Oligarchies in Judgment Aggregation

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Abstract

In the general framework of abstract binary Arrowian aggregation introduced by Wilson (1975), we characterize the aggregation problems in which the only Arrowian aggregators are oligarchic.

In a range of different settings, social choice theory has produced a large number of impossibility results driven by Arrow-like independence conditions. Frequently, these impossibilities take the form of a dictatorship, as in Arrow's classical contribution. Early papers by Wilson (1975) and Rubinstein and Fishburn (1986) attempted to unify the literature by formulating the Arrowian aggregation problem in an abstract setting and providing sufficient combinatorial conditions that yielded a version of Arrow's original result as a special case. Under a natural monotonicity assumption, the first characterization in this framework was obtain by Nehring and Puppe (2002,2005) (stated there in a property-space framework in the context of an analysis of strategy-proof social choice); by dropping this monotonicity assumption, this result was subsequently generalized by Dokow and Holzman (2005).

Dictatorial impossibilities cover only a fraction of the important (near-)impossibility results in the literature. Frequently, the structure of the aggregated space immediately gives rise to minimal possibilities in the form of a unanimity or, more generally, an oligarchic rule. For example, in the aggregation of partial rather than weak orders, one can consistently aggregate individual preferences by asserting a preference between two alternatives at the social level if and only if a fixed group of agents (the "oligarchy") asserts this preference. In this case, the interesting possibility question is whether there exist Arrowian aggregation rules different from these obvious, yet rather degenerate ones. Quite often, as in the case of partial orders, see Gibbard (1969), the answer is negative, leading to an oligarchic impossiblity result. Other instances occur in the aggregation of equivalence relations (Mirkin 1975, Fishburn-Rubinstein 1986) and in judgment aggregation (Nehring-Puppe 2005b, Dietrich-List 2006, Dokow-Holzman 2006).

In this note, we offer a general characterization when such oligarchic impossibilities arise, providing a counterpart to the characterization of dictatorial impossibilities in

Nehring and Puppe (2002,2005).

As in that paper, we adopt a property space formulation.

Definition 1 *A **property space** is a pair $(X, \mathcal{H})$, where X is a finite set of "valuations", and $\mathcal{H}$ is a collection of subsets of X satisfying*

H1) $H \neq \varnothing$ *for all* $H \in \mathcal{H}$;

H2) $H \in \mathcal{H}$ *implies* $H^c \in \mathcal{H}$;

H3) *for all* $x \neq y$ *there exists* $H \in \mathcal{H}$ *such that* $x \in H$ *and* $y \notin H$.

Property spaces can be identified with subsets of hypercubes $Z \subseteq \{0,1\}^K$; in particular, the pair $(Z, \mathcal{H})$ defines a property space, with $\mathcal{H}$ given by the family of all sets of the form $\{z\} \times \{0,1\}^{K \setminus k}$, for $k \in K$ and $z \in \{0,1\}$. Satisfaction of the conditions H1 through H3 is easily verified.

Let $N = \{1, ..., n\}$ be a finite set of agents, with $n > 1$. An aggregator f is a mapping from X^N to X.

Definition 2 *The aggregator $f : X^N \to X$ is **Arrowian** if it satisfies*

(Unanimity) $f(x, ..., x) = x$ *for all* $x \in X$;

(Independence) *If* $f(x_1, ..., x_n) \in H$ *and, for all* $i \in N$, $x_i \in H$ *iff* $y_i \in H$, *then* $f(y_1, ..., y_n) \in H$;

(Monotonicity) *If* $f(x_1, ..., x_n) \in H$, $y \in H$ *and* $j \in J$, *then* $f(y_1, ..., y_n) \in H$, *where* $y_i = x_i$ *if* $i \neq j$ *and* $y_j = y$.

Note that the conjunction of Independence and Monotonicity is equivalent to the following condition of Monotone Independence:

> **(Monotone Independence)** If $f(x_1, ..., x_n) \in H$ and, for all $i \in N$, $y_i \in H$ whenever $x_i \in H$, then $f(y_1, ..., y_n) \in H$.

An aggregator f is **oligarchic** if there exists $x^* \in X$ (the default) and $J \subseteq N$ (the oligarchy) such that, $f(x_1, ..., x_n) = x^*$ whenever $x^* = x_i$ for some $i \in J$. Clearly, for any x^*, J, there exists at most one oligarchic aggregator f that is Arrowian; it is given by the condition

$$\text{for all } H \in \mathcal{H} \text{ with } x^* \in H : f(x_1, ..., x_n) \in H \text{ iff } x_i \in H \text{ for some } i \in J; \qquad (1)$$

if an aggregator satisfying (1) exists, it will be referred to as $f_{x^*,J}$. An aggregator f is **dictatorial** if it is oligarchic with singleton oligarchy $J = \{i\}$.

The property space $(X, \mathcal{H})$ is **oligarchic** (respectectively **dictatorial**) if all its Arrowian aggregators are oligarchic (resp. dictatorial). It is **properly oligarchic** if it is oligarchic but not dictatorial. The purpose of this note is the characterization of oligarchic and properly oligarchic property spaces.

To do so, we will rely on the combinatorial approach developed in Nehring-Puppe (2006) and Nehring-Puppe (2005), henceforth NP1 and NP2, respectively; we refer the reader to these papers for more detailed explanations and illustrations of the various concepts.

A *critical family* is a minimal subset $\mathcal{G} = \{H_1, ..., H_m\} \subseteq \mathcal{H}$ with empty intersection. Let Γ denote the set of all critical families in $(X, \mathcal{H})$. Say that $H \geq_0 G$ ("G conditionally entails H") iff there exists a critical family $\mathcal{G}$ containing H and G^c; let $\geq$ denote the transitive closure of $\geq_0$, with symmetric component $\equiv$ and asymmetric component $>$. The property space $(X, \mathcal{H})$ is **semi-blocked** if, for all $G, H \in \mathcal{H}$, $G \equiv H$ or $G \equiv H^c$. The following is the main result of this note.

Theorem 3 *The following two statements are equivalent:*

1. $(X, \mathcal{H})$ *is oligarchic.*
2. $(X, \mathcal{H})$ *is semi-blocked.*

For comparison and further reference, we cite two adaptations of central central results of earlier work. These require three additional definitions.

The property space $(X, \mathcal{H})$ is **totally blocked** if, for all $G, H \in \mathcal{H}$, $G \geq H$. It is **unblocked** if, for no $H \in \mathcal{H}$, $H \equiv H^c$.

For any $x \in X$, let $\mathcal{H}_x = \{H : H \ni x\}$. Say that x is a **median point** if it meets every critical family at most once, that is, if $\#(\mathcal{G} \cap \mathcal{H}_x) \leq 1$ for all $\mathcal{G} \in \Gamma$. In NP2, a geometric definition of median points is given and shown to be equivalent to the present one (Lemma 5 of NP2). Property spaces with a non-empty set of median points are called **quasi-median spaces.**

Theorems 1 and 3 of NP2 translate into the following results on Arrowian aggregators.

Theorem 4 *The following two statements are equivalent:*

1. $(X, \mathcal{H})$ *is dictatorial.*
2. $(X, \mathcal{H})$ *is totally blocked.*

Theorem 5 *The following three statements are equivalent:*

1. $(X, \mathcal{H})$ *admits non-dicatorial oligarchic Arrowian aggregators.*
2. $(X, \mathcal{H})$ *is unblocked.*
3. $(X, \mathcal{H})$ *is a quasi-median space.*

The link between oligarchic Arrowian aggregators and median points implicit in the equivalence between (2) and (3) is fleshed out by the following proposition (c.f. Proposition 5.1 in NP2).

Proposition 6 *The following three statements are equivalent for an element x of a property space $(X, \mathcal{H})$:*

1. *x is a median point;*
2. *there exist a non-singleton J such that the oligarchic rule $f_{x,J}$ is well-defined.*
3. *for all $J \subseteq N$, the oligarchic rule $f_{x,J}$ is well-defined.*

Proof of Theorem 3.

We will make frequent use of the representation of Arrowian aggregators as consistent "voting by properties" with voting structure $\mathcal{W}$. A *structure (of winning coalitions)* $\mathcal{W}$ assigns to every property $H \in \mathcal{H}$ a family of non-empty subsets of agents ("winning coalitions") $\mathcal{W} = (\mathcal{W}_H)_{H\in\mathcal{H}}$ such that $W \in \mathcal{W}_H$ iff $W \notin \mathcal{W}_{H^c}$.

The following proposition follows from adapting results in NP1 (Propositions 3.1 and 3.4).

Proposition 7 *The following two statements are equivalent:*

1. *The aggregator f is Arrowian.*
2. *There exists a structure of winning coalitions $\mathcal{W}$ satisfying*
   (a) *for all $H \in \mathcal{H}$, i) $N \in \mathcal{W}_H$ and ii) $W \in \mathcal{W}_H$ and $W' \supseteq W$ imply $W' \in \mathcal{W}_H$,*
   (b) ***(Intersection Property)*** *for all critical families $\mathcal{G}$ and all selections $\{W_G\}_{G\in\mathcal{G}}$ such that $W_G \in \mathcal{W}_G$, $\bigcap_{G\in\mathcal{G}} W_G \neq \varnothing$,*
   (c) *for all $H \in \mathcal{H}$, $f(x_1, ..., x_n) \in H$ iff $\{i \in N : x_i \in H\} \in \mathcal{W}_H$.*

It is easily seen that, given an Arrowian aggregator f, the associated $\mathcal{W}$ is unique, and that, conversely, given structure of winning coalitions $\mathcal{W}$ satisfying a) and b), c) defines a unique Arrowian aggregator $f_{\mathcal{W}}$.

The Intersection Property implies the following two basic facts (cf. NP1, (3.2), and NP2, Fact 4.1, respectively).

Fact 8 *$W \in \mathcal{W}_H$ if and only if $W \cap W' \neq \varnothing$ for all $W \in \mathcal{W}_{H^c}$.*

Fact 9 *$G \geq H$ implies $\mathcal{W}_G \subseteq \mathcal{W}_H$.*

(2) implies (1).

Suppose that $(X, \mathcal{H})$ is semi-blocked. Then $\mathcal{H}$ can be bi-partitioned into complementary families $\mathcal{H}^+$ and $\mathcal{H}^-$ such that for all $G, H \in \mathcal{H}^+$, $G \equiv H$, and likewise, for all $G, H \in \mathcal{H}^-$, $G \equiv H$. Evidently, $(X, \mathcal{H})$ must contain at least one critical family of cardinality greater than two $\mathcal{G} = \{H_1, H_2, H_3, ...\}$. W.l.o.g. $H_1, H_2 \in \mathcal{H}^+ \geq 2$. Hence $H_1 \geq H_2^c$. By semi-blockedness therefore

$$H \geq H' \text{ for all } H \in \mathcal{H}^+ \text{ and } H' \in \mathcal{H}^-. \tag{2}$$

In view of Theorem 4, we are done if $(X, \mathcal{H})$ is totally blocked. Thus suppose that $(X, \mathcal{H})$ is not totally blocked. Then by (2) and semi-blockedness, for all $\mathcal{G} \in \Gamma$,

$$\#\left(\mathcal{G} \cap \mathcal{H}^-\right) \leq 1. \tag{3}$$

We now claim that there exists at least one critical family $\mathcal{G}$ such that $\#\mathcal{G} \geq 3$ and $\mathcal{G} \cap \mathcal{H}^- \neq \varnothing$. Indeed, if this was false, in view of (3), all comparisons of the form $H_1 \geq_0 H_2$ for $H_1, H_2 \in \mathcal{H}^-$ would be based on critical families of the form $\{H_1, H_2^c\}$, i.e. of inclusions $H_1 \subseteq H_2$. By semi-blockedenss, this implies that in fact $H_1 = H_2$ for all $H_1, H_2 \in \mathcal{H}^-$, an evident impossibility.

Take a critical family of the form just asserted, $\mathcal{G} = \{G_1, G_2, G_3, ...\}$ with $G_1, G_2 \in \mathcal{H}^+$ and $G_3 \in \mathcal{H}^-$. Consider any Arrowian aggregator f with committee structure $\mathcal{W}$. By semi-blockedness and Fact 9,

$$\mathcal{W}_{G_1} = \mathcal{W}_{G_2} \text{ and } \mathcal{W}_{G_1^c} = \mathcal{W}_{G_3}. \tag{4}$$

Take any $W, W' \in \mathcal{W}_{G_1}$. By the Intersection Property and (4), for all $W'' \in \mathcal{W}_{G_1^c}$,

$$(W \cap W') \cap W'' \neq \varnothing. \tag{5}$$

By Fact 8, it follows that $W \cap W' \in \mathcal{W}_{G_1}$.

Thus $\mathcal{W}_{G_1}$ is closed under intersection; it therefore contains a single minimal coalition J; by semi-blockedness, this implies at once that f is oligarchic with oligarchy J.

(1) implies (2).

Suppose $(X, \mathcal{H})$ is oligarchic but not semi-blocked. Since a fortiori $(X, \mathcal{H})$ cannot be totally blocked, by Theorem 2 it must be properly oligarchic. By Theorem 3, $(X, \mathcal{H})$ must therefore contain a median point x^*. Let $\mathcal{H}^- = \mathcal{H}_{x^*}$ and let $\mathcal{H}^+ = \{H : H^c \in \mathcal{H}_{x^*}\}$.

By transitivity of $\geq$, there exists $H_1 \in \mathcal{H}^+$ such that, for all $G \in \mathcal{H}^+$, $G \geq H_1$ implies $H_1 \equiv G$. Let $\mathcal{H}_1^+ := \{H' \in \mathcal{H}^+ : H' \equiv H_1\}$ and $\mathcal{H}_2^+ = \mathcal{H}^+ \backslash \mathcal{H}_1$, $\mathcal{H}_1^-$ and $\mathcal{H}_2^-$ denote the associated families of complements. $\mathcal{H}_2^+$ is non-empty since $(X, \mathcal{H})$ is not semi-blocked by assumption.

Define a non-oligarchic choice rule f in terms of the following committee structure $\mathcal{W}$ for some arbitrary $J \subseteq N$ with $J \neq N$.

$$\mathcal{W}_H := \begin{cases} \{N\} & \text{if } H \in \mathcal{H}_1^+; \\ \{W : W \supseteq J\} & \text{if } H \in \mathcal{H}_2^+; \\ \{W : W \neq \varnothing\} & \text{if } H \in \mathcal{H}_1^-; \\ \{W : W \cap J \neq \varnothing\} & \text{if } H \in \mathcal{H}_2^-. \end{cases}$$

We need to verify that $\mathcal{W}$ is consistent, i.e. that it satisfies the Intersection Property.

First of all, since x^* is a quasi-median point, for all critical families $\mathcal{G}$,

$$\#\mathcal{G} \cap \mathcal{H}^- \leq 1.$$

If $\mathcal{W}$ is inconsistent, i.e. if it violates the Intersection Property, there must exist a critical family $\mathcal{G}$ containing $G_1 \in \mathcal{H}_1^-$ and $G_2 \in \mathcal{H}_2^+$. But this means that $G_2 \geq_0 G_1^c$, and thus by the definition of $\mathcal{H}_1^+$, $G_2 \in \mathcal{H}_1^+$, contradicting the assumption that $G_2 \in \mathcal{H}_2^+$. ■

Theorem 3 has the following corollary.

Corollary 10 *The following three statements are equivalent:*

i) $(X, \mathcal{H})$ *is properly oligarchic;*

ii) $(X, \mathcal{H})$ *is semi-blocked but not totally blocked;*

iii) $(X, \mathcal{H})$ *is semi-blocked but unblocked.*

The equivalence of i) and ii) follows immediately from Theorems 3 and 4, that of i) and iii) from Theorems 3 and 5. Finally, the non-trivial implication "ii) implies iii)" follows from (2) in the proof of Theorem 3. □

This step in the proof also implies that if $(X, \mathcal{H})$ is oligarchic, all non-dictatorial oligarchic Arrowian aggregators must share the same default.

Corollary 11 *If* $f_{x,J}$ *and* $f_{x',J'}$ *are non-dictatorial oligarchic Arrowian aggregators for the oligarchic space* $(X, \mathcal{H})$*, then* $x = x'$*. In particular, any properly oligarchic space* $(X, \mathcal{H})$ *contains exactly one median point.*

Indeed, it is a straightforward consequence of Fact 9 that if $f_{x,J}$ is an oligarchic Arrowian aggregator for $(X, \mathcal{H})$ and if $x \in H$ with $H \geq H^c$, $f_{x,J}$ must be dictatorial. By modus tollens, in view of (2), if $f_{x,J}$ is non-dictatorial, $x \in H$ for all $H \in \mathcal{H}^-$. By H3, this uniquely pins down the default x. By Proposition 6, this also implies that $(X, \mathcal{H})$ contains exactly one median point. □

In view of Corollary 11, one might conjecture that the properly oligarchic spaces are those with exactly one median point. But this conjecture turns out to be false. A trivial counterexample is the cartesian product of two property spaces with unique median point. Then the product space has a unique median point but is not oligarchic since the properties of the two component spaces can be determined by different oligarchies. But the conjecture turns out to be false even when product spaces are excluded. A counterexample in the 13-dimensional hypercube (i.e. $\#\mathcal{H}$=26) can be given; while this is most likely not the simplest possible, it seems probable that all counterexamples will be fairly contrived, and that in applications, the existence of a unique median point will prove to be effectively necessary and sufficient for proper oligarchicity.

Theorem 3 can be viewed as an engine for churning out oligarchic impossibility results. If the property space is in fact oligarchic, this will typically be easy, especially since one does not need to characterize the entire set of critical families but only a sufficiently large subset. As an example, consider the classical problem of aggregating partitions (see Mirkin (1975) and Fishburn-Rubinstein (1986)). Let Z be a finite set of elements, and $X = \mathfrak{F}(Z)$ the set of all partitions on Z, i.e. all non-empty families $\mathcal{F}$ of pairwise disjoint subsets of Z whose union is Z. Z is naturally made into a property space by characterizing partitions in terms of the joint membership of the elements of Z. Formally, let $\mathcal{H}^+ = \{H_{z,z'}\}_{z\neq z'\in Z}$, with $\mathcal{F} \in H_{z,z'}$ if there exists $S \in \mathcal{F}$ such that $\{z, z'\} \subseteq S$, and let $\mathcal{H} = \mathcal{H}^+ \cup \{H^c : H \in \mathcal{H}^+\}$. This move can viewed as reducing the problem of aggregating partitions to one of aggregating equivalence relations.

Proposition 12 ***(Mirkin 1975)*** $(\mathfrak{F}(Z), \mathcal{H})$ *is properly oligarchic.*

For verification, it is easily seen that the critical families are all families $\mathcal{G}$ of the form $\mathcal{G} = \{H_{z^1,z^2}, H_{z^2,z^3}, .., H_{z^{k-1},z^k}, H^c_{z^1,z^k}\}$, reflecting the transitivity of the "joint membership" relation. It follows immediately that the partition $\mathcal{F}^* = \{\{z\}\}_{z\in Z}$ is the unique median point in $(\mathfrak{F}(Z), \mathcal{H})$, as $\mathcal{F}^*$ is characterized by the property that, for all $H \ni \mathcal{F}^*$, $H \notin \mathcal{H}^+$. Thus $(\mathfrak{F}(Z), \mathcal{H})$ admits non-dictatorial, oligarchic Arrowian aggregators.

To see that $(\mathfrak{F}(Z), \mathcal{H})$ is semi-blocked, hence oligarchic, note that $H \geq_0 H'$ for all $H, H' \in \mathcal{H}^+$, which by the definition of $\geq_0$ also implies $H \geq_0 H'$ for all $H, H' \in$

$\mathcal{H}\backslash\mathcal{H}^+$. In particular, for $H = H_{z^1,z^2}$ and $H' = H_{z^3,z^4}$ with $\{z^1, z^2\} \cap \{z^3, z^4\} = \varnothing$, $H \geq_0 H'$ follows from noting that $\{H_{z^4,z^1}, H_{z^1,z^2}, H_{z^2,z^3}, H^c_{z^3,z^4}\}$ is a critical family. □

Most of the results on oligarchies in the literature obtain proper oligarchies as in the case of Gibbard's (1969) and Mirkin's (1975) classical results. A partial exception is the characterization of consistent Arrowian aggregators in problems of truth-functional judgment aggregation in Nehring-Puppe (2005b) which are shown to be oligarchic in general. In a second step, truth-functional judgment aggregation problems are then classified as properly oligarchic or dictatorial.

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