Abstract Arrowian Aggregation*

Klaus Nehring
Department of Economics, University of California at Davis
Davis, CA 95616, U.S.A.
kdnehring@ucdavis.edu

and

Clemens Puppe
Department of Economics, University of Karlsruhe
D – 76128 Karlsruhe, Germany
puppe@wior.uni-karlsruhe.de

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Abstract *In the general framework of abstract binary Arrowian aggregation introduced by Wilson (1975), we characterize aggregation problems in terms of the (monotone) Arrowian aggregators they admit. Specifically, we characterize the problems that enable non-dictatorial, locally non-dictatorial, anonymous and neutral aggregators, respectively.*

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1 Introduction

The model of abstract binary Arrowian aggregation introduced by Wilson (1975), and further developed by Rubinstein and Fishburn (1986), provides a general framework for studying the problem of aggregating sets of logically interconnected propositions, a problem that has recently received some attention in the literature on judgement aggregation, following List and Pettit (2002). A great variety of aggregation problems can be analyzed in this framework, among them the classical preference aggregation problem and the problem of strategy-proof social choice on generalized single-peaked domains, see Section 4 below for a few examples.

Within this framework, we characterize the problems that admit non-dictatorial, locally non-dictatorial, anonymous and neutral Arrowian aggregators, respectively. A "problem" is characterized by a set of social states which are described in terms of a family of binary properties, or equivalently, in terms of yes/no-issues. Each state corresponds to a unique combination of properties, or yes/no-evaluations. Crucially, the issues are logically interrelated so that some evaluations are ruled out. An aggregator maps profiles of such evaluations to a collective evaluation. An aggregator is called Arrowian if it satisfies the familiar independence condition and respects unanimity. Throughout, we will also assume a natural and desirable condition of non-negative responsiveness ("monotonicity"). Monotonicity enables a unified characterization of the class of all Arrowian aggregators in terms of a simple combinatorial condition, the "Intersection Property" that is not available without monotonicity (see Appendix B).

Our first main result, Theorem 1, derives a condition called "total blockedness" that is both necessary and sufficient for a problem to admit only dictatorial monotone Arrowian aggregators. Many, but by far not all, interesting problems are totally blocked; examples are provided in Section 4.

While this result ensures that if a problem is not totally blocked non-dictatorial monotone Arrowian aggregators exist, those may still be "almost dictatorial" by giving almost all decision power to a single agent, or by giving all decision power on some issues to one agent and all decision power on all other issues to another agent. Thus, the negation of total blockedness cannot be viewed as securing genuine possibility results. The second main result of the paper, Theorem 2, therefore characterizes those problems that admit anonymous and monotone Arrowian aggregators, ensuring that all agents have equal influence on the chosen outcome. It turns out that the problems that admit anonymous aggregators are exactly those that admit locally non-dictatorial aggregators.

As illustrated by an example, the characterizing condition for the existence of anonymous monotone Arrowian aggregators is necessarily complex. The complexity results from the existence of rather contrived cases in which anonymous aggregation rules exist only for an odd number of agents. A simpler and more satisfying characterization is obtained for problems admitting anonymous monotone Arrowian aggregators for an arbitrary number of agents (Theorem 3).

While anonymous aggregation rules treat agents symmetrically, they typically treat social states asymmetrically, for instance by applying different quotas to different issues. We therefore finally characterize the circumstances under which monotone Arrowian aggregation is compatible with different notions of neutrality, i.e. symmetric treatment of social states (Theorem 4).

The remainder of this paper is organized as follows. The following Section 2 introduces our framework and notation. In Section 3, we state the main results. Section 4 presents and discusses applications and examples. All proofs are collected in the appendices.

2 Framework and Notation

2.1 Property Spaces: Definition

A *property space* is a pair $(X, \mathcal{H})$, where X is a finite universe of social states or social alternatives, and $\mathcal{H}$ is a collection of subsets of X satisfying

H1 $H \in \mathcal{H} \Rightarrow H \neq \emptyset$,
H2 $H \in \mathcal{H} \Rightarrow H^c \in \mathcal{H}$,
H3 for all $x \neq y$ there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$,

where, for any $S \subseteq X$, $S^c := X \setminus S$ denotes the complement of S in X. The elements $H \in \mathcal{H}$ are referred to as the *basic properties* (with the understanding that a property is extensionally identified with the subset of all social states possessing that property). A pair (H, H^c) is referred to as an *issue*.

Property spaces can be identified with subsets $Z \subseteq \{0,1\}^K$ of hypercubes satisfying $proj_k Z = \{0,1\}$. Specifically, any property space $(X, \mathcal{H})$ with $\mathcal{H} = \{H_1, H_1^c, H_2, H_2^c, ..., H_K, H_K^c\}$ naturally defines a subset Z of $\{0,1\}^K$ by

$$Z := \{(z^1, ..., z^K) \in \{0,1\}^K : [(\cap_{k:z^k=0} H_k^c) \cap (\cap_{k:z^k=1} H_k)] \neq \emptyset\}.$$

Conversely, any subset $Z \subseteq \{0,1\}^K$ canonically defines a property space $(Z, \mathcal{H})$ with $\mathcal{H}$ given by the family of all sets of the form $\{z \in Z : z^k = 0\}$ or $\{z \in Z : z^k = 0\}$, for all $k = 1, ..., K$; satisfaction of conditions H1-H3 is easily verified.[1]

The notion of a property space is also closely related to the notion of an *agenda* in the literature on judgement aggregation, see List and Puppe (2007) for a recent survey of that literature. Specifically, any issue (H, H^c) can be identified with a proposition/negation pair and the elements of X with complete and consistent judgements on these; equivalently, the elements of X can be identified with the consistent truth-value assignments on the propositions.[2]

2.2 Arrowian Aggregation on Property Spaces

Let $N = \{1, ..., n\}$ be a set of individuals, where $n \geq 2$. An *aggregator* is a mapping $f : X^n \to X$. The following conditions on such mappings play a fundamental role in our analysis.

Unanimity $f(x, ..., x) = x$, for all $x \in X$.

Independence If $f(x_1, ..., x_n) \in H$ and, for all $i \in N$, $[x_i \in H \Leftrightarrow y_i \in H]$, then $f(y_1, ..., y_n) \in H$.

Monotonicity If $f(x_1, ..., x_i, ..., x_n) \in H$ and $y_i \in H$, then $f(x_1, ..., y_i, ..., x_n) \in H$.

[1] Note that the property space formulation identifies "isomorphic" subsets, not only in a hypercube of given dimension, but also across hypercubes of different dimensions. E.g., the sets $\{(0, ..., 0), (1, ..., 1)\} \subseteq \{0,1\}^K$ give rise to the same property space for all K.

[2] The abstract aggregation and property space framework can be adapted to model judgment aggregation problems also beyond classical two-valued logic, see e.g. Dokow and Holzman (2006).

An aggregator is called *Arrowian* if it satisfies unanimity and independence. In this paper, we will be concerned with Arrowian aggregators that satisfy in addition the monotonicity condition. Note that under monotonicity, unanimity can be deduced from the weaker requirement that the aggregator is surjective, i.e. that any element of X is in the range of f. Note also that the conjunction of independence and monotonicity is equivalent to the following single condition.

Monotone Independence If $f(x_1, ..., x_n) \in H$ and, for all $i \in N$, $[x_i \in H \Rightarrow y_i \in H]$, then $f(y_1, ..., y_n) \in H$.

Besides its evident appeal as a condition on satisfactory aggregation, the main advantage of assuming monotonicity is the existence of a unified characterization of all monotone Arrowian aggregators in terms of "voting by issues" satisfying a simple combinatorial condition, the "Intersection Property" (see Appendix B). A comparable characterization of all Arrowian aggregators without monotonicity is not known.

2.3 Conditional Entailment and (Total) Blockedness

In this subsection, we describe the key conditions on the logical complexity of a property space that will later allow us to classify these in terms of the monotone Arrowian aggregators they admit.

Say that a family $\mathcal{G} \subseteq \mathcal{H}$ of basic properties is a *critical family* if $\cap\mathcal{G} = \emptyset$ and for all $G \in \mathcal{G}$, $\cap(\mathcal{G} \setminus \{G\}) \neq \emptyset$. A critical family $\mathcal{G} = \{G_1, ..., G_l\}$ thus describes the exclusion of the combination of the corresponding basic properties in the sense that $G_1, ..., G_l$ cannot be jointly realized. "Criticality" (i.e. minimality) means that this exclusion is not implied by a more general exclusion in the sense that the basic properties in any proper subset of $\mathcal{G}$ are jointly realizable. Observe that all pairs $\{H, H^c\}$ of complementary basic properties are critical; they are referred to as the *trivial* critical families.

The following "entailment" relation is crucial. Say that H *conditionally entails* G, written as $H \geq^0 G$ if $H \neq G^c$ and there exists a critical family containing both H and G^c. Intuitively, $H \geq^0 G$ thus means that, given *some* combination of other basic properties, the basic property H entails the basic property G. More precisely, let $H \geq^0 G$, i.e. let $\{H, G^c, G_1, ..., G_l\}$ be a critical family; then with $A = \cap_{j=1}^{l} G_j$ one has both $A \cap H \neq \emptyset$ ("property H is compatible with the combination A of properties") and $A \cap G^c \neq \emptyset$ ("property G^c is compatible with A as well") but $A \cap H \cap G^c = \emptyset$ ("properties H and G^c are jointly incompatible with A"). Note that $H \geq^0 G \Leftrightarrow G^c \geq^0 H^c$. We write $\geq$ for the transitive closure of $\geq^0$, and $\equiv$ for the symmetric part of $\geq$.

It turns out that all subsequent characterization results can be stated in terms of this conditional entailment relation. Its key role in our analysis derives from the following observation. For any monotone Arrowian aggregator f, say that a coalition W of agents is *winning for H* if the agents in W can jointly force the outcome under f to be an element of H. If $H \geq G$ then any a coalition that is winning for H must also be winning for G (see Lemma 1 in Appendix C).

Say that $(X, \mathcal{H})$ is *totally blocked* if, for all $H, G \in \mathcal{H}$, $H \geq G$, i.e. if there exists a sequence of conditional entailments from every basic property to every other basic property. Total blockedness is a quite demanding condition as it requires a large number of entailments. As we shall see in Section 4 below, it is nevertheless satisfied by a number of interesting aggregation problems. Say that H is *blocked* if $H \equiv H^c$, i.e. if there exists a sequence of conditional entailments from H to its complement H^c, and vice versa. Call a property space $(X, \mathcal{H})$ *blocked* if some $H \in \mathcal{H}$ is blocked; otherwise, if no H is blocked, $(X, \mathcal{H})$ is called *unblocked.* Finally, for each $G \in \mathcal{H}$, let $\mathcal{H}_{\equiv G} := \{H \in \mathcal{H} : H \equiv G\}$, and say that a property space is *quasi-unblocked* if for any $G \in \mathcal{H}$ and any critical family $\mathcal{G}$, $\#(\mathcal{H}_{\equiv G} \cap \mathcal{G}) \leq 2$, whenever G is blocked. Evidently, quasi-unblockedness is intermediate in strength between not total blockedness and unblockedness.

3 Characterization Results

3.1 Non-Dictatorial Aggregation

An aggregator f is called *dictatorial* if there exists an individual i such that, for all $x_1, ..., x_n$, $f(x_1, ..., x_n) = x_i$.

Theorem 1 *A property space $(X, \mathcal{H})$ admits non-dictatorial and monotone Arrowian aggregators if and only if it is not totally blocked.*

To use Theorem 1 to show that a given domain is dictatorial is typically fairly straightforward, as it involves coming up with sufficiently many instances of conditional entailment; in particular, it is not necessary to determine the set of critical families exhaustively. By contrast, in order to show that a domain is non-dictatorial, in principle one needs to determine the transitive hull of the entire conditional entailment relation; this may be difficult. However, an easily verifiable and frequently applicable sufficient condition is that there be at least one basic property not contained in any non-trivial critical family. Indeed, if H is only contained in the trivial critical family $\{H, H^c\}$, one has $H \not\geq^0 G$ for all G, and therefore $H \not\geq H^c$, which implies that the underlying property space is not totally blocked.

3.2 Locally Non-Dictatorial and Anonymous Aggregation

As a possibility result, Theorem 1 is not completely satisfactory since non-dictatorial aggregation rules can still be rather degenerate, e.g. by giving almost all decision power to one agent, or by specifying different "local" dictators for different issues. In this subsection, we therefore characterize the problems for which locally non-dictatorial monotone Arrowian aggregators exist. It turns out that this is also exactly the class of problems for which anonymous monotone Arrowian aggregators exist.

An aggregator f is called *locally dictatorial* if there exists an individual i and an issue (H, H^c) such that, for all $x_1, ..., x_n$, $f(x_1, ..., x_n) \in H \Leftrightarrow x_i \in H$. Note that there may exist several local dictators (over different issues). An aggregator f is called *anonymous* if it is invariant under permutations of individuals. An anonymous aggregator is necessarily locally non-dictatorial.

Theorem 2 *Let $(X, \mathcal{H})$ be a property space. The following are equivalent.*
(i) $(X, \mathcal{H})$ *admits locally non-dictatorial and monotone Arrowian aggregators.*
(ii) $(X, \mathcal{H})$ *admits anonymous and monotone Arrowian aggregators.*
(iii) $(X, \mathcal{H})$ *is quasi-unblocked.*

In Appendix A, we show that there are spaces that are quasi-unblocked yet blocked. However, these appear quite contrived, and it seems unlikely that they are relevant in applications. Moreover, on such spaces anonymous aggregation rules exist only for an odd number of agents; hence, the possibility obtained in these cases is not robust.

A cleaner and more satisfying characterization is obtained for property spaces admitting anonymous rules for an arbitrary number of agents. This characterization hinges on the following characterization of unblocked spaces as quasi-median spaces, as follows. For each $x \in X$, denote by $\mathcal{H}_x := \{H \in \mathcal{H} : x \in H\}$; an element $\hat{x} \in X$ is called a *median point* if, for any critical family $\mathcal{G}$, $\#(\mathcal{H}_{\hat{x}} \cap \mathcal{G}) \leq 1$. Thus, a state is a median point if every critical family contains at most one of its constituent properties. The set of all median points is denoted by $M(X)$, and a property space $(X, \mathcal{H})$ is called a *quasi-median space* if $M(X) \neq \emptyset$. A space is called a *median space* if every element is a median point, i.e. if $M(X) = X$. It is easily verified that $(X, \mathcal{H})$ is a median space if and only if all critical families have exactly two elements. Median spaces are well-studied in combinatorial mathematics and related fields (see, e.g., van de Vel, 1993); their important role in the theory of aggregation is highlighted in Nehring and Puppe (2007), where it is shown among other things that issue-by-issue majority voting is consistent if and only if the underlying property space is a median space.

The concept of quasi-median space has been introduced by Nehring (2004) and seems to be new to the literature. Nehring (2004) provides a geometric characterization of median points which explains terminology, and the following fundamental result.

Proposition 3.1 *A property space $(X, \mathcal{H})$ is unblocked if and only if $(X, \mathcal{H})$ is a quasi-median space.*

Median points play a central role in our present context, because they are canonically associated with unanimity rules. Unanimity rules, in turn, are the canonical examples anonymous rules for an arbitrary number of agents. An Arrowian aggregator f is called a *unanimity rule* if there exists $\hat{x} \in X$ such that for all $H \in \mathcal{H}_{\hat{x}}$,

$$f(x_1, ..., x_n) \in H \Leftrightarrow x_i \in H \text{ for some } i \in N. \tag{3.1}$$

Clearly, a state $\hat{x}$ such that (3.1) is satisfied for all $H \in \mathcal{H}_{\hat{x}}$ is uniquely determined and is referred to as the *status quo*. Henceforth, we denote the unanimity rule with status quo $\hat{x}$ by $f_{\hat{x}}$. Note that the basic properties determined in (3.1) may not be jointly compatible, so that an Arrowian unanimity rule of the form $f_{\hat{x}}$ may or may not exist.

Proposition 3.2 *A property space $(X, \mathcal{H})$ admits an Arrowian unanimity rule of the form $f_{\hat{x}}$ if and only if $\hat{x} \in M(X)$.*

The following result summarizes the two characterizations of quasi-median spaces entailed by Propositions 3.1 and 3.2, and shows that quasi-median spaces are also exactly the spaces that admit anonymous monotone Arrowian aggregators for any number of individuals.

Theorem 3 *Let $(X, \mathcal{H})$ be a property space. The following are equivalent.*
(i) *$(X, \mathcal{H})$ admits anonymous and monotone Arrowian aggregators for some even n.*
(ii) *$(X, \mathcal{H})$ admits anonymous and monotone Arrowian aggregators for all $n \geq 2$.*
(iii) *$(X, \mathcal{H})$ admits some Arrowian unanimity rule.*
(iv) *$(X, \mathcal{H})$ is unblocked.*
(v) *$(X, \mathcal{H})$ is a quasi-median space.*

3.3 Neutral Aggregation

Let x_i, y_i, for all i, and two basic properties H and H' be given such that, for all i, $x_i \in H \Leftrightarrow y_i \in H'$. An aggregator f is called *neutral with respect to* H *and* H' if in this situation $f(x_1, ..., x_n) \in H \Leftrightarrow f(y_1, ..., y_n) \in H'$. An aggregator f is called *neutral within issues* if, for all H, f is neutral with respect to H and H^c; moreover, f is called *neutral across issues* if, for all H and H', f is neutral with respect to H and H' or with respect to H^c and H'; finally, f is called *(fully) neutral* if it is neutral with respect to all H and H'.

Examples of aggregators that are neutral across but not within issues are the unanimity rules, or more generally, supermajority rules with a uniform quota $> 1/2$ for each issue. An example of an aggregator that is neutral within but not across issues is *weighted* issue-by-issue majority voting where the weights differ across issues. Specifically, let $\mathcal{H} = \{H_1, H_1^c, ..., H_K, H_K^c\}$; for all k and i, denote by $w_i^k \geq 0$ the weight of voter i in issue k, and assume that $\sum_i w_i^k = 1$ for all $k = 1, ..., K$. Weighted issue-by-issue majority voting is defined by

$$f(x_1, ..., x_n) \in H_k :\Leftrightarrow \sum_{i: x_i \in H_k} w_i^k > 1/2.$$

The difference in weights across issues may be the natural result of voters having different stakes and/or different expertise in different dimensions.

Theorem 4 *Let $(X, \mathcal{H})$ be a property space.*
a) *$(X, \mathcal{H})$ admits monotone Arrowian aggregators that are non-dictatorial and neutral across issues if and only if $(X, \mathcal{H})$ is a quasi-median space.*
b) *$(X, \mathcal{H})$ admits monotone Arrowian aggregators that are locally non-dictatorial and neutral within issues if and only if $(X, \mathcal{H})$ is a median space.*
c) *$(X, \mathcal{H})$ admits monotone Arrowian aggregators that are non-dictatorial and (fully) neutral if and only if $(X, \mathcal{H})$ is a median space.*

Note that, by part b), non-dictatorial aggregators that are neutral within issues may exist also outside the class of median spaces. However, if the underlying space is "indecomposable" then neutrality within issues is just as demanding as full neutrality, as shown by the following result. Say that $(X, \mathcal{H})$ is *decomposable* if $\mathcal{H}$ can be partitioned into two non-empty subfamilies $\mathcal{H}_1$ and $\mathcal{H}_2$ such that each critical family is either entirely contained in $\mathcal{H}_1$ or entirely contained in $\mathcal{H}_2$; otherwise, $(X, \mathcal{H})$ is called *indecomposable*. One can easily show that a property space is decomposable if and only if it can be represented as the Cartesian product of (at least) two property spaces.

Proposition 3.3 *Suppose that $(X, \mathcal{H})$ is indecomposable. Then, any monotone Arrowian aggregator that is neutral within issues is also neutral across issues, hence fully neutral.*

4 Examples and Applications

4.1 Aggregation of Preferences

It is well-known that the classical problem of preference aggregation is a special case of the binary aggregation framework considered here; see Wilson (1975), Nehring (2003), Dokow and Holzman (2005). Specifically, let $A = \{a, b, ...\}$ be a finite set of alternatives and let $\mathcal{R}$ be a family of binary relations on A. For each pair $a, b \in A$ let $H_{(a,b)}$ denote the basic property "aRb," or more formally,

$$H_{(a,b)} := \{R \in \mathcal{R} : aRb\}.$$

A binary relation on A can thus be identified with a certain combination of basic properties, and the family $\mathcal{R}$ can thus be embedded in a $(\#A)^2$-dimensional hypercube. Different requirements on the members of the family $\mathcal{R}$ give rise to different subsets. For instance, transitivity of the binary relations in $\mathcal{R}$ implies that $\{H_{(a,b)}, H_{(b,c)}, H^c_{(a,c)}\}$ is an excluded combination of properties, i.e. forms a critical family. If the elements of $\mathcal{R}$ are all irreflexive, then $\mathcal{R}$ can be embedded in a hypercube of dimension $\#A \cdot (\#A - 1)$. If the elements of $\mathcal{R}$ are in addition asymmetric and connected, then $\mathcal{R}$ can even be embedded in a hypercube of dimension $\frac{\#A\cdot(\#A-1)}{2}$.

For instance, the set $\mathcal{L}in(A)$ of all strict linear orderings on A can be embedded in the 3-hypercube if $\#A = 3$, as shown in Figure 1; in general, the embedding of $\mathcal{L}in(A)$ in the $\frac{\#A\cdot(\#A-1)}{2}$-hypercube is known as the $\#A$-permutahedron.

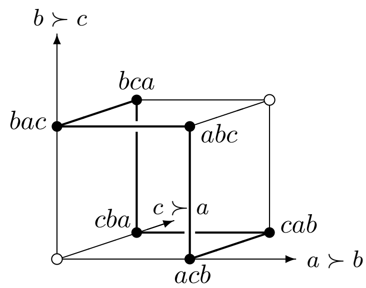

Figure 1: The 3*-Permutahedron*

Nehring (2003) has shown that $\mathcal{L}in(A)$ is totally blocked whenever $\#A \geq 3$, and used Theorem 1 above to show that therefore all monotone Arrowian aggregators on $\mathcal{L}in(A)$ are dictatorial if $\#A \geq 3$. This is a monotone version of Arrow's theorem for strict preferences.

Another preference domain for which our results have an immediate implication is the set of strict partial orderings $\mathcal{P}art(A)$. The domain $\mathcal{P}art(A)$ is a quasi-median space with a unique median point given by the empty relation according to which no pair of alternatives can be compared. By Theorem 3, there exist anonymous Arrowian aggregators on $\mathcal{P}art(A)$, and by Proposition 3.2 there is a unique unanimity rule among these, which is given by taking the intersection of all individual orderings.

4.2 Discursive Dilemma

A special class of aggregation problems arises by considering a set of binary propositions that can be split into a set of "premises" and a set of "conclusions" which depend on the evaluation of the premises. A simple example arises by taking a conclusion that is logically equivalent to the conjunction of its premises. The so-called "discursive dilemma" (see Pettit, 2001, Kornhauser and Sager, 1986) consists in the observation that satisfactory aggregation methods, such as proposition-wise majority voting, may yield inconsistent collective judgements. The results presented here can be used to classify the monotone Arrowian aggregators admitted in such contexts.

For simplicity, we consider only the case with a single conclusion. Suppose also that the conclusion is uniquely determined by a complete evaluation of the premises (the "truth-functional case"). It can be shown that the corresponding space is a quasi-median space if and only if the conclusion or its negation can be written as the conjunction of the premises or their respective negations. In this "conjunctive" case, there is a unique median point, and hence the problem admits anonymous Arrowian aggregators in form of unanimity rules. In the non-conjunctive case, by contrast, the underlying space is totally blocked and thus only admits dictatorial aggregators (see Nehring and Puppe, 2005a).

4.3 Admitting Applicants

Consider the K-dimensional hypercube and the subset $X_{(K;k,k')} \subseteq \{0,1\}^K$ of all binary sequences with at least k and at most k' coordinates having the entry 1, where $0 \leq k \leq k' \leq K$. A possible interpretation is that each coordinate corresponds to an applicant for a number vacant positions of which at least k *have* to be filled, and at most k' *can* be filled. A binary evaluation in $X_{(K;k,k')}$ specifies which applicants should be admitted (those having entry 1).[3]

If $k = 0$ and $k' = K$, we obtain the full hypercube in which the only critical families are the trivial critical families of the form $\{H_0^k, H_1^k\}$ for $k = 1, ..., K$, where H_0^k (resp. H_1^k) denotes the set of binary evaluations with 0 (resp. 1) in coordinate k. Evidently, the full hypercube is a median space; in particular, it is not totally blocked.

Next, assume $k > 0$. If $k' = K$, the non-trivial critical families of the resulting space are exactly the subsets of $\{H_0^1, H_0^2, ..., H_0^K\}$ with $K - k + 1$ elements. The interpretation of such a critical family is that, if already $K - k$ applicants have been rejected, then all of the remaining applicants must be admitted. The resulting space is not a median space, but it is a quasi-median space and consequently admits both anonymous Arrowian aggregators and aggregators that are neutral across issues. For instance, the unanimity rule according to which a candidate is admitted unless all voters reject her is consistent since the point $(1, ..., 1)$ is a median point. More generally, one can show that the aggregator according to which an applicant is admitted as soon as at least a fraction of $1/(K - k + 1)$ voters approve her is consistent.

Let now $0 < k \leq k' < K$. Then, in addition to all subsets of $\{H_0^1, H_0^2, ..., H_0^K\}$ with $K - k + 1$ elements also any subset of $\{H_1^1, H_1^2, ..., H_1^K\}$ with $k' + 1$ elements forms a critical family. It is easily verified that the corresponding spaces are totally blocked whenever $K \geq 3$. By Theorem 1, any monotone Arrowian aggregator is dictatorial.

As a variation of this example, consider a non-empty subset $J \subseteq \{1, ..., K\}$ representing a subgroup of applicants, and suppose that at least one applicant has to be admitted, but at most m out of the subgroup J, where $1 \leq m \leq \#J$. Denote the corresponding subspace by $X_{(K;m,J)}$. If $\#J < K$, none of the spaces $X_{(K;m,J)}$ is totally blocked. Indeed, for all $k \notin J$, the property "applicant k is admitted" is not an element of any non-trivial critical family. Thus, by the remark after Theorem 1 above, the space

[3]Another possible interpretation arises when the set of applicants is the set of voters themselves. The resulting problem is known in the literature as the *group identification* problem, following Kasher and Rubinstein (1997), see also List (2006) for a formulation in the judgement aggregation model.

is not totally blocked. On the other hand, if $\#J > 2$, then the subspace corresponding to the coordinates in J is totally blocked. It can be shown that, therefore, all monotone Arrowian aggregators on $X_{(K;m,J)}$ are locally dictatorial whenever $2 < \#J < K$. If $\#J = 2$ the corresponding spaces are quasi-median spaces.

As another variation suppose that l of the K candidates are women and that a regulation requires that at least as many women be hired as men, so that again not all points of the hypercube represent possible states. Evidently, the state in which all women and no men are admitted is a median point, so that the underlying space is a quasi-median space. There may be other median points, but in general the space is not a median space; for instance, the space that results from taking $l = 2$ and $K = 3$ is isomorphic to the space $X_{(3;1,3)}$ above. It is also easily verified that the class of all anonymous and monotone Arrowian aggregators that treat all women and all men symmetrically is a one-dimensional family with the extreme points $(1, \frac{1}{m})$ and $(\frac{m}{m+1}, \frac{1}{m+1})$, where the first entry is the quota for hiring a man, and the second the quota for hiring a woman. Note the extent to which the regulation biases the hiring in favor of women.

4.4 Strategy-Proof Social Choice

The results of the present paper allow one to derive corresponding results on the existence of strategy-proof social choice functions on a large class of domains, the "generalized single-peaked domains" introduced in Nehring and Puppe (2002, 2007). There is a one-to-one correspondence between the class of strategy-proof social choice functions defined on a rich domain of generalized single-peaked preferences on a given property space $(X, \mathcal{H})$ and the class of monotone Arrowian aggregators on $(X, \mathcal{H})$. The characterization results derived here thus apply via this correspondence to strategy-proof social choice functions on such domains. In particular, the Gibbard-Satterthwaite theorem is easily obtained as a special case of Theorem 1 above (see Nehring and Puppe, 2005b).

5 Related Literature

In the present paper, we have classified aggregation problems in terms of the monotone Arrowian aggregators they admit. The monotonicity condition is crucial in order to obtain a simple unified characterization of the class of all Arrowian aggregators for an arbitrary problem. While the monotonicity condition seems conceptually uncontroversial, mathematically, it may be interesting to explore the aggregation possibilities without it. Some results have already been established in that direction. Dokow and Holzman (2005) identify a condition ("non-affineness") that together with total blockedness characterizes the aggregation problems on which all Arrowian aggregators (with or without monotonicity) are dictatorial. In a similar vein, Dietrich and List (2007a,b) show how Theorem 4b) and 4c) can be adapted to the non-monotone case. Finally, Dietrich and List (2007c) show that Theorem 3 remains valid also without the monotonicity requirement.

Appendix A: Anonymity without Median Points

Consider the subspace $X \subseteq \{0,1\}^5$ shown in Figure 2 below. The two cubes to the right correspond to a "1" in coordinate 4 (i.e. to the basic property H_1^4), similarly, the two top cubes correspond to a "1" in coordinate 5 (i.e. to H_1^5). Missing points of the 5-hypercube are indicated by blank circles. For the purpose of better illustration, the edges connecting different points across the four subcubes have been omitted in the figure.

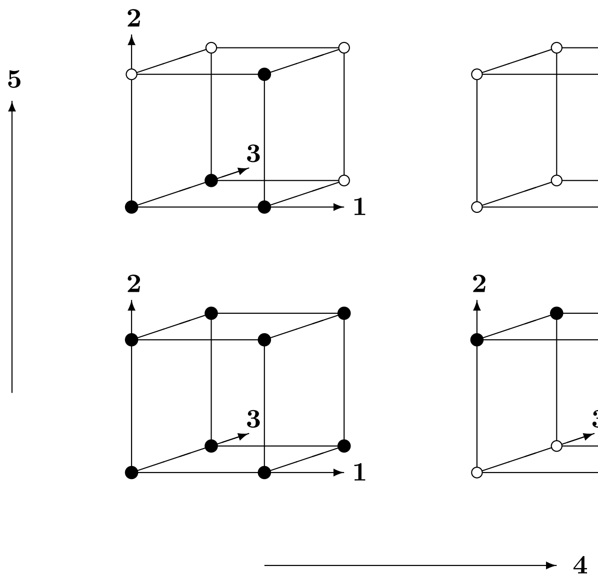

Figure 2: A quasi-unblocked space without median points

This space is characterized by the following critical families: $\mathcal{G}_1 = \{H_1^1, H_0^3, H_1^4\}$, $\mathcal{G}_2 = \{H_1^1, H_1^3, H_1^5\}$, $\mathcal{G}_3 = \{H_0^1, H_0^2, H_1^4\}$, $\mathcal{G}_4 = \{H_0^1, H_1^2, H_1^5\}$, $\mathcal{G}_5 = \{H_0^2, H_0^3, H_1^4\}$, $\mathcal{G}_6 = \{H_1^2, H_1^3, H_1^5\}$ and $\mathcal{G}_7 = \{H_1^4, H_1^5\}$. For instance, the criticality of $\{H_1^4, H_1^5\} = \mathcal{G}_7$ reflects the fact that the top-right cube contains no element of X, and is a maximal subcube with this property. As is easily verified, one has $H_0^k \equiv H_1^k$ for $k = 1, 2, 3$, i.e. the first three coordinates are blocked; in particular, by Proposition 3.1, the underlying space admits no median points. Nevertheless, denoting by q_1^k the quota corresponding to H_1^k, the following anonymous choice rule is easily seen to be consistent if the number of voters is odd: The final outcome lies in the top left subcube if and only if all voters endorse the basic property H_1^5 (i.e. $q_1^5 = 1$); similarly, the choice is in the bottom right subcube if and only if all voters endorse H_1^4 (i.e. $q_1^4 = 1$). In all other cases, the outcome lies in the bottom left subcube ($q_0^5 = q_0^4 = 0$). In addition, the location of the outcome within any of the three admissible subcubes is decided by majority vote in each of the first three coordinates ($q_1^1 = q_1^2 = q_1^3 = \frac{1}{2}$). Using the anonymous version of the Intersection Property (see Nehring and Puppe, 2007, Fact 3.4) is is easily verified that this rule is in fact the *only* anonymous and monotone Arrowian aggregator in the present example. Note in particular that, in accordance with Theorem 3, there is no anonymous rule for an even number of voters.

Appendix B: Characterization of the Class of all Monotone Arrowian Aggregators

The proofs of our results use the following general characterization of the class of all monotone Arrowian aggregators on an arbitrary property space from Nehring and Puppe (2002, 2007). A family of *winning coalitions* is a non-empty family $\mathcal{W}$ of subsets of the set N of all individuals satisfying $[W \in \mathcal{W}$ and $W' \supseteq W] \Rightarrow W' \in \mathcal{W}$. A *structure of winning coalitions* on $(X, \mathcal{H})$ assigns a family of winning coalitions $\mathcal{W}_H$ to each property satisfying the following condition,

$$W \in \mathcal{W}_H \Leftrightarrow (N \setminus W) \notin \mathcal{W}_{H^c}. \tag{B.1}$$

In words, a coalition is winning for H if and only if its complement is not winning for the negation of H. Using (B.1) and the fact that families of winning coalitions are closed under taking supersets, we obtain

$$\mathcal{W}_{H^c} = \{W \subseteq N : W \cap W' \neq \emptyset \text{ for all } W' \in \mathcal{W}_H\}. \tag{B.2}$$

A mapping $f : X^n \to X$ is called *voting by issues* if for some structure of winning coalitions and all $H \in \mathcal{H}$,

$$f(x_1, ..., x_n) \in H \Leftrightarrow \{i : x_i \in H\} \in \mathcal{W}_H.$$

A structure of winning coalitions satisfies the *Intersection Property* if for any critical family $\{H_1, ..., H_l\} \subseteq \mathcal{H}$, and any selection $W_j \in \mathcal{W}_{H_j}$,

$$\bigcap_{j=1}^{l} W_j \neq \emptyset.$$

The following result is proved in Nehring and Puppe (2002; 2007, Theorem 3).

Theorem *A mapping $f : X^n \to X$ is a monotone Arrowian aggregator if and only if it is voting by issues satisfying the Intersection Property.*

Appendix C: Proofs

For the subsequent proofs, we always assume that a monotone Arrowian aggregator is described as voting by issues with associated structure $\{\mathcal{W}_H\}_{H \in \mathcal{H}}$ of winning coalitions. The fundamental role played by the conditional entailment relation derives from the following simple observation.

Lemma 1 (Inclusion Lemma) *If $\{\mathcal{W}_H\}_{H \in \mathcal{H}}$ satisfies the Intersection Property and $H \geq G$, then $\mathcal{W}_H \subseteq \mathcal{W}_G$.*

Proof of Lemma 1 By transitivity, it suffices to show that $H \geq^0 G \Rightarrow \mathcal{W}_H \subseteq \mathcal{W}_G$. Thus, suppose that $\{H, G^c\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. By the Intersection Property, $W \cap W' \neq \emptyset$ for any $W \in \mathcal{W}_H$ and any $W' \in \mathcal{W}_{G^c}$. By (B.2), this implies $\mathcal{W}_H \subseteq \mathcal{W}_G$.

The following "veto lemma" gives a simple sufficient condition for the existence of an individual who can veto a certain property (by being winning alone for the complementary property).

Lemma 2 (Veto Lemma) *Suppose the structure of winning coalitions satisfies the Intersection Property and that* $\{G_1, G_2, G_3\} \subseteq \mathcal{G}$ *for some critical family* $\mathcal{G}$*. If* $\mathcal{W}_{G_1^c} \subseteq \mathcal{W}_{G_2}$*, then* $\{i\} \in \mathcal{W}_{G_3^c}$ *for some* $i \in N$*.*

Proof of Lemma 2 Let $\tilde{W}_1$ be a minimal element of $\mathcal{W}_{G_1}$, and let $i \in \tilde{W}_1$. By (B.2) and minimality of $\tilde{W}_1$, one has $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}_{G_1^c}$. By assumption, $\mathcal{W}_{G_1^c} \subseteq \mathcal{W}_{G_2}$, hence $(\tilde{W}_1^c \cup \{i\}) \in \mathcal{W}_{G_2}$. Now consider any $W_3 \in \mathcal{W}_{G_3}$. By the Intersection Property, $\cap_{j=1}^3 W_j \neq \emptyset$ for any selection $W_j \in \mathcal{W}_{G_j}$. In particular, $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) \cap W_3 \neq \emptyset$. Since $\tilde{W}_1 \cap (\tilde{W}_1^c \cup \{i\}) = \{i\}$, this means $i \in W_3$ for all $W_3 \in \mathcal{W}_{G_3}$. By (B.2), this implies $\{i\} \in \mathcal{W}_{G_3^c}$.

Proof of Theorem 1 Suppose that $(X, \mathcal{H})$ is totally blocked. By Lemma 1, $\mathcal{W}_H = \mathcal{W}_0$ for some $\mathcal{W}_0$ and all H. Moreover, it is easily verified that any totally blocked space admits at least one critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$. By Lemma 2, $\{i\} \in \mathcal{W}_{G_3^c} = \mathcal{W}_0$; but then voter i is a dictator.

Suppose then that $(X, \mathcal{H})$ is not totally blocked. To construct a non-dictatorial strategy-proof social choice function partition $\mathcal{H}$ as follows.

$$\begin{aligned}
\mathcal{H}_0 &:= \{H \in \mathcal{H} : H \equiv H^c\},\\
\mathcal{H}_1^+ &:= \{H \in \mathcal{H} : H > H^c\},\\
\mathcal{H}_1^- &:= \{H \in \mathcal{H} : H^c > H\},\\
\mathcal{H}_2 &:= \{H \in \mathcal{H} : \text{ neither } H \geq H^c \text{ nor } H^c \geq H\}.
\end{aligned}$$

For future reference we note the following facts about this partition of $\mathcal{H}$. Part c) of the following lemma will only be used later in the proofs of Theorem 2 and Proposition 3.1 below.

Lemma 3 a) *For any critical family* $\mathcal{G}$*, if* $G \in \mathcal{G} \cap \mathcal{H}_1^-$*, then* $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$*.*
b) *For any critical family* $\mathcal{G}$*, if* $\mathcal{G} \cap \mathcal{H}_0 \neq \emptyset$*, then* $\mathcal{G} \subseteq \mathcal{H}_0 \cup \mathcal{H}_1^+$*.*
c) *Take any* $\tilde{H} \in \mathcal{H}_2$*. Then there exists a partition of* $\mathcal{H}_2$ *into* $\mathcal{H}_2^-$ *and* $\mathcal{H}_2^+$ *with* $\tilde{H} \in \mathcal{H}_2^-$ *such that* $G \in \mathcal{H}_2^- \Leftrightarrow G^c \in \mathcal{H}_2^+$*, and for no* $G \in \mathcal{H}_2^-$ *and* $H \in \mathcal{H}_2^+$*,* $G \geq H$*.*

Proof of Lemma 3 a) Suppose $G \in \mathcal{G} \cap \mathcal{H}_1^-$, i.e. $G^c > G$. Consider any other $H \in \mathcal{G}$. We have $H \geq G^c > G \geq H^c$, hence $H > H^c$, i.e. $H \in \mathcal{H}_1^+$.
b) Suppose $G \in \mathcal{G} \cap \mathcal{H}_0$ and let $H \in \mathcal{G}$ be different from G. We have $H \geq G^c \equiv G \geq H^c$, hence $H \geq H^c$. But this means $H \in \mathcal{H}_0 \cup \mathcal{H}_1^+$.
c) The desired partition into $\mathcal{H}_2^- = \{G_1, ..., G_l\}$ and $\mathcal{H}_2^+ = \{G_1^c, ..., G_l^c\}$ will be constructed inductively. Set $G_1 = \tilde{H}$, and suppose that $\{G_1, ..., G_r\}$, with $r < l$, is determined such that $G_j \ngeq G_k^c$ for all $j, k \in \{1, ..., r\}$. Take any $H \in \mathcal{H}_2 \setminus \{G_1, G_1^c, ..., G_r, G_r^c\}$ and set

$$G_{r+1} := \begin{cases} H & \text{if for no } j \in \{1, ..., r\} : \quad G_j \geq H^c \\ H^c & \text{if for some } j \in \{1, ..., r\} : \quad G_j \geq H^c \end{cases}$$

First note that $G_{r+1} \ngeq G_{r+1}^c$ since $H \in \mathcal{H}_2$. Thus, the proof is completed by showing that for no $k \in \{1, ..., r\}$, $G_k \geq G_{r+1}^c$ (and hence also not $G_{r+1} \geq G_k^c$). To verify this, suppose first that $G_{r+1} = H$; then, the claim is true by construction. Thus, suppose $G_{r+1} = H^c$; by construction, there exists $j \leq r$ with $G_j \geq H^c$, hence also $H \geq G_j^c$. Assume, by way of contradiction, that $G_k \geq G_{r+1}^c$, i.e. $G_k \geq H$. This would imply $G_k \geq H \geq G_j^c$, in contradiction to the induction hypothesis.

Proof of Theorem 1 (cont.) If $\mathcal{H}_1^+ \cup \mathcal{H}_1^-$ is non-empty, set $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all

$H \in \mathcal{H}_1^-$ and $\mathcal{W}_H = \{N\}$ for all $H \in \mathcal{H}_1^+$; moreover, choose a voter $i \in N$ and set $\mathcal{W}_G = \{W \subseteq N : i \in W\}$ for all other $G \in \mathcal{H}$. Clearly, the corresponding voting by issues is non-dictatorial. It also satisfies the Intersection Property. Indeed, the only problematic case is when a critical family $\mathcal{G}$ contains elements of $\mathcal{H}_1^-$. However, by Lemma 3a), if $G \in \mathcal{G} \cap \mathcal{H}_1^-$, we have $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$, in which case the Intersection Property is clearly satisfied.

Next, suppose that $\mathcal{H}_1^+ \cup \mathcal{H}_1^-$ is empty, and consider first the case in which both $\mathcal{H}_0$ and $\mathcal{H}_2$ are non-empty. By Lemma 3b), no critical family $\mathcal{G}$ can meet both $\mathcal{H}_0$ and $\mathcal{H}_2$. Hence, we can specify two different dictators on $\mathcal{H}_0$ and $\mathcal{H}_2$, respectively, by setting $\mathcal{W}_H = \{W : i \in W\}$ for all $H \in \mathcal{H}_0$ and $\mathcal{W}_G = \{W : j \in W\}$ for all $G \in \mathcal{H}_2$ with $i \neq j$. Clearly, the Intersection Property is satisfied in this case.

Now suppose that $\mathcal{H}_2$ is also empty, i.e. $\mathcal{H} = \mathcal{H}_0$. Since $(X, \mathcal{H})$ is not totally blocked, $\mathcal{H}$ is partitioned in at least two equivalence classes with respect to the equivalence relation $\equiv$. Since, obviously, no critical family can meet two different equivalence classes, we can specify different dictators on different equivalence classes while satisfying the Intersection Property.

Finally, if $\mathcal{H}_0$ is empty, $(X, \mathcal{H})$ is a quasi-median space by Proposition 3.1, hence the existence of non-dictatorial monotone Arrowian aggregators follows as in the proof of Proposition 3.2 below.

Proof of Theorem 2 Obviously, (ii) implies (i). Thus, it suffices to show that (i) implies (iii), and that (iii) implies (ii).

"(i) $\Rightarrow$ (iii)" We prove the claim by contraposition. Assume that $(X, \mathcal{H})$ is not quasi-unblocked. This means that there exists $G \in \mathcal{H}$ with $G \equiv G^c$ and some critical family $\mathcal{G}$ such that $(\mathcal{H}_{\equiv G} \cap \mathcal{G}) \supseteq \{H, H', H''\}$ for three distinct H, H', H''. Consider a structure of winning coalitions satisfying the Intersection Property. By Lemma 1, $\mathcal{W}_H = \mathcal{W}_G$ for all $H \in \mathcal{H}_{\equiv G}$. By Lemma 2, applied to the critical family $\mathcal{G} \supseteq \{H, H', H''\}$, there exists i, such that $\{i\} \in \mathcal{W}_H$ for all $H \in \mathcal{H}_{\equiv G}$. Hence, i is a dictator on $\mathcal{H}_{\equiv G}$, which proves the claim.

"(iii) $\Rightarrow$ (ii)" We will construct an anonymous Arrowian aggregator by specifying an appropriate structure of winning coalitions, provided that $(X, \mathcal{H})$ is quasi-unblocked. Partition $\mathcal{H}$ as above, i.e.

$$\begin{aligned}
\mathcal{H}_0 &:= \{H \in \mathcal{H} : H \equiv H^c\}, \\
\mathcal{H}_1^+ &:= \{H \in \mathcal{H} : H > H^c\}, \\
\mathcal{H}_1^- &:= \{H \in \mathcal{H} : H^c > H\}, \\
\mathcal{H}_2 &:= \{H \in \mathcal{H} : \text{ neither } H \geq H^c \text{ nor } H^c \geq H\}.
\end{aligned}$$

Furthermore, partition $\mathcal{H}_2$ according to Lemma 2c) into $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$. Let n be odd, and set

$$\begin{array}{lll}
\mathcal{W}_H = \{W : \#W > n/2\} & \text{if} & H \in \mathcal{H}_0, \\
\mathcal{W}_H = 2^N \setminus \{\emptyset\} & \text{if} & H \in \mathcal{H}_1^- \cup \mathcal{H}_2^-, \\
\mathcal{W}_H = \{N\} & \text{if} & H \in \mathcal{H}_1^+ \cup \mathcal{H}_2^+.
\end{array}$$

Clearly, this structure of winning coalitions is anonymous; we will show that it satisfies the Intersection Property. Let $\mathcal{G}$ be a critical family; we distinguish three cases.

Case 1: $\mathcal{G} \cap (\mathcal{H}_1^- \cup \mathcal{H}_2^-) \neq \emptyset$. If $G \in \mathcal{G} \cap \mathcal{H}_1^-$, then by Lemma 3a), $\mathcal{G} \setminus \{G\} \subseteq \mathcal{H}_1^+$, and the Intersection Property is clearly satisfied. Thus, suppose that there exists $H \in \mathcal{G} \cap \mathcal{H}_2^-$. By Lemma 3b), we must have $\mathcal{G} \cap \mathcal{H}_0 = \emptyset$, and by Lemma 3a), $\mathcal{G} \cap \mathcal{H}_1^- = \emptyset$.

Hence, if there exists $H' \in \mathcal{G} \setminus \{H\}$ with $\mathcal{W}_{H'} \neq \{N\}$, we must have $H' \in \mathcal{H}_2^-$. But then $H \geq (H')^c$ contradicts the construction of $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ in Lemma 3c). Thus, if $H \in \mathcal{G} \cap \mathcal{H}_2^-$, one has $\mathcal{W}_{H'} = \{N\}$ for any other element $H' \in \mathcal{G}$, in which case the Intersection Property is satisfied.

Case 2: $\mathcal{G} \cap \mathcal{H}_0 \neq \emptyset$. First, observe that $G_1 \equiv G_2$ whenever $\{G_1, G_2\} \subseteq \mathcal{G} \cap \mathcal{H}_0$. Indeed, $G_1 \equiv G_2$ follows at once from $G_1 \geq G_2^c$, $G_2 \geq G_1^c$, $G_1 \equiv G_1^c$ and $G_2 \equiv G_2^c$. Thus, by quasi-unblockedness, $\mathcal{G}$ can contain at most two elements of $\mathcal{H}_0$. By Lemma 3b), for any $H \in \mathcal{G} \setminus \mathcal{H}_0$ one has $\mathcal{W}_H = \{N\}$. Hence, the Intersection Property is also satisfied in Case 2.

Case 3: If $\mathcal{G}$ does not meet $\mathcal{H}_0$, $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$, then $\mathcal{G} \subseteq (\mathcal{H}_1^+ \cup \mathcal{H}_2^+)$, in which case the Intersection Property is trivially satisfied. This completes the proof of Theorem 2.

Proof of Proposition 3.1 Suppose that for all $H \in \mathcal{H}$, $H \not\equiv H^c$. Partition $\mathcal{H}$ into $\mathcal{H}_1^-$, $\mathcal{H}_1^+$, $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ as above, where $\mathcal{H}_2^-$ and $\mathcal{H}_2^+$ are determined according to Lemma 3c). Then, any critical family $\mathcal{G}$ can meet $\mathcal{H}_1^- \cup \mathcal{H}_2^-$ at most once. Indeed, by Lemma 3a), $H \in \mathcal{G} \cap \mathcal{H}_1^-$ implies $\mathcal{G} \setminus \{H\} \subseteq \mathcal{H}_1^+$. Furthermore, if $\{H, H'\} \subseteq \mathcal{G} \cap \mathcal{H}_2^-$, one would obtain $H' \geq H^c$ which contradicts the construction of $\mathcal{H}_2^-$. But this implies that $\cap(\mathcal{H}_1^- \cup \mathcal{H}_2^-)$ is non-empty (otherwise it would contain a critical family), and by H3, it consists of a single element, say $\hat{x}$. By definition, $x \in M(X)$.

Conversely, let $\hat{x} \in M(X)$, and consider any $H \in \mathcal{H}_{\hat{x}}$. Then, $H \geq^0 G$ implies $G \in \mathcal{H}_{\hat{x}}$. Indeed, by definition, $H \geq^0 G$ means that $\{H, G^c\} \subseteq \mathcal{G}$ for some critical family $\mathcal{G}$. Since $\hat{x} \in M(X)$, $\mathcal{G}$ contains at most one element of $\mathcal{H}_x$, hence $G^c \notin \mathcal{H}_x$, which implies $G \in \mathcal{H}_x$. This observation immediately implies $H \not\equiv H^c$.

Proof of Proposition 3.2 Let $f_{\hat{x}}$ be an Arrowian unanimity rule and consider the set $\mathcal{H}_{\hat{x}}$ of all properties possessed by $\hat{x}$. As is easily verified, $f_{\hat{x}}$ corresponds to voting by issues with $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_{\hat{x}}$ and $\mathcal{W}_H = \{N\}$ for all $H \notin \mathcal{H}_{\hat{x}}$. Suppose that there exists a critical family $\mathcal{G}$ and two distinct H, H' with $H, H' \in \mathcal{H}_{\hat{x}} \cap \mathcal{G}$; then one can choose $W \in \mathcal{W}_H$ and $W' \in \mathcal{W}_{H'}$ with $W \cap W' = \emptyset$, violating the Intersection Property. Thus, $\#(\mathcal{H}_{\hat{x}} \cap \mathcal{G}) \leq 1$ for every critical family $\mathcal{G}$, i.e. $\hat{x} \in M(X)$.

Conversely, it is immediate from the Intersection Property that for any median point $\hat{x} \in M(X)$ voting by issues with $\mathcal{W}_H = 2^N \setminus \{\emptyset\}$ for all $H \in \mathcal{H}_{\hat{x}}$ and $\mathcal{W}_H = \{N\}$ for all $H \notin \mathcal{H}_{\hat{x}}$ is well-defined (and coincides with the Arrowian unanimity rule $f_{\hat{x}}$).

Proof of Theorem 3 The equivalences "(iv) $\Leftrightarrow$ (v)" and "(iii) $\Leftrightarrow$ (v)" follow at once from Propositions 3.1 and 3.2, respectively. The implications "(iii) $\Rightarrow$ (ii)" and "(ii) $\Rightarrow$ (i)" are evident. Thus, the proof is completed by verifying the implication "(i) $\Rightarrow$ (iv)." This is done by contraposition. Thus, assume that H is blocked, i.e. $H \equiv H^c$. By Lemma 1 this implies $\mathcal{W}_H = \mathcal{W}_{H^c}$ for any structure of winning coalitions satisfying the Intersection Property. Under anonymity, this implies, using (B.1), $\mathcal{W}_H = \mathcal{W}_{H^c} = \{W \subseteq N : \#W > 1/2\}$, which is compatible with (B.2) only if the number of voters is odd.

Proof of Theorem 4 a) By Proposition 3.2, any quasi-median space admits at least one Arrowian unanimity rule, and any such rule is neutral across issues and non-dictatorial.

Conversely, let $f : X^n \to X$ be voting by issues satisfying the Intersection Property. We show by contraposition that if f is non-dictatorial and neutral across issues, then $(X, \mathcal{H})$ must be a quasi-median space. Thus, suppose that $(X, \mathcal{H})$ is not a quasi-median space. By Proposition 3.1, there exists a basic property H that is blocked, i.e. $H \equiv H^c$.

By Lemma 1, this implies $\mathcal{W}_H = \mathcal{W}_{H^c}$, hence f is fully neutral, i.e. $\mathcal{W}_H = \mathcal{W}_0$ for all H and some fixed $\mathcal{W}_0$. Since $(X, \mathcal{H})$ is not a median space, there exists a critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$. By Lemma 2, $\{i\} \in \mathcal{W}_{G_3^c} = \mathcal{W}_0$, i.e. voter i is a dictator.

b) Median spaces are characterized by the property that all critical families have cardinality two. Using the Intersection Property this implies that, e.g., issue-by-issue majority voting with an odd number of agents is consistent on any median space, and evidently, issue-by-issue majority voting with an odd number of agents is neutral, in particular neutral within issues.

Conversely, let $f : X^n \to X$ be voting by issues satisfying the Intersection Property. We show by contraposition that if f is locally non-dictatorial and neutral within issues, then $(X, \mathcal{H})$ must be a median space. Thus, suppose that $(X, \mathcal{H})$ is not a median space. Then there exists a critical family $\mathcal{G}$ with at least three elements, say $\mathcal{G} \supseteq \{G_1, G_2, G_3\}$, in particular, $G_j \geq G_k^c$ for distinct $j, k \in \{1, 2, 3\}$. By Lemma 1, $\mathcal{W}_{G_j} \subseteq \mathcal{W}_{G_k^c}$ for distinct $j, k \in \{1, 2, 3\}$. Under neutrality within issues this implies at once that $\mathcal{W}$ assigns identical committees to G_1, G_2, G_3 and their respective complements. By Lemma 2 above, $\{i\} \in \mathcal{W}_{G_3^c}$, i.e. voter i is a local dictator.

c) As in part b), an underlying median space guarantees the existence of a fully neutral aggregator. The converse follows from part b) together with the observation that, under full neutrality, a local dictator must even be a global dictator.

Proof of Proposition 3.3 Suppose that $(X, \mathcal{H})$ is indecomposable, and consider any fixed $H \in \mathcal{H}$. Then, for any $G \in \mathcal{H}$, at least one of the following holds, $G \geq H$, $G \geq H^c$, $G^c \geq H$, or $G^c \geq H^c$. Indeed, otherwise the subfamilies $\mathcal{H}_1 := \{G \in \mathcal{H} : G \geq H, G \geq H^c, G^c \geq H, \text{ or } G^c \geq H^c\}$ and $\mathcal{H}_2 := \mathcal{H} \setminus \mathcal{H}_1$ form a decomposition, as is easily verified. The claim follows immediately from this observation using Lemma 1.

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