$$f(i+1) - f(i) = \nabla f(i) \geq \frac{n-1-i}{n-1} \cdot \nabla f(0).$$

Summing these inequalities, one obtains

$$f(n) = \sum_{i=0}^{n-1} [f(i+1) - f(i)] \geq \left(\sum_{i=0}^{n-1} \frac{n-1-i}{n-1} \right) \cdot f(1) = \frac{n}{2} f(1),$$

i.e. (A.1).

Proof of Fact 3.1 We show that, for any $A \in \Lambda$ and any $x \in A$, $x\!\uparrow \subseteq A$. Fix any $x \in X$, and consider the family $\mathcal{B}_x := \{A : x \in A, \exists y \geq_c x \text{ with } y \notin A\}$. It will be shown that $\lambda_A = 0$ for all $A \in \mathcal{B}_x$ by induction over $\#A$. First recall that by (2.4), for all S,

$$c(S \cup \{x\}) - c(S) = \lambda(\{A : x \in A \subseteq S^c\}). \tag{A.2}$$

Let $A \in \mathcal{B}_x$ with $\#A = 1$, i.e. $A = \{x\}$. By assumption, one has $c(\{x\}^c \cup \{x\}) - c(\{x\}^c) = 0$; hence, taking $S = \{x\}^c$ in (A.2) one obtains $\lambda_{\{x\}} = 0$.

Now consider any $A \in \mathcal{A}_x$ and suppose that $\lambda_B = 0$ for all $B \in \mathcal{B}_x$ with $\#B < \#A$. By assumption, $c(A^c \cup \{x\}) - c(A^c) = 0$; hence, using (A.2) with $S = A^c$, one obtains

$$0 = \lambda(\{B : x \in B \subseteq A\}) = \lambda(\{B : x \in B \subset A\}) + \lambda_A,$$

thus, by the induction hypothesis, $\lambda_A = 0$.

Proof of Proposition 3.1 We prove that SSP implies $\Lambda \subseteq \{x\!\uparrow : x \in X\}$ by a contradiction argument. Thus, suppose that $A \in \Lambda$ is not of the form $A = x\!\uparrow$ for some $x \in X$, and assume that A is of minimal cardinality with that property. Let $x \in A$ be a $\geq_c$-minimal element of A. By Fact 3.1, $A \supseteq x\!\uparrow$. By assumption there exists $z \in A \setminus (x\!\uparrow)$. By (2.5) and minimality of A one obtains,

$$\text{syn}_{\{x,z\}}(A^c) = \lambda(\{B : \{x,z\} \subseteq B \subseteq A\}) = \lambda_A.$$

By assumption, $\lambda_A \neq 0$, hence by SSP there must exist some $y \notin (A^c)\!\downarrow$ with $x \geq_c y$ and $z \geq_c y$. But $y \notin (A^c)\!\downarrow$ in particular implies $y \in A$, which is not possible by the choice of x and z.

Conversely, suppose that $\Lambda \subseteq \{x\!\uparrow : x \in X\}$. This implies

$$\begin{aligned} \text{syn}_{\{x,z\}}(S) &= \lambda(\{A : \{x,z\} \subseteq A \subseteq S^c\}) \\ &= \lambda(\{y\!\uparrow : x \geq_c y, z \geq_c y \text{ and } y \notin S\!\downarrow\}), \end{aligned}$$

and hence SSP. Finally, formula (3.2) follows at once from $\Lambda \subseteq \{x\uparrow : x \in X\}$.

Proof of Proposition 4.1 For each $k \in Y$, fix $c_k(1)$ arbitrarily such that $0 \leq c_k(1) \leq c(\{x_k\})$, where $x_k \in X_k$. For any k choose $x_k \in X_k$, and define $F : 2^Y \to \mathbf{R}$ as follows. For all $Y' \subseteq Y$,

$$F(Y') := c(\{x_k : k \in Y'\}) - \sum_{k \in Y'} c_k(1).$$

By condition (ii), this does not depend on the choice of the x_k. The functions c_k are inductively defined as follows. Choose any set S that contains exactly $i \geq 2$ elements of X_k, and let $x_k \in S \cap X_k$. Then, set

$$c_k(i) := c_k(i-1) + [c(S) - c(S \setminus \{x_k\})].$$

By conditions (i) and (ii), this definition does not depend on the choice of S and x_k. It is easily verified that the functions F and c_k yield the desired decomposition.

Proof of Fact 4.2 By definition of $h(\omega)$, one has $c_\omega(\{x\}) = 0$ for all $x \notin h(\omega)$, and by the submodularity, in fact $\nabla_x c_\omega(S) = 0$ for all $x \notin h(\omega)$. Moreover, if c_ω is of type I, one has $c_\omega(\{x, y\}) - c_\omega(\{y\} = 0$ for all $x, y \in h(\omega)$, by definition. This implies $c_\omega(\{x\}) = c_\omega(\{y\}) =: \bar{c}$ for all $x, y \in h(\omega)$, and therefore, again by submodularity, $c_\omega(S) = \bar{c}$ for any S with $S \cap h(\omega) \neq \emptyset$. From this it is immediate that the c.m.i. of c_ω is given by $\lambda_{h(\omega)} = \bar{c}$ and $\lambda_A = 0$ for all $A \neq h(\omega)$.

References

[1] Arrow, K. (1962), The Economic Implications of Learning by Doing, *Review of Economic Studies* **29**, 155-173.

[2] Caballé, J. and A. Pomansky (1996), Mixed Risk Aversion, *Journal of Economic Theory* **71**, 485-513.

[3] Chateauneuf, A. and J.-Y. Jaffray (1989), Some Characterizations of Lower Probabilities and Other Monotonic Capacities through the Use of Moebius Inversion, *Mathematical Social Sciences* **17**, 263-283.

[4] Choquet, G. (1953), Theory of Capacities, *Annales de l'Institut Fourier* **5**, 131-295.

[5] Doignon, J.-P. and J.-C. Falmagne (1985), Spaces for the Assessment of Knowledge, *International Journal of Man-Machine Studies* **23**, 175-196.

[6] Doignon, J.-P. and J.-C. Falmagne (1999), *Knowledge Spaces*, Springer, New York.

[7] Kimball, M.S. (1990), Precautionary Saving in the Small and in the Large, *Econometrica* **58**, 53-73.

[8] Marshall, A. (1890/1920), *Principles of Economics*, First published 1890, Reprint of the eighth edition 1920, Macmillan, New York.

[9] Moulin, H. (1988), *Axioms of Cooperative Decision Making*, Cambridge University Press, Cambridge.

[10] Nehring, K. (1999), Diversity and the Geometry of Similarity, *mimeo.*

[11] Nehring, K. and C. Puppe (1999), Modelling Economies of Scope, SFB Discussion Paper A-607, University of Bonn.

[12] Nehring, K. and C. Puppe (2002), A Theory of Diversity, *Econometrica* **70**, 1155-1198.

[13] Panzar, J.C. (1989), Technological Determinants of Firm and Industry Structure, Ch. 1 in: *Handbook of Industrial Organization*, ed. by R. Schmalensee and R.D. Willig, Elsevier.

[14] Panzar, J.C. and R.D. Willig (1981), Economies of Scope, *American Economic Review* **71**, 268-272.

[15] Rota, G.C (1964), Theory of Moebius Functions, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **2**, 340-368.

[16] Shapley, L.S. (1953), A Value for n-Person Games, in: *Contributions to the Theory of Games II (Annals of Mathematics Studies 28)*, ed. by H.W.Kuhn and A.W.Tucker, Princeton University Press, Princeton.

[17] Scarf, H. (1981), Production Sets with Indivisibilities. Part I: Generalities, and Part II: The Case of Two Activities, *Econometrica* **49**, 1-32, and 395-423.

[18] Scarf, H. (1994), The Allocation of Resources in the Presence of Indivisibilities, *Journal of Economic Perspectives* **8**, 111-128.

[19] Young, H.P. (1994), *Equity. In Theory and Practice*, Princeton University Press, Princeton.

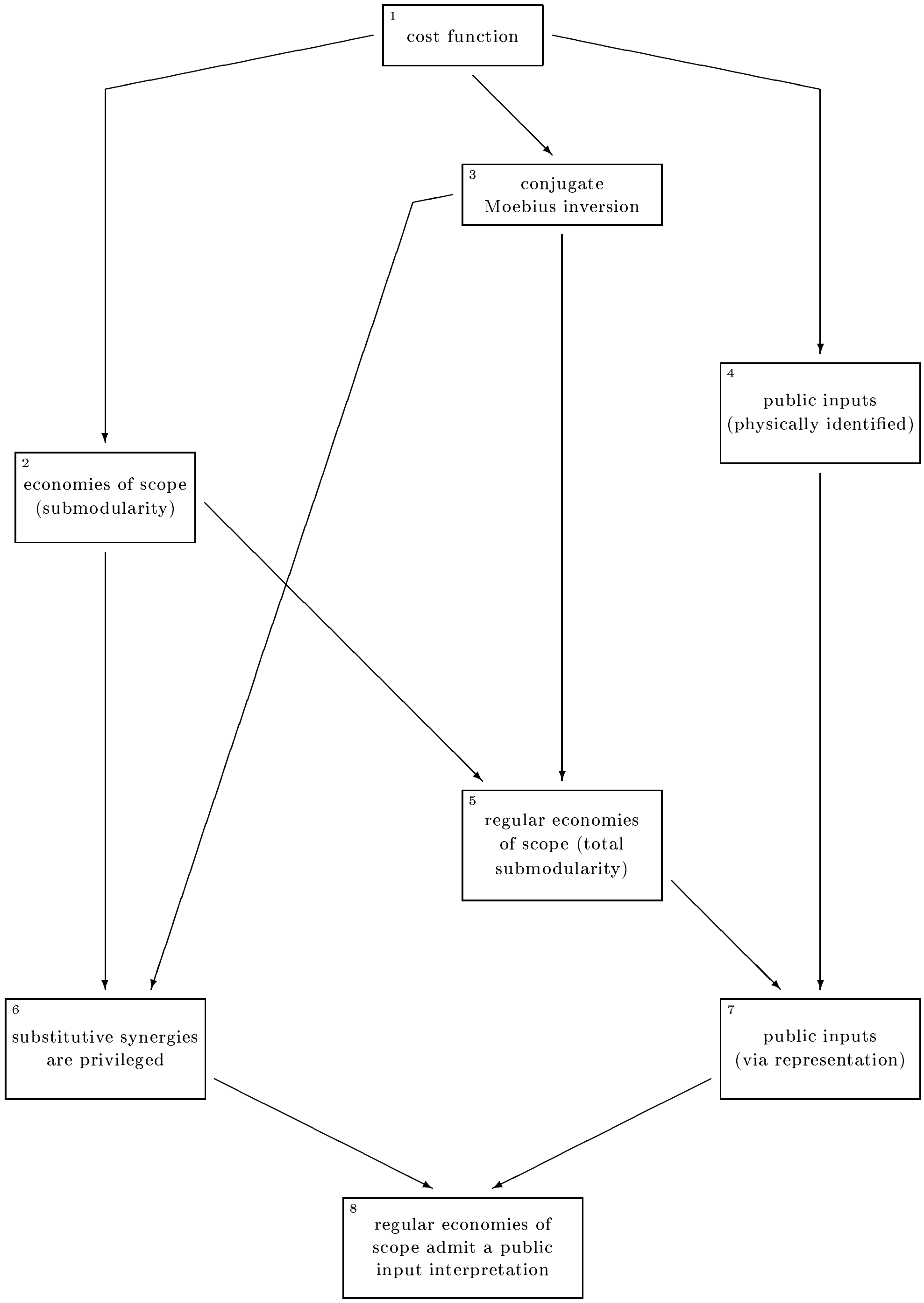

Figure 5: The presupposition relation of Section 2

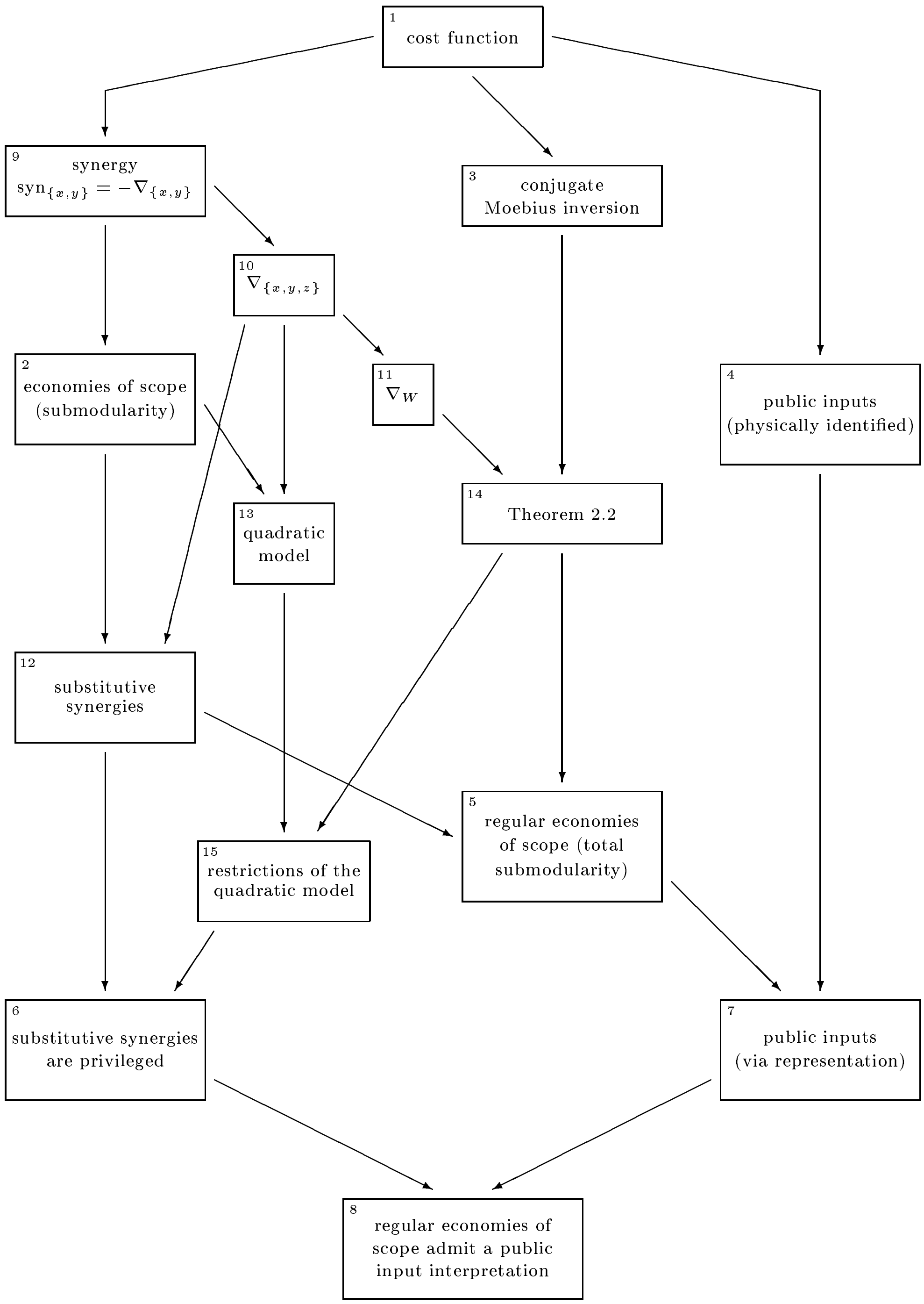

Figure 7: The presupposition relation of Section 2 revisited. SSP satisfied on an enlarged domain