Asset Pricing with Unforeseen Contingencies

Alan Kraus Jacob S. Sagi*

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Abstract

Data suggest that macro asset pricing models based solely on aggregate consumption perform poorly. Yet, there is no consensus over what alternative primitive pricing factors should accompany or take the place of aggregate consumption. We assume an economy where agents are like economists: they know that non-derivative factors other than aggregate consumption have welfare implication, but they can’t specifically and explicitly state what these factors are. Instead, agents attribute the additional source of uncertainty to ‘unforeseen contingencies’.

We investigate an economy of heterogeneous agents that cannot specify all exogenous welfare-relevant events, and provide a parametric model in which aggregating unforeseen contingencies across agents leads to non-consumption pricing factors. To fit the stylized facts, (i) non-consumption factors must dominate the pricing kernel and are closely related to the wealth-consumption ratio, (ii) markets are incomplete, (iii) agents’ preferences with respect to unforeseen contingencies are non-expected utility, and (iv) although non-consumption pricing factors can be conditionally uncorrelated with aggregate consumption shocks, they must be correlated with shocks to expected consumption growth.

JEL Classification: D51, D52, D58, G12, G13

*Address all correspondence to Jacb S. Sagi, Haas School of Business, University of California at Berkeley, 545 Student Services Building, Berkeley, California, 94720-1900. email: sagi@haas.berkeley.edu, phone: (510)-642-3442
1 Introduction

The macroeconomic asset-pricing literature is motivated by the failure of neo-classical models to describe empirical moments of macroeconomic variables, namely: the equity premium puzzle of Mehra and Prescott (1985), the volatility puzzle (Campbell (2000)), and the risk-free rate puzzle of Weil (1989). Dissonance between theory and data is partly due to the necessary compromises inherent in model selection (in the simplest case, a representative agent who is indifferent to the timing of resolution of uncertainty), and partly due to the fact that consumption is much smoother than stock prices and performs poorly as a cross-sectional pricing factor. Recent work on habit formation (Campbell and Cochrane (1999, 2000)) seems promising in helping to establish a consistent story based on aggregate consumption only. In the end however, it seems that all theoretical models based solely on aggregate consumption have to somehow ratchet the aggregate absolute risk aversion to levels that are not ‘esthetically’ pleasing while they continue to perform poorly in cross-sectional tests. On the other hand, there is recent empirical work by Lettau and Ludvigson (2001ab, 2002) that indicates that the wealth-consumption ratio plays an important role in the pricing of risk. Overall, there seems to be a consensus (see the review by Kocherlakota (1996)) that an asset pricing model where the fundamental shocks are those corresponding to aggregate consumption cannot fully capture the litany of stylized macroeconomic facts.

The aim of this paper is to motivate the addition of non-consumption based factors to the standard consumption based general equilibrium asset pricing model. The main difficulty is simple enough to state: if agents care about fundamental\textsuperscript{3} systematic economic factors other than consumption or derivatives of consumption\textsuperscript{4}, then what are these factors and how can one identify them? The answer is not obvious and we are no better than our colleagues in identifying additional primitive or non-derived factors directly;\textsuperscript{5} instead, we attempt to

\textsuperscript{1}A list of references includes, but is not limited to Abel(1990), Constantinides (1990), Epstein and Zin (1989), and Sudaresan (1989). A frequently missed early paper on consumption based asset pricing is Rubinstein (1976).

\textsuperscript{2}Although the aggregate of agents need not please anyone esthetically, one should also keep in mind that it is risk tolerance and not risk aversion that is aggregated in equilibrium (Wilson (1968)). In other words, one theoretically expects the economy’s risk attitudes to be dominated by those who are least risk averse. For example, in a frictionless economy, the presence of a single classically risk-neutral agent is enough to drive the aggregate risk aversion of the entire economy to zero.

\textsuperscript{3}In referring to fundamental factors, we mean primitives of choice. This includes consumption but excludes derived quantities such as wealth, interest rates and prices.

\textsuperscript{4}Derivatives of consumption include factors that depend on consumption history (as in habit formation) or any other functions of past and future (e.g., forecasts of) consumption.

\textsuperscript{5}Candidates may include higher moments of income distribution (status), the changes in the constituents of the
shed light on the issue indirectly by assuming that agents in the economy themselves realize that their future welfare depends on things other than consumption, but they cannot fully characterize the additional variables in terms of fundamentals or non-derived quantities. Such agents know that something can and will affect them in the future but cannot specify what it will be in terms of ethically neutral Savage-type events. In the decision theory literature, these events are said to be ‘unforeseen’ since they cannot be exogenously specified within a standard information filtration.

To model our agents’ preferences we make use of recent work in axiomatic decision theory under unforeseen contingencies (see Kreps (1979, 1992), Dekel, Lipman and Rustichini (2001), and Kraus and Sagi (2002)). A general conclusion of this literature is that an agent making rational decisions in the face of unforeseen contingencies will act as if she experiences private tastes shocks. In other words, since agents cannot point to specific external events that affect their welfare, they instead postulate ‘utility states’. Moreover, these agents will not generally obey the standard expected utility axioms with respect to their private states. In this paper, we consider the aggregate behavior of agents along the lines suggested by this literature. Within our parametric framework the derived price of risk is determined by the dynamics of three factors: per capita aggregate consumption, a measure that aggregates the ‘utility states’ across agents, and a variable that jointly measures the presence of market incompleteness plus the departure of preferences from expected utility theory. At first blush, the two ‘new’ variables are something of an ‘expected’ disappointment since they are empirically unobserved. The news is not entirely bad, however, as we can show that, under various conditions, changes in these variables are directly related to changes in the wealth-consumption ratio; the latter, in principle, is observable. Moreover, to fit the stylized asset-pricing facts, the ‘new’ variables cannot be completely unrelated to aggregate consumption: they must be contemporaneously correlated with shocks to expected consumption growth. Overall, our work suggests that non-consumption based exogenous factors that impact aggregate preferences, not surprisingly, must be sensitive to the business cycle, and can be proxied by the wealth-consumption ratio and some term structure variable.

Our economy is populated by heterogeneous agents who experience private taste shocks. In aggregate, these can impact prices, and since we allow agents to trade any Arrow-Debreu security on consumption or price related event, some component of taste shocks can be hedged. Part of our innovation is to provide a framework in which an equilibrium endo-commodity bundle comprising ‘aggregate consumption’, non-tradeable endowment shocks, etc. Moreover, it may very well be the case that it is not any particular mix of such candidate variables that comprises the ‘missing factor’, but that different variables contribute to asset pricing at different times.
nously determines both the events that are observable and the consequent set of tradeable assets. After aggregating over agents’ consumption/investment decisions, the three-factor equilibrium briefly described above and nesting the Lucas (1978) model obtains. If unforeseen contingencies enter agents’ preferences in a manner different from ‘objective expected utility’, the wealth-consumption ratio must appear explicitly in state prices and markets are incomplete. Thus, although some states cannot be hedged (since they cannot be publically observed), their existence does have a non-trivial impact on macro-economic variables.

In contrast with other models, we find that the equity premium depends on the relative volatility of an aggregate measure of risk aversion as well as its level. Allowing this volatility to be high results in a high market price of risk even when the level of aggregate risk aversion is low. In fact, to fit stylized facts within our framework, it is the volatility in changing risk attitudes that is most material to the market price of risk. We argue that the ‘excess’ volatility of aggregate risk aversion should come from ‘correlated unforeseen contingencies’ - events that are not directly related to consumption. Attempting to fit to stylized facts (such as the real risk-free rate and the observed Sharpe ratio) otherwise leads to an ‘equity premium’ puzzle. The non-consumption pricing factor that arises after aggregation over unforeseen contingencies can be conditionally uncorrelated with aggregate consumption shocks to the degree that the market price of risk for consumption is negligible. However, to be consistent with stylized facts, the non-consumption pricing factor must be correlated with shocks to expected consumption growth.

Section 2 discusses the intuition and literature behind the economic model. Section 3 of the paper outlines the general model and presents our aggregation theorem. Section 4 provides further discussion and results on how our model may be calibrated to stylized facts. Section 5 concludes with an approximate and testable version for the pricing kernel that consists solely of observed macro economic variables: aggregate consumption, the wealth-consumption ratio, and the price of a console bond.

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6Recently, several researchers have begun to investigate the aggregation of agents with heterogeneous preferences. Some references include Dumas (1989), Wang (1996) and Chan and Kogan (2000). Our method of aggregation and choice of preferences is closest in spirit to Rubinstein (1974).

7The presence of wealth in the pricing kernel is not truly an innovation. This is also a property of the Epstein-Zin (1989) model (and even the Lucas (1978) Euler equations) can be written in terms of wealth. These models however, are entirely consumption based, and as Koehlerlakota (1996) points out, wealth is actually linear in contemporary aggregate consumption. Thus, strictly speaking, wealth in the Epstein-Zin (1989) model cannot proxy for non-consumption based factors since it is perfectly correlated with consumption by construction.
2 Discussion of setting and literature

We now informally describe the economic setting. In addition to facing the usual uncertainty, each agent in our single good exchange economy has to contend with the fact that there are other contingencies (i.e., unrelated to aggregate consumption) that might affect her. However, like us, the agent cannot explicitly list all future contingencies that can affect her. Recent literature on unforeseen contingencies\(^8\) gives normative insight into agents’ choice behavior under such circumstances: the impact of unforeseen contingencies can be consistently taken into account by incorporating additional *subjective utility states* into the already existing objective exogenous filtration. Since this is likely to be unfamiliar to most financial economists, we provide a simple illustration: consider an agent facing an exogenously given filtration of states as in Figure 1. Squares (circles) denote decision (chance) nodes and \(z_{i,j}\) is the bundle consumed at date \(i\) and branch \(j\). In the conventional approach to dynamically consistent choice, the agent can determine what her choice will be at \(x_{1,4}\) even though she is presently at date 0. For instance, if she knows that she will prefer the upper branch of \(x_{1,4}\), then in considering her date 0 choices she can simplify the tree and ‘erase’ the lower branch at \(x_{1,4}\) - this is essentially a pre-commitment to the upper branch. If there are unforeseen contingencies, however, the agent may be reluctant to ‘erase’ a future choice, or in other words, to pre-commit to a future choice branch. How does one correctly represent such behavior? The answer is illustrated in Figure 2. There the agent is willing to pre-commit to the upper branch of \(x_{1,4}\), but is reluctant to do so at any of the branches of \(x_{1,1}\). She behaves as if there are at least two unlisted ‘states’: one in which she will prefer the upper branch, and one in which she will prefer the lower branch. The unlisted ‘states’ are not exogenous states, but subjective preference states that can only be inferred by the agent’s aversion to commitment, and they can be represented by two different utility states at date 1 (these are illustrated by a bold chance node in Figure 2). Thus, unforeseen contingencies are synonymous with changing tastes or private taste shocks.

To provide additional motivation for the relevance of such endogenous preference states, consider the following exercise in introspection: would you (would anyone) commit to a contingent stream of consumption for all future dates based only on what you know today about future consumption/production opportunities? In particular, consider that a contingent consumption stream delivers a basket or index of goods at each point in time that

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\(^8\)See Kreps (1979, 1992), Dekel, Lipman and Rustichini (2001) and Kraus and Sagi (2002). The term *unforeseen contingencies* is now a technical one that defines exogenous events that are not adapted to the current exogenous filtration as seen by the agent.
Figure 1: A consumption-decision tree. Squares (circles) denote decision (chance) nodes and $z_{i,j}$ is the bundle consumed at date $i$ and branch $j$. 
Figure 2: Assume that in Figure 1 the agent knows at date 0 that she will prefer the upper branch of $x_{1,4}$, but is reluctant to precommit to either branch in $x_{1,1}$. The induced tree, shown above, ‘erases’ the lower branch at $x_{1,4}$, but transforms the choice node at $x_{1,1}$ into a chance node corresponding to preference states.
depends on exogenous events that the agent can prespecify. By committing to a particular stream of commodity baskets today, one is foregoing the possibility of enjoying alternative baskets containing goods of which one cannot currently conceive. Although a rational agent may not conceive of all future available goods, she does recognize her own limitations and refuses to commit to a contingent consumption stream. Instead, she values flexibility, reasoning that something can happen that will cause her to deviate from a contingent plan. Although neither she nor outside observers can name what event might do so, such deviation from a plan can be viewed as a ‘taste shock.’

Uncertainty with respect to future preferences is neither new nor unreasonable, and there is a literature on ‘changing tastes’ and associated utility for flexibility dating to the 60’s. Alternatively, one can view the preference structure of each of our agents as that of reduced-form utility. In this context, the private preference states of Figure 2 can be interpreted as proxying for residual utility possibilities that remain after imperfectly aggregating a consumer’s bundle of goods into a single index that is homogeneous across agents. In other words, the agent feels differently about her allotment of the aggregate good in node $x_{1,1}$ because she is actually consuming a different amount of some unlisted commodities in each of these branches. Ultimately, though, we defend our use of heterodox preferences by claiming that it is unreasonable to suggest that agents can tell now how they will feel later about some future consumption contingency.

The figures illustrate the decision problem of a single agent. In a multi-agent economy such as ours, it is sensible to assume that certain contingencies relevant to all agents are also largely unforeseen by all of them; these can be viewed as private taste shocks that share a common component. We are particularly interested in the possibility that common components of taste shocks are not correlated with aggregate consumption: it is mainly in this way that the model departs from the conventional paradigm of state dependent preferences (e.g., habit formation models) and ‘non-consumption’ factors are introduced into the pricing kernel.

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10We also note that although our agents have non-standard preferences, they are inter-temporally consistent – the preference structure is recursive as in Kreps and Porteus (1978, 1979), Chew and Epstein (1989, 1991), Epstein and Zin (1989) Machina (1989), Grant, Kajji and Polak (1998, 2000ab), and Skiadas (1997, 1998) – and therefore perfectly rational. The decision theoretic framework for such preferences is established in Kraus and Sagi (2002). We note that it is largely the fact that our agents take account of their changing preferences (i.e., they are inter-temporally consistent and thus immune to manipulation) that results in non-trivial implications for state prices.
The preference states (i.e., the branches at $x_{1,1}$ in Figure 2) of individual agents are known only to them. However, in a general equilibrium, related components of agents’ taste shocks may have an impact on prices.\textsuperscript{11} Such price impact allows agents to infer the common component of private taste shocks (although the identities of agents experiencing the different shocks cannot generally be determined). We allow for a security market where every aggregate consumption or price-related event can be hedged.\textsuperscript{12} Since prices reveal the ‘correlated’ parts of private taste shocks, the set of observable states must be endogenously determined by trading, and thus \textit{the asset mix in the economy is determined by the equilibrium}. It is not hard to see that if every unforeseen contingency is common to all agents, no truly private state exists and the market is complete - a classic exchange economy follows with the possibility of state dependent aggregate risk aversion. On the other hand, market incompleteness arises from the failure of some or any private shock to have macroeconomic significance. The last few statements clarify that an economy with unforeseen contingencies ought to explicitly aggregate heterogeneous agents’ preferences to arrive at a pricing kernel. This is especially true if the market is incomplete, since a representative agent does not, in general, exist.

3 General Equilibrium With Unforeseen Contingencies

The assumed economy is one of pure exchange, consisting of $N$ agents with state dependent preferences.\textsuperscript{13} There are $T$ periods and one perishable consumption good that is produced according to some exogenous process (all of the good produced at date $t$ must be consumed at date $t$). There are two basic types of states: nature states that determine the amount of aggregate good available each period, and individual preference states corresponding to private taste shocks. Nature and preference states give rise to \textit{macro} and \textit{micro} events, depending on whether or not the event under consideration is observable by all agents.

\textsuperscript{11}E.g., if the same unforeseen contingency increases the appetites of all agents for mangoes, then such an event will have an impact on the price of mangoes.

\textsuperscript{12}Note that it is possible that the source of the event (e.g., development of the internet) is not foreseen, yet the impact on welfare is anticipated by prices (e.g., something \textit{might} improve everyone’s ability to communicate and pool information). By betting on technology stocks, one can hedge breakthroughs in technology even though one cannot possibly characterize what these breakthroughs might actually be.

\textsuperscript{13}One can consider a countable number of agents with only a slight change in subsequent notation.


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Before we define precisely what it is that we mean by ‘macro’ or observable states, we introduce some technical definitions and notation. Let Ω be a finite set and \( \mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_T) \) be a sequence of partitions of \( \Omega \) such that \( \mathcal{F}_1 = \{\Omega\} \), \( a_{t+1} \in \mathcal{F}_{t+1} \Rightarrow a_{t+1} \subseteq a_t \) for one and only one \( a_t \in \mathcal{F}_t \), and \( \mathcal{F}_T \) is identified with the set of all singleton subsets of \( \Omega \). \( \mathcal{F} \) is an information filtration and we refer to elements of \( \mathcal{F}_t \) as date- \( t \) ‘states’ or ‘events’.

**Definition 1.** A state-price system adapted to \( \mathcal{F} \) is a strictly positive real-valued process, \( \phi(\cdot) \), adapted to \( \mathcal{F} \) and an associated set at each date \( t \in \{1, \ldots, T\} \) and event \( a_t \in \mathcal{F}_t \):

\[
\{\phi(a_s|a_t)\}_{a_s \subseteq a_t},
\]

where

\[
\phi(a_s|a_t) = \begin{cases} 
\frac{\phi(a_s)}{\phi(a_t)} & s \leq T \text{ and } a_s \subseteq a_t \\
0 & \text{otherwise}
\end{cases}
\]  

Intuitively, a state-price system represents a set of Arrow-Debreu prices for \( \mathcal{F} \) at each date \( t \) and each state \( a_t \). The \( \phi(a_t) \)'s represent the date-1 Arrow-Debreu prices, and \( \phi(a_s|a_t) \) represents the price of an Arrow-Debreu claim that pays in state \( a_s \) given that the current state is \( a_t \). Note that for now this is simply an interpretation; we have said nothing about the contents of \( \Omega \), nor established that an equilibrium set of prices exists.

**Definition 2 (Observable/Macro States).** Given a state-price system generated by \( \phi(\cdot) \), \( \mathcal{F} \) is a filtration of observable/macro states if and only if the following three properties are satisfied:

i) Per capita aggregate consumption, \( C(\cdot) \) is adapted to \( \mathcal{F} \), and

ii) for any \( a_t, a'_t \in \mathcal{F}_t \) such that \( a_t \cup a'_t \subseteq a_{t-1} \) for some \( a_{t-1} \in \mathcal{F}_{t-1} \), \( a_t \neq a'_t \) if and only if per-capita consumption in state \( a_t \) is different from per-capita consumption in state \( a'_t \) or if \( \{\phi(a_s|a_t)\}_{a_s \subseteq a_t} \neq \{\phi(a_s|a'_t)\}_{a_s \subseteq a'_t} \).

iii) \( \mathcal{F} \) is adapted to the information filtration generated by pooling over all agents’ private information sets.
In other words, two date-\(t\) states with the same history (i.e., \(a_t \cup a_t' \subseteq a_{t-1}\) for some \(a_{t-1} \in \mathcal{F}_{t-1}\)) are observationally distinguishable (i.e., \(a_t \neq a_t'\)) if and only if each can be distinguished by a different realization of aggregate consumption and/or system of state prices. The third requirement excludes situations in which the ‘observable’ filtration contains more information than that contained in the pooled filtration. The definition is recursive: at date \(T\), the price system for future state claims is degenerate, thus observable states simply index the possible realizations of aggregate consumption; at date \(T-1\), the possible states index both different realizations of aggregate consumption and different sets of state prices for hedging date \(T\) aggregate consumption. Note that \(\mathcal{F}\) is at least as fine as the coarsest filtration to which aggregate consumption is adapted; i.e., \(\mathcal{F}\) contains at least as many states as can be distinguished via aggregate consumption history.

The set of ‘macro’ states, those that can be observed by everyone, includes all of the nature (i.e., aggregate consumption) states, but it also includes anything else that affects prices; in particular, any aggregated taste shocks that are not correlated with aggregate consumption and yet impact prices will induce additional states. Some preference states have no impact on prices because they are not correlated with aggregated shifts in preferences - those are assumed to be unobservable and are labeled as ‘micro’ states. Note that the definition of observable states is somewhat circular: to identify a filtration of observable states, one requires a set of prices for Arrow-Debreu claims on \(\mathcal{F}\)-events; the set of Arrow-Debreu claims over \(\mathcal{F}\), in turn, depends on which states are identified as belonging to \(\mathcal{F}\). In other words, \(\mathcal{F}\) must be determined \textit{endogenously} as part of a general equilibrium. In large measure, it is this property that differentiates our approach from that of the standard complete market exchange economy (where the filtration of states is exogenously specified).

For ‘micro’ states to be meaningful, one must consider an economy with heterogeneous agents. We assume a preference structure that is amenable to closed form solutions. To motivate our assumptions, it is first helpful to consider the case in which there are no micro states (i.e., all states are observable to all agents). Our benchmark is time-separable expected utility preferences under objective risk. Assume a filtration of observable states, \(\mathcal{F}\), exists. Agents’ preferences are constructed as follows: at date \(t\) and macro state \(a_t \in \mathcal{F}_t\), agent \(i\) must think ahead and consider the indirect utility generated from future wealth, \(w_{t+1}^i\) in state \(a_{t+1}\), where \(a_{t+1} \subseteq a_t\) refers to a date \(t+1\) macro state. For a particular pair of date \(t\) consumption and date \(t+1\) realization of wealth, \((c_t, w_{t+1}^i)\), the contribution to the agent’s utility in state \(a_{t+1}\) is,

\[
U_t(c_t, w_{t+1}^i, a_{t+1}) = u(c_t) + \beta V^*_t(w_{t+1}^i, a_{t+1})
\]
where $0 < \beta < 1$, $V_{t+1}^*$ is the indirect utility function for date $t+1$, and $u(c_t)$ is the agent’s utility of date $t$ consumption\footnote{We assume that $u(\cdot)$ is time independent.}. Note that we have assumed time separability and that the indirect utility function is state dependent (as is usually the case when the investment opportunity set is time varying).

Assuming expected utility with respect to macro-economic states (i.e., the $a_{t+1}$’s), the agent’s utility over the pair of current consumption and random future wealth, $(c_t, \tilde{w}_{t+1}^i)$, is

$$
\sum_{a_{t+1} \leq a_t} \pi(a_{t+1}|a_t) U_t(c_t, w_{t+1}^i, a_{t+1}) = u(c_t) + \beta \sum_{a_{t+1} \leq a_t} \pi(a_{t+1}|a_t) V_{t+1}^*(w_{t+1}^i, a_{t+1})
$$

(3)

where $\pi(a_{t+1}|a_t)$ is a conditional ‘probability’ for the occurrence of the macro-economic state $a_{t+1}$. To elaborate on $\pi(a_{t+1}|a_t)$: we assume that associated with each final state, $\omega \in \Omega$, is an unconditional probability, $\pi(\omega) > 0$. The unconditional probability of an event at date $t$, $a_t \in F_t$ is simply $\pi(a_t) = \sum_{\omega \in a_t} \pi(\omega)$. For $t < u$ and any two events, $a_t \in F_t$ and $a_u \in F_u$ such that $a_u \subseteq a_t$. Bayes’ Rule gives the probability of the event $a_u$ conditional on the occurrence of $a_t$ as $\pi(a_u|a_t) = \pi(a_u)/\pi(a_t)$. We assume that macro state probabilities are ‘objective’ and that all agents agree on their values.

Finally, the indirect utility is,

$$
V_t^*(w_t^i, a_t) \equiv \max_{(c_t, \tilde{w}_{t+1}^i) \in B(w_t^i)} \sum_{a_{t+1} \leq a_t} \pi(a_{t+1}|a_t) U_t(c_t, w_{t+1}^i, a_{t+1})
$$

(4)

where $B(w_t^i)$ is the agent’s budget set.

We now introduce micro states. Specifically, suppose agent $i$ has different possible preference realizations in a given macro state. The normative theory of unforeseen contingencies\footnote{See Kraus and Sagi (2002).} gives an indication of how time consistent preferences incorporating subjective (i.e., micro) preference states should be constructed: let $s^i_t$ denote agent $i$’s current preference state, and $V_{t+1}^{s^i_j}(w_{t+1}^i, a_{t+1})$ denote the date $t + 1$ indirect utility function associated with the subjective micro state $s^j_t$, $j \in 1, ..., n$, and the objective macro state, $a_{t+1}$; consistent preferences are achieved by positively aggregating all the date $t + 1$ indirect utility functions given $a_{t+1}$ and indexed by subjective micro states. That is, instead of Eqn. (2) one writes:

$$
U_t^{s^i_t}(c_t, w_{t+1}^i, a_{t+1}) = u_{s^i_t}(c_t) + \beta \varphi_t^{s^i_t}\left(V_{t+1}^{s^i_1}, V_{t+1}^{s^i_2}, ..., V_{t+1}^{s^i_n}\right)
$$

(5)

where $s^i_t$ denotes the current (i.e., date $t$) preference state and $s^i_1, ..., s^i_n$ denote the possible event $a_{t+1}$ preference states.\footnote{We could, and perhaps should, index the date $t + 1$ preference states with a time index (i.e., $t + 1$) and the agent’s identity (i.e., $j$). Since, however, we later assume that agents have the same possible set of preference states, we feel consistent that a single index suffices.} $\varphi_t^{s^i_t}$ is increasing in all its arguments. Note that the subjective
preference state affects attitudes towards consumption, inter-temporal substitution, and risk.

In what follows, we assume a particular form for $\varphi_{s_{i}t}$ that reduces to Eqn. (2) when preference do not evolve. One advantage to our recursive form is parsimony, while the other is the obvious similarity with other forms in the literature (see, especially, Epstein and Zin (1989)); specifically, consider

$$U_{s}^{i}(c_{t}, w_{t+1}, a_{t+1}) = u_{s_{i}}(c_{t}) + \beta v_{s_{i}}\left(\mathbb{E}_{s'}^{i}\left[v^{-1}_{s'}(V_{s_{i}}) | a_{t+1}\right]\right)$$

(6)

The operator, $\mathbb{E}_{s'}^{i}[\cdot | a_{t+1}]$ is a positive weighting function for the date $t + 1$ micro states (micro states are indexed by $s' \in \{s_{1}', s_{2}', ...,\}$). In particular, it has the following properties:

$$\mathbb{E}_{s'}^{i}\left[1 + f(s') | a_{t+1}\right] = 1 + \mathbb{E}_{s'}^{i}\left[f(s') | a_{t+1}\right]$$

$$\mathbb{E}_{s'}^{i}\left[af(s') | a_{t+1}\right] = a\mathbb{E}_{s'}^{i}\left[f(s') | a_{t+1}\right]$$

for any real-valued function, $f(s')$ over the micro states, and

$$\mathbb{E}_{s'}^{i}\left[f(s') | a_{t+1}\right] > \mathbb{E}_{s'}^{i}\left[g(s') | a_{t+1}\right]$$

whenever $f(s') \geq g(s')$ for each $s'$ with strict inequality over a measurable set of micro states. In Eqn. (6) the indirect utility function for state $s_{j}'$ is first converted to a consumption ‘certainty equivalent’ via the transformation, $v^{-1}_{s_{j}}(V_{s_{j}}^{*t+1})$. The certainty equivalents are then aggregated by the weighting function, $\mathbb{E}_{s'}^{i}$. The result is then transformed back to a consumption utility equivalent via $v_{s_{i}}(\cdot)$. Note that if there is only a single micro state, $s' \equiv s_{i}$, Eqn. (6) reduces to Eqn. (2). There are several possibilities to consider:

1. $\mathbb{E}_{s'}^{i}$ is linear and the weights correspond to objective probabilities.

2. $\mathbb{E}_{s'}^{i}$ is linear but the weights are not ‘objective’ probabilities.

3. $\mathbb{E}_{s'}^{i}$ is non-linear (e.g., a Choquet integral) and may utilize ‘objective’ probabilities or otherwise.

The first interpretation is valid when the agent’s tastes change, but the distribution of tastes is stationary; the second interpretation is valid whenever tastes are not stationary - realizations of past tastes give little or no information about the future distribution of tastes that the additional notation does not justify the resulting confounding notation. By contrast, referring to the current or realized preference state of the agent (i.e., $s_{i}$) is necessary.
(e.g., the agent’s tastes change in different ways as she ages). In the last interpretation the agent is simply not an expected utility maximizer when considering micro states - she responds differently to different sources of uncertainty (i.e., micro versus macro states). These distinctions are far from inconsequential. Later we show that there are two necessary conditions for calibrating the model to empirical stylized facts: (i) the market is incomplete, and (ii) either agents’ preferences with respect to unforeseen contingencies are non-expected utility, or they cannot deduce objective probabilities for the evolution of their own preferences (due perhaps to non-stationarity).

The analogous expression for Eqn. (3) is
\[
\sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1} | a_t) U_t^{s_i}(c_t, w_{t+1}, a_{t+1}) = u_{s_t}(c_t) + \beta \sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1} | a_t) v_{s_i}\left( E_{s'}\left[ u_{s'}^{-1}(V_{s'}^{t+1} | a_{t+1}) \right] \right)
\]

This state-dependent preference structure is consistent with the general recursive utility form\(^{17}\) as well as the utility for flexibility discussed in the ‘unforeseen contingencies’ and ‘changing tastes’ literature\(^{18}\). In particular, since the preferences are not necessarily ‘expected utility’ with respect to preference-states, this formulation belongs to the class of non-Expected Utility general equilibrium models.

### 3.1 Equilibrium

Although an agent, by introspection, can observe her own current preferences, we assume that her actions (e.g., investment and consumption) and wealth are unobservable to others. Under these assumptions, moral hazard prevents anyone from insuring herself against a micro state - the intuition is that no agent can hedge against a mood change that is not related to aggregate sentiments because her private actions are not observable. Markets can be said, however, to be pseudo-complete in a practical sense: contracts on all macro states are available and marketed. All publically observed risk but not all private uncertainty can be hedged. In particular, agents whose tastes do not change can completely hedge their future consumption by purchasing a set of contingent claims at the first session of trade.

Prior to the beginning of each period, public and private uncertainty is resolved. Following the realization of preference and nature states, each agent begins period \(t\) with a set


\(^{18}\)See, for example, Kreps (1979, 1991), Dekel, Lipman and Rustichini (1999) and Kraus and Sagi (2002).
of claims over current consumption and a portfolio of claims to future (macro state contingent) amounts of the consumption good. The total value of each agent’s endowment of claims (in terms of current consumption) forms that agent’s budget. Following this but still during period $t$, a market opens for trading current and future claims. If the markets are incomplete, in that there are unobserved micro states\(^\text{19}\), agents will typically continue to trade after the initial round (period 1). By trading, agents can revise their holdings subject to the budget constraint by choosing a desired amount of consumption for period $t$ and a set of state contingent claims for consumption in the remaining periods. After trade, agents consume their share of current production of the consumption good.

Before trading begins at date $t$ and state $a_t$, agent $i$ has $c_{t-1}^i(a_u)$ units of the state contingent claim that pays out in the event $a_u \in F_u (u \geq t)$. These are traded between the agents when markets subsequently open to achieve the new equilibrium allocations of state claims, $c_t^i(a_u)$ for $u > t$, plus a current consumption allotment, $c_t^i$. It is important to note that, in the presence of non-degenerate micro states, the portfolio holdings and personal consumption for each agent will not be adapted to $F$ (hence the additional time-subscript). Equilibrium is achieved when at each date every agent trades to maximize her current period utility function from Eqn. (7) and the following market clearing conditions are met:

$$C(a_u) = \begin{cases} \frac{1}{N} \sum_{i=1}^{N} c^i_t(a_u) & \forall \ u > t \\ \frac{1}{N} \sum_{i=1}^{N} c^i_t & u = t \end{cases} \quad (8)$$

where $C(a_t)$ is the economy’s per-capita aggregate consumption of the good produced in the event $a_t$. Note that although $c_t^i$ need not be adapted to the filtration of macro states, this is clearly required of the aggregate of all date $t$ allocations regardless of micro state realizations.

**Definition 3.** An equilibrium is a quadruplet $\left(F, \{\phi(\cdot)\}, \{C(\cdot)\}, \{e_i\}_{i=1}^{N}\right)$ where,

i) $\{\phi(\cdot)\}$ is a complete set of state prices adapted to $F$.

ii) $F$ is a filtration of observable states given the state-price system $\{\phi(\cdot)\}$.

iii) $\{C(\cdot)\}$ is an aggregate consumption process adapted to $F$.

iv) $\{e_i\}_{i=1}^{N}$ is an initial endowment of state claims satisfying the market clearing condition (Eqn. (8)).

\(^{19}\)It is assumed that the moral hazard problem that will accompany contracts over unobserved states cannot be solved.
v) at each date every agent maximizes a recursive utility function of the form in Eqn. (7) by trading in \( F \)-Arrow-Debreu state claims subject to her budget constraint.

vi) at each date agents’ consumption and portfolio holdings satisfy the market clearing equation, Eqn. (8).

There are two main differences between our equilibrium concept and the standard one. First, \( F \) is determined endogenously and consistently with the set of state prices adapted to it; second, the evolution of any individual agent’s optimal consumption or portfolio holdings need not be adapted to \( F \) (since an agent’s individual consumption may depend on both the macro event and her micro state).

At this stage, we do not have general results on the existence or efficiency of an equilibrium. We expect that a competitive equilibrium with macro-complete markets is not generally Pareto efficient relative to all possible macro-complete market equilibria. If agents are forced to trade in a different pattern than implied by a competitive equilibrium, it may be possible to reveal more private states and thereby introduce more macro states; the new hedging opportunities may make everyone better off. For now, we restrict ourselves to investigating the existence of an equilibrium in a parametric model.

### 3.2 Equilibrium with HARA Preferences

We now demonstrate the existence of an equilibrium under specific parametric assumptions. Assume that agent \( i \) has preferences given by Eqn. (7) with:

\[
\begin{align*}
\hspace{1em} u_{s_i}^i(x) &= \frac{(s_i^i x + 1)^{1-\gamma}}{s_i^i (1-\gamma)} \Delta \quad \gamma > 0, \gamma \neq 1 \\
\hspace{1em} v_{s_i}^{-1}(x) &= u_{s_i}^{-1}(x) + \frac{1}{s_i^i}
\end{align*}
\]  

(9)  

(10)

All agents have utility of current consumption given by (9)-(10) with identical power, \( \gamma \), but different proportional displacement, \( s_i^i \).\(^{20}\) The displacement \( s_i^i \) parameterizes the date \( t \) realization of tastes for agent \( i \). The constant \( \Delta \) represents the time interval between decisions; one can arrive at a continuous time limit by taking \( \Delta \to 0 \). The assumption over \( v_{s_i}^{-1}(x) \) is dictated by both parsimony and tractability - note that it introduces no additional

\(^{20}\)The case \( \gamma = 1 \) (i.e., log utility) can be treated with minor modification. The same is true of constant absolute risk aversion (CARA) utility.
parameters other than those already incorporated into $u_{i_t}(x)$. The final assumption needed concerns the nature of the weighting operator, $E_{s_t'}$. We assume little about it save for the following:

$$E_{s_t'}^i \left[ \frac{1}{s_{t+1}^i} \mid a_{t+1} \right] = \frac{1}{F(a_{t+1})s_t^i}$$

where $F(a_{t+1}) > 0$ depends only on the macro state $a_{t+1}$ and is identical across individuals. As shall soon be apparent, this suffices to identify an equilibrium. Moreover, this assumption is consistent with a variety of hypotheses regarding the evolution of tastes: agents can be probabilistically sophisticated (Machina and Schmeidler (1992)) or ambiguity averse (Gilboa and Schmeidler (1989)) about future tastes. There are two ways to view $F(a_{t+1})$, and its interpretation is crucial for specifying the evolution of the pricing kernel. If $E_{s_t'}$ is a linear weighting scheme that coincides with objective probabilities for the evolution of tastes, then $F(a_{t+1})$ should be viewed as the forecast or objective expected value of $s_{t+1}^i$. In other words, $F(a_{t+1})$ gives the direction in which the average investor’s tastes drift. If, on the other hand, $E_{s_t'}$ is non-linear or non-probabilistic – an interpretation that is particularly valid if past shifts in taste are not useful for predicting future shifts, or if agents’ attitudes towards micro states is not ‘expected utility’ – then $F(a_{t+1})$ may have little to do with the actual evolution of tastes, and simply reflects an aspect of agents’ preferences. We will shortly return to these interpretations.

When $\gamma < 1$, for instance, Eqn. (6) becomes

$$U^i_{s_t^i}(c_{i_t}, w_{i_{t+1}}, a_{t+1}) = \frac{(s_{i_t}^i c_{i_t} + 1)^{1-\gamma}}{s_{i_t}^i (1 - \gamma)} \Delta + \beta E_{s_t'}^i \left[ \left( \frac{s_{t+1}^{i+1}}{s_t^i} \right)^{\frac{\gamma}{1 - \gamma}} \left( V_{s_{t+1}^i}^{i+1} \right)^{\frac{1}{1 - \gamma}} \mid a_{t+1} \right]^{1-\gamma}$$

The above form is somewhat reminiscent of Epstein-Zin (1989) recursive utility. Note that in their formalism, the operator $E_{s_t'}$ also need not be expected utility. There are important differences, however. Our indirect utility is additionally weighed by the state-dependent term, $\left( \frac{s_{i_t}^i}{s_{i_t}^{i+1}} \right)^{\frac{\gamma}{1 - \gamma}}$, and the aggregation is only over micro (i.e., preference) states and not over consumption states.

21This is only possible if agents have had enough time to ‘learn’ the evolution process of their own tastes, not to mention that the process can be ‘learned’ in the first place (i.e., a stationarity assumption).

22When $\gamma > 1$, this becomes slightly more complicated because of the change of sign in the utility function:

$$U^i_{s_t^i}(c_{i_t}, w_{i_{t+1}}, a_{t+1}) = \frac{(s_{i_t}^i c_{i_t} + 1)^{1-\gamma}}{s_{i_t}^i (1 - \gamma)} \Delta + \frac{1}{1 - \gamma} \beta E_{s_t'}^i \left[ \left( \frac{s_{t+1}^{i+1}}{s_t^i} \right)^{\frac{\gamma}{1 - \gamma}} \left( (1 - \gamma) V_{s_{t+1}^i}^{i+1} \right)^{\frac{1}{1 - \gamma}} \mid a_{t+1} \right]^{1-\gamma}$$
At date $T$ each agent is assumed to have the following utility for consumption:

$$U^s_T (c^i_T) = \frac{(s^i_T c^i_T + 1)^{1-\gamma}}{s^i_T(1-\gamma)} \Delta$$

(13)

The assumptions on the agents can be used to calculate their optimal portfolios given a filtration of macro states, $\mathcal{F}$, and a complete set of state claims over the macro states.

**Proposition 1.** Under assumptions (9)-(11), an agent with wealth $w^i_t$ at date $t$ optimally consumes $c^i_t$:

$$c^i_t = \rho(a^t) - \frac{1}{z(a^t)} w^i_t + \frac{\rho(a^t) - \frac{1}{s^i_t}}{s^i_t - 1}$$

(14)

leading to a date $t$ indirect utility function given by:

$$V^s_t (w^i_t, a^t) = \rho(a^t) z(a^t) \frac{(s^i_t w^i_t + 1)^{1-\gamma}}{s^i_t(1-\gamma)}$$

(15)

where $z(a^t)$ and $\rho(a^t)$ are defined recursively via

$$z(a^t) \equiv \Delta + \sum_{a_{t+1} \leq a^t} \frac{\phi(a^t_{t+1} | a^t)}{F(a^t_{t+1})} z(a^t_{t+1})$$

(16)

$$\rho(a^t) \frac{1}{z(a^t)} \equiv \Delta + \sum_{a_{t+1} \leq a^t} \phi(a^t_{t+1} | a^t) \left( \frac{\phi(a^t_{t+1} | a^t)}{\beta_i(a^t_{t+1} | a^t)} \right)^{-1/\gamma} \rho(a^t_{t+1}) \frac{1}{z(a^t_{t+1})}$$

(17)

**Proof:**

See appendix. \(\square\)

Agent $i$’s wealth is calculated as the present value of the state claims she owns at the beginning of period $t$. Each of $z(a^t)$ and $\rho(a^t) \equiv \rho(a^t) \frac{1}{z(a^t)}$ can be interpreted as the value of a unit coupon consol bond under an adjusted set of state prices. In particular, if $F(a^t_{t+1})$ is always identically equal to one, then $z(a^t)$ is the price of a consol bond. Both $z(a^t)$ and $\rho(a^t)$ depend only on the macro states (i.e., the investment opportunity set). The micro state-dependence of optimal consumption and the indirect utility function is due to the explicit dependence on $s^i_t$.

One can aggregate the marginal rates of substitution across agents to derive state prices:
Proposition 2. Under assumptions (9)-(11), if an equilibrium exists then state prices satisfy Eqn. (1) and 

\[ \phi(a_{t+1}|a_t) = \begin{cases} \beta \pi(a_{t+1}|a_t) \rho(a_{t+1}) \left( \frac{G(t)w(a_{t+1}) + \frac{1}{G(t)C(a_t) + 1}}{F(a_{t+1})} \right)^{-\gamma} & t < T \\ 0 & t = T \end{cases} \]  

(18)

where,

\[ \frac{1}{G(t)} \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_i^t} \]  

(19)

and

\[ w(a_t) \equiv C(a_t) \Delta + \sum_{a_{t+1} \leq a_t} \phi(a_{t+1}|a_t)w(a_{t+1}) \]  

(20)

Proof:

See appendix.

\[ \square \]

\( w(a_t) \) is aggregate wealth: the present value of current and all future per-capita consumption. \( G(t) \) is a measure of aggregate risk aversion across investors and is the only quantity in Eqn. (18) that directly depends on the realization of investors’ utility functions (i.e., the \( s_i^t \)'s). Note also that prices do not depend on the distribution of initial endowments across agents.

Theorem 1. Assume (9)-(11) and that \( G(t) \) has a finite number of possible realizations at each date \( t \). Let \( \{e_i^t\}_{i=1}^{N} \) be the set of initial endowments, \( \mathcal{F} \) be the filtration of states generated by per-capita consumption at all dates and by realizations of \( G(t) \) for dates \( t < T \), and let \( \{\phi(\cdot)\} \) be the state price system in (18). Then for any set of market clearing initial endowments and almost every parameterization of \( F(a_{t+1}), (\mathcal{F}, \{\phi(\cdot)\}, \{C(\cdot)\}, \{e_i^t\}_{i=1}^{N}) \) is an equilibrium.

Proof:

By definition, aggregate consumption and the realizations of \( G(t) \) are adapted to \( \mathcal{F} \). Defining \( \phi(a_s) \equiv \phi(a_s|a_1) \), then a simple induction argument shows that \( \phi(\cdot) \) is adapted to
The intuitions behind the result is as follows: under almost every parameterization of $F(a_{t+1})$, all realizations of aggregate risk aversion can be inferred by their direct impact on state prices of future observable states. The proof of the theorem also establishes that in the few (i.e., zero measure) situations where this is not the case, state claims that pay in the following (i.e., date $t+1$) period are valued independent of current risk attitudes - see footnote 23. Thus we hasten to add that in addition to applying to almost all parameterizations of $F(a_{t+1})$, the Theorem applies to all economically compelling models for $F(a_{t+1})$.

From here on assume that for every $a_t \in \mathcal{F}_t$ there is some $a_{t+1} \subseteq a_t$ with $\frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) \neq C(a_t)$. Thus the proof of Theorem 1 also establishes that $\{G(t)\}_{t=1}^T$ is adapted to the macro state filtration and $G(t)$ is observable at date $t$ through state prices.\textsuperscript{24} We can therefore refer

\footnotetext[23]{This is equivalent to requiring $\phi(a_{t+1}|a_t) = \beta F(a_{t+1})^\gamma \rho(a_{t+1}) \pi(a_{t+1}|a_t)$. The latter expression is independent of $G(t)$ and thus independent of agents’ risk attitudes. To see this note that if some agent is risk neutral (e.g., $s_t \to 0$ implies the indirect utility for wealth in (15) exhibits risk neutrality), $G(t) = 0$ and the same pricing kernel derives; however, when $\frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) = C(a_t)$ the risk neutral prices arise whether or not there is a risk-neutral agent.}

\footnotetext[24]{We have assumed that the number of macro states is finite. This places a constraint on the topology of realizations.
to realizations of $G$ as $G(a_t)$. In essence, all information about individuals’ tastes washes out upon aggregation, save for the average risk tolerance $\frac{1}{G(a_t)}$. It is only through this factor that correlated preference states can give rise to macro-economic or priced states, and these must be part of the partition, $\mathcal{F}_t$. For example, suppose consumption is deterministic, but agents’ tastes are random; then the macro-state partition, $\mathcal{F}_t$, consists of all possible configurations of risk tolerance, $\frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_i}$. Clearly, this filtration need not be as large as the product filtration of individuals’ preference states. In fact, given enough agents with independent preference shocks to their risk aversion, $\frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_i}$ can be degenerate. In general, however, the market here is incomplete in that not all preference states are revealed (i.e., there can be some dispersion in individuals’ micro states that fails to show up in macro prices).

There is one final issue to address. Can anything be said about the evolution of $G(a_t)$? The answer depends on the meaning ascribed to the micro-state weighting function, $\mathbb{E}_{s_t}^i$. Suppose, for instance, that the operator corresponds to an expectation using objective probabilities. Since $\frac{1}{G(a_{t+1})}$ does not depend on micro states, it must be that

$$
\frac{1}{G(a_{t+1})} = \mathbb{E}_{s_t}^i \left[ \frac{1}{G(a_{t+1})} \middle| a_{t+1} \right] = \mathbb{E}_{s_t}^i \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_{i+1}} \middle| a_{t+1} \right] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{s_t}^i \left[ \frac{1}{s_{i+1}} \middle| a_{t+1} \right] = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_{i+1}} \frac{1}{F(a_{t+1})}
$$

where the second equality is due to the assumption that $\mathbb{E}_{s_t}^i$ is both linear and objective, rendering the agent specific superscript redundant. Thus, the assumption of objective and linear weights in $\mathbb{E}_{s_t}^i$ completely pins down the evolution of $G(a_t)$ in terms of existing model parameters. By contrast, if $\mathbb{E}_{s_t}^i$ is not an objective expected utility functional (i.e., if beliefs about future tastes are either not probabilistic or beliefs cannot be separated from utility in the agent’s preferences) then it need not be the case that $G(a_{t+1}) = G(a_t)F(a_{t+1})$. Since we know that $G(a_{t+1})$ is $\mathcal{F}$-measurable, we can write, without loss of generality:

$$
G(a_{t+1}) = G(a_t)F(a_{t+1})e^{-\delta(a_{t+1})}
$$

(21)

$\delta(a_{t+1})$ reflects the degree of deviation from objective expected utility in the agent’s attitudes towards micro states. Once again, we stress that there is nothing of $G(t)$. An extension to the case of a continuum of states or one where $T \to \infty$ introduces additional technical difficulties that we do not want to address in this paper. Even even if the number of agents is large (or infinite), only mild conditions need be placed on the $s_i$’s so that the number of configurations of $G(a_t)$ is small (e.g., a ‘law of large numbers’ in the case where $N$ is infinite).
inherently wrong or irrational about non-expected utility preferences so long as they are dynamic programming consistent (which is the case here by construction).

We can now summarize our results in the following:

**Theorem 2.** Assume (9)-(11) and that $G(t)$ has a finite number of possible realizations at each date $t$. Let $\{e_i^t\}_{i=1}^N$ be the set of initial endowments, $\mathcal{F}$ be the filtration of states generated by realizations of $G(t)$ for $t < T$ and by per-capita consumption at all dates, and let $\{\phi(\cdot)\}$ be the state price system in (18). Finally assume that for every $a_t \in \mathcal{F}$ there is some $a_{t+1} \subseteq a_t$ with $\frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) \neq C(a_t)$. Then $\left(\mathcal{F}, \{\phi(\cdot), \{C(\cdot)\}, \{e_i^t\}_{i=1}^N\right)$ is an equilibrium and for any $t < T, \frac{1}{G(a_t)} \equiv \frac{1}{N} \sum_{i=1}^N \frac{1}{N} \beta_i$ and $C(a_t)$ are adapted to the $\mathcal{F}_t$ filtration of macro states. The pricing kernel is given by:

$$
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t) \beta \left( \frac{G(a_{t+1})C(a_{t+1}) + 1}{G(a_t)C(a_t) + 1} \left[ 1 - \frac{1 - e^{-\delta(a_{t+1})}}{G(a_{t+1})w(a_{t+1}) + 1} \right] \right)^{-\gamma} (22)
$$

where $\delta(a_{t+1}) \equiv \ln F(a_{t+1}) + \ln G(a_t) - \ln G(a_{t+1})$ is adapted to $\mathcal{F}$.

**Proof:**

Only Eqn. (22) needs derivation. This is done in the Appendix.

There are several observations that remain to be made.

**Remark 1.** If $\delta(a_{t+1}) = 0$, corresponding to expected utility with objective probabilities, then the pricing kernel only features the term $\frac{G(a_{t+1})C(a_{t+1}) + 1}{G(a_t)C(a_t) + 1}$ of $G(a_t)$. This is exactly the result that would be obtained if we postulated a representative agent with time-separable utility function

$$
U_t = E_t \left[ \sum_{j=0}^{\infty} \beta^j \left( C(a_{t+j}) + \frac{1}{G(a_{t+j})} \right)^{1-\gamma} \right]
$$

$1/G(a_t)$ above is a factor that summarizes the impact of unforeseen contingencies on aggregated tastes. The impact is identical to that of a ‘substitute’ for current consumption. A similar ‘substitution’ factor can arise due to the formation of habit\(^25\), or due to the presence

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\(^25\)One can nearly recover an Abel-like (1990) external habit formation utility for the representative agent by assuming that the habit level is inversely proportional to $G(a_t)$.  

21
of consumption bundle components that are not captured when constructing an aggregate consumption index.

We stress that a true representative agent (i.e., in the sense of efficient allocations) does not necessarily exist for this economy when \( \delta(a_{t+1}) = 0 \). That is because the market may still be incomplete in the sense that each agent may not be able to hedge all of her taste shocks. As far as the macro-economy is concerned, however, such incompleteness is immaterial whenever \( \delta(a_{t+1}) = 0 \).

**Remark 2.** Note that the relative risk aversion of this representative agent in the last remark (with respect to date \( t \) consumption) is

\[
R_{R}(a_{t}) = \frac{\gamma G(a_{t})C(a_{t})}{G(a_{t})C(a_{t}) + 1}
\]

and the pricing kernel can be written as

\[
\phi(a_{t+1}|a_{t}) = \pi(a_{t+1}|a_{t})\beta\left(\frac{C(a_{t+1})}{C(a_{t})}\right)^{-\gamma}\left(\frac{R_{R}(a_{t+1})}{R_{R}(a_{t})}\right)^{\gamma}
\]

(23)

All but the last term are standard, while the last reflects the impact of time varying risk aversion. In particular, the case \( \frac{G(a_{t+1})}{G(a_{t})} = F(a_{t+1}) \equiv \frac{C(a_{t})}{C(a_{t+1})} \forall t \) reduces to the Lucas (1978) model. Note that relative risk aversion for consumption may be small in level (e.g., if \( G(a_{t})C(a_{t}) \) is small in level), and yet the standard deviation of state prices can be high (if the relative volatility of \( R_{R}(a_{t}) \) is high).26 Thus the volatility of relative risk aversion can contribute to the equity premium equally or even a great deal more than the level of relative risk aversion.

**Remark 3.** The case \( \delta(a_{t+1}) \neq 0 \) corresponds to non-expected utility preferences (or to expected utility without objective probabilities) with respect to micro-states. The result is a pricing kernel that is explicitly wealth dependent: both the consumption wealth ratio and the ‘term-structure’ variable, \( z(a_{t}) \), become explicit components of the kernel. Moreover, in this case one cannot derive state-prices from the marginal utility of a representative agent.

If there are no micro states (i.e., all preference states are revealed) then the weighting operator, \( E_{\delta}^{a} \), by definition, gives the forecast for the evolution of tastes. Thus, perforce, \( \delta(a_{t+1}) \) is set to zero. A direct implication is that the presence of a non-trivial \( \delta(a_{t+1}) \) is

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26Unless the process for \( R_{R}(a_{t}) \) exhibited a high degree of mean reversion, there is no guarantee of maintained low relative risk aversion. With sufficiently high mean reversion, however, shocks to relative risk aversion could be large in percentage terms, while the overall level always kept low.
sufficient to imply the presence of micro state dispersion in agents’ preferences - i.e., markets are incomplete. micro states, in this case, do not show up in a verifiable manner and are therefore not priced and cannot be hedged. However, their impact can be felt in the economy (through a non-zero $\delta(a_{t+1})$).

An important conclusion is that it is not merely the presence of incompleteness in the market that contributes to the more exotic aspects of the pricing kernel. Incompleteness is necessary, but an additional and crucial requirement is that un-hedgeable states enter agents’ preferences in a non-standard (i.e., non-Expected utility) manner.

In summary, the price of risk in this economy is determined by the dynamics of three factors: per capita aggregate consumption, a measure that aggregates the ‘utility states’ across agents (i.e., $G(a_t)$), and a variable that jointly measures the presence of market incompleteness plus the departure of preferences from expected utility theory (i.e., $\delta(a_t)$).

4 Asset Pricing

Potentially, the pricing kernel in Eqn. (22) can describe anything, so long as the random variables $G(a_t)$ and $\delta(a_t)$ are chosen appropriately. This is because one of the effects of these variables, if their innovations are not perfectly correlated with the consumption process, is to ‘add’ new priced states to the economy. Such freedom is actually a burden unless there is some economic guideline for deciding what macro-economic variables other than aggregate consumption affect prices. To deal with this difficulty, the literature has implicitly set a standard for consumption based models of time varying risk premia: innovations to the macroeconomic variables should be spanned by shocks to consumption. In other words, the only news that is priced is news about consumption. This rules out injecting new states by introducing variables that do not exhibit conditionally perfect correlation with aggregate consumption. In our setting, this limits the choice of the parameterization of $G(a_t)$ and $\delta(a_t)$ to functions of current and past consumption. In the next subsection we argue that forcing $G(a_t)$ and $\delta(a_t)$ to be adapted to the aggregate consumption filtration always results in a type of ‘equity premium puzzle’. We then argue that under some circumstances $G(a_t)$ can actually be proxied by a power function of the wealth consumption ratio, thus making a case for generically incorporating such a variable explicitly in the pricing kernel even if one does not know how $G(a_t)$ fundamentally arises. Finally, we study a simple two state model that captures many of the macroeconomic stylized facts using a two state non-consumption
variable. An important and robust result of our analysis is that, although we may not know the source of fluctuations in $G(a_t)$, its dynamics cannot be completely arbitrary. In particular, shocks to $C(a_t)G(a_t)$ must be correlated with shocks to the expected growth rate of aggregate consumption. Moreover, we argue that the stylized facts force $\delta(a_{t+1}) \neq 0$, thus both incomplete markets and non-standard preferences are required, at least in our model.

4.1 The Consumption-Pricing Puzzle

Here we argue that restricting $G(a_t)$ and $\delta(a_t)$ to be adapted to the aggregate consumption filtration always results in ‘puzzles’ aside from the usual conundrum regarding preference parameters for a representative agent. The argument is fairly simple. The maximum Sharpe ratio in the economy is calculated as the mean excess return on an asset whose payoffs have unit standard deviation and are perfectly (negatively) correlated with the state price deflator. Explicitly, setting

$$\frac{1}{R_f} = \sum_{a_{t+1} \leq a_t} \phi(a_{t+1}|a_t) = \text{mean}(\frac{\phi}{\pi})$$

The maximum Sharpe Ratio is given by

$$\text{Sharpe Ratio} = R_f \text{stdev}(\frac{\phi}{\pi})$$

Now, the observed real interest rate is close to zero (Campbell and Cochrane (1999) argue that it may not even be distinguishable from 0). Moreover, the quarterly maximum Sharpe ratio is $\approx \frac{1}{4}$ (some trading strategies (e.g., momentum) can achieve Sharpe ratios that are higher - unadjusted for trading costs). Assuming $R_f \approx 1$, the average of $\phi/\pi$ is 1; since state prices are assumed to be adapted only to the aggregate consumption filtration, a typical ‘bad’ state of nature will have to be priced at a 25% premium relative to its likelihood, and a typical ‘good’ state of nature will be priced at a 25% discount. For instance, if there are only two equally likely states of aggregate consumption growth at each point in the filtration, the good state claim costs $0.375$ and the bad state claim costs $0.625$. This has to be considered against the fact that the ‘bad’ consumption state in reality corresponds to a roughly 0.5% reduction in aggregate consumption (with the good state corresponding to 1.5% increase).\(^{27}\)

Assuming consumption in a good quarter grows by 1.5% and drops by 0.5% in a bad quarter, the consumption smoothing insurance premium that would be paid by the more

\(^{27}\text{This assumes a quarterly mean and standard deviation for consumption growth of, respectively, 0.5\% and 1.0\%.}
risk averse market participants to those how are less risk averse corresponds to the difference between the mean and certainty equivalent value of next period’s consumption:

$$1.005 - \left( 1.015 \times 0.375 + 0.995 \times 0.625 \right) = 0.25\%$$

Although this ‘premium’ does not seem big, one has to consider that it corresponds to insurance profits that are half of the average quarterly growth ($\approx 0.5\%$), and on an annual basis, roughly 1% of real GDP – the amount that the more risk averse agents in the economy would give up to smooth consumption if they held all producing assets – in other words, as much as 100 Billion dollars of profit per year. This is an amount that is larger than the profits of the entire insurance industry (including medical and life insurance). With such sizeable profits at stake, are there active insurance markets for aggregate consumption? One possibility is that there is not enough dispersion in risk aversion to warrant this much risk sharing. However, even if risk sharing implies the trading of only 5% of GDP - an assumption that requires only moderate dispersion in relative risk aversion across individuals - the resulting average profits from premia correspond to an average of $5 Billion per year - still an attractive sum. Another possibility is that the stock market provides this service and reaps the sizeable rewards (that, in fact would be the argument behind a high equity premium). The problem with that argument is that the stock market correlation with aggregate consumption is only, at best, about $\rho_{MC} = 0.4$, and the consequent maximal reduction in standard deviation of aggregate consumption that such a correlation affords is roughly $1 - \sqrt{1 - \rho_{MC}^2} \approx 10\%$ (assuming futures contracts are used).

Consumption based asset pricing models that are fit to stylized facts (e.g., the Sharpe ratio and risk free rate) imply that significant profits can be made by providing aggregate consumption insurance. Another way to state the equity premium puzzle is therefore in terms of the absence of obvious security markets for hedging aggregate consumption risk given that the model implied risk premium for aggregate consumption risk is significant. This is particularly conspicuous due to the presence of obvious hedging instruments for every other factor that is associated with an observed empirical risk premium. Moreover, there is no evidence that the market price of risk for empirically priced factors arises due to covariance with aggregate consumption (e.g., Chen, Roll and Ross (1986), and a more recent paper by Duffee (2002)); or equivalently, there is little proof that aggregate consumption performs well in cross-sectional asset pricing tests. In particular, this suggests that priced

28e.g., aggregate industrial production, inflation, default spreads, term-structure spreads, and oil. Oil consumption, in particular, is approximately a sixth of real personal consumption. There is evidence for time-varying risk premium associated with oil price risk (see, for instance, Chen, Roll and Ross (1986)).
economic factors do not provide a good hedge for aggregate consumption, thus that there is something other than aggregate consumption that drives the equity risk premium.

4.2 Aggregate Risk Aversion and the Wealth-Consumption Ratio

If \( G(a_t) \) is not entirely dependent on consumption states then what does it depend on? In this subsection we argue that, at least under some circumstances, one can relate \( G(a_t) \) to the wealth-consumption ratio, \( f(a_t) \equiv \frac{w(a_t)}{C(a_t)} \), via:

\[
G(a_t)C(a_t) \propto f(a_t)^\epsilon
\]  

(24)

Suppose, for now, that \( \delta \) in (22) is zero and that \( C(a_t)G(a_t) \) is mean stationary and much smaller than 1. One can therefore describe the economy via a representative agent (see Remark 2) whose relative risk aversion is

\[
\gamma \frac{G(a_t)C(a_t)}{G(a_t)C(a_t) + 1} \approx \gamma G(a_t)C(a_t) \ll 1
\]

The representative agent is not very risk averse with respect to variation in consumption. This assumption is consistent with the empirical fact that there are no active markets for hedging aggregate consumption risk. On the other hand, the agent’s relative risk aversion with respect to changes in \( 1/G(a_t) \) is roughly \( \gamma \). In other words, the representative agent’s hedging needs are concerned mostly with the effect various states have on \( G(a_t) \). To relate \( G(a_t) \) to observable variables, we need to assume some structure. To prove our point, it is sufficient to set \( \gamma = 1 \) and suppose that \( x(a_t) \equiv \ln \left( G(a_t)C(a_t) \right) \) is a Markov process. From Eqn. (23), the pricing kernel is approximately

\[
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t)\beta \left( \frac{C(a_{t+1})}{C(a_t)} e^{x(a_t) - x(a_{t+1})} \right)^{-1}
\]

By letting \( \Delta \ll 1 \), \( \beta \equiv e^{-r_0 \Delta} \) and \( T \gg 1 \), we can make continuous time approximations. In particular, assuming \( x_t \) exhibits mean reversion, one can model it as a continuous time Ornstein-Uhlenbeck process: \( dx_t = \kappa(\mu_X - x_t)dt + \sigma_X dW_t \), with \( dW_t \) a Wiener process. The wealth-consumption ratio can be approximated using Eqn. (20) and \( f(a_t) \equiv \frac{w(a_t)}{C(a_t)} \) to give:

\[
f(a_t) \approx \mathbb{E}_t \left[ \int_0^\infty e^{-r_0 \tau - x_t + x_{t+\tau}} d\tau \right]
\]

\[
\int_0^1 \frac{\tau}{\kappa} e^{-(x_t - \mu_X)(1-y) + \sigma_X^2 \frac{(1-y^2)}{2\kappa}} dy = Ae^{-(1-y^*)x_t}
\]  

(25)
where $y^* \in (0, 1)$ depends on $x_t$. However, plotting $\ln f(x_t)$ against $x_t$ reveals that $y^*$ is nearly constant (i.e., independent of $x_t$). Consequently, one can approximate $G(a_t)C(a_t)$ as in (24). Most importantly, the wealth-consumption ratio is (roughly) observable (e.g., Lettau and Ludvigson (2001ab)), allowing us to approximate the pricing kernel as

$$
\phi(a_{t+1}|a_t) \approx \pi(a_{t+1}|a_t)\beta \left( \frac{C(a_{t+1})}{C(a_t)} \left( \frac{f(a_{t+1})}{f(a_t)} \right)^{\frac{1}{1-y^*}} \right)^{-1}
$$

thereby giving rise to Euler equations that can be used to estimate $y^*$. Note that a corollary of the model described above is that the wealth-consumption ratio must be time varying, else a risk premium would only be applied to consumption states and one obtains an ‘equity premium’ puzzle once more. Equation (26) characterizes a simple and testable asset pricing model with the same number of parameters as the standard Lucas Model with constant relative risk aversion. Unfortunately, the model is guaranteed to be rejected because it features a ‘risk-free rate puzzle’: in this approximate model high volatility in the wealth-consumption ratio will yield large fluctuations in the risk-free rate. As we show in the next section, the problem can be traced to having assumed standard expected utility in how agents treat micro states. In other words, one can hope to alleviate this by setting $\delta(a_{t+1}) \neq 0$.

Regardless, the above argument is simply a motivating illustration of the fact that $G(a_t)$ need not be perfectly adapted to the aggregate consumption filtration, and yet ‘observed’ in the wealth-consumption ratio. Moreover, it lends additional theoretical support to the direct use of Lettau and Ludvigson’s (2001ab) CAY variable as a pricing factor.

### 4.3 A Simple Parameterization

In the previous section we motivated the notion that the unobserved state variable, $X(a_t) = G(a_t)C(a_t)$, is approximately related to the wealth consumption ratio via a simple power law. We now consider a simple model where such a relationship is generically true. Before proceeding, however, note that $X(a_t)$ is a monotonic transform of relative risk aversion when $\delta(a_t) = 0$ (i.e., when one can deduce state prices from a representative agent - see Remark 2). From here on we will refer to $X(a_t)$ simply as ‘relative risk aversion’, whether representative agent pricing is justified or not.

**Assumption 1.** Aside from aggregate consumption, the only other macro-state variable is aggregate relative risk aversion: $X(a_t)$. $X(a_t)$ follows a Hamilton (1989) style two-state Markov switching process. The states at date $t$, are denoted as $s_t \in \{s_+, s_\}$.
Note that this assumption only relates to the evolution of relative risk aversion, \( X(a_t) \). We abuse notation by writing \( X(a_t) \) as \( X(s_t) \), and view \( s_t \) as a label for a two-member partition of the date \( t \) set of possible macro states. Each of these partitions contains states corresponding to different realization of aggregate consumption. We stress that \( X(s_t) \) cannot be directly observed - its variation is due to taste shocks or unforeseen contingencies that are correlated across agents. We cannot say what it is that causes such aggregate shifts, but if they exist we can infer them from prices. It will soon be clear, however, that, as with other models with time varying risk aversion (e.g., habit formation), changes in \( X(a_t) \) are associated with the business cycle.

**Assumption 2.** Consumption growth at date \( t \) follows a random walk with a trend that can depend on \( s_t \). The date \( t + 1 \) shocks to the growth rate are independent of \( s_t \).

The second assumption allows for two important properties: (i) conditionally, \( X(s_t) \) and \( C(a_t) \) are uncorrelated, while (ii) unconditionally, \( X(s_t) \) and \( C(a_t) \) may be correlated.

**Assumption 3.** The pricing kernel takes the form given in Eqn. (22):

\[
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t)\beta \left( \frac{C(a_{t+1})X(s_{t+1}) + 1}{C(a_t)X(s_t) + 1} \left[ 1 - \frac{(1 - e^{-\delta(s_{t+1}|s_t)})}{X(s_{t+1})x(a_{t+1}) + 1} \right] \right)^{-\gamma}
\]

with \( \delta(a_{t+1}) \) a function of \( s_{t+1} \) and \( s_t \) only.

Assumption 3 requires the pricing kernel to be history independent, and in particular, requires \( \delta(a_{t+1}) \) to be independent of contemporaneous consumption shocks. This is both to simplify the analysis, as well as to help differentiate our approach from the literature on habit formation.

Finally we constrain the pricing kernel by requiring that the resulting economy conform to a number of stylized fact

**Assumption 4.**

i) According to the NBER, the typical duration of a U.S. recession is between 4-6 quarters (post-war recessions have been shorter). Typical expansion duration is between 9-17 quarters (post-war expansions have been longer).

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29The ‘stylized’ facts strongly depend on the time period in which they are documented. For instance, Campbell (1999) reports that the mean and standard deviation for the real U.S. risk free rate was, respectively, 1% (2%) and 2% (9%) for the period 1947-1996 (1891-1995).
ii) The unconditional mean of the annual maximum Sharpe ratio is between 0.33-0.66 (half that number per quarter). The maximum Sharpe ratio is counter-cyclical (Campbell (1999)) and is close to twice as much during recessions as during expansions (Lettau and Ludvigson (1999)).

iii) The unconditional mean for U.S. quarterly consumption growth is around 0.5% and the standard deviation is between 0.5-1.5% (Campbell (1999)).

iv) The risk premium on a portfolio that mimics aggregate consumption growth is not statistically different from zero (see the argument in Section 4.1).

v) The quarterly unconditional standard deviation of changes in log aggregate wealth is approximately twice that as for consumption growth and the correlation between log aggregate consumption and log wealth is approximately 50% (Lettau and Ludvigson (1999)).

vi) Virtually all unpredictable changes to aggregate consumption are permanent, while only 15% of unpredictable changes to aggregate wealth are permanent. Moreover, the wealth consumption ratio has far more explanatory power for asset returns than does consumption alone (Lettau and Ludvigson (2001)).

Assumption 5.

i) The unconditional mean and standard deviation for the U.S. quarterly real risk free rate is, respectively, between 0.25 - 0.75% and 0.5 - 4.0% (post-war rates have been low and smooth - Campbell (1999)).

ii) The correlation between the U.S. equity market and real interest rates is low and negative (Campbell (1999)).

Holding $s_t$ constant, the random walk assumption for aggregate consumption implies that the pricing kernel at date $t$ does not depend on the current or previous levels of consumption. In particular, this means that all date $t$ macroeconomic variables are functions of $s_t$ only. In a two state model this has restrictive implications: any two $s_t$-dependent variables will be positively or negatively perfectly correlated. Thus, it is impossible to accommodate the fact that both real interest rates and the wealth consumption ratio vary, while at the same

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30The factor of two change is based on, (i) Lettau and Ludvigson (1999) who report that a one standard deviation change in their predictive variable, CAY, leads to a 220 basis point change in expected excess returns, and (ii) on Campbell (1999) who notes that conditional volatility does not vary much over quarterly horizons.
time allowing for their correlation to be close to zero (i.e., we cannot accommodate both Assumption 4 and Assumption 5). Our compromise solution is to stick to the two state model and ignore Assumption 5 by setting the real rate to be a constant between 0.25 - 0.75%.  

Our first result establishes that, although we have said nothing about what causes variation in aggregate relative risk aversion, \(X(s_t)\), the stylized facts imply that there must be some non-trivial relationship between \(X(s_t)\) and consumption growth. Under assumption 2 this means that the trend in consumption growth is different in the two \(X(s_t)\) states.

**Theorem 3.** Given Assumptions 1-4 and constant interest rates, \(\mu(s_+) \neq \mu(s_-)\). Moreover, the difference between \(\mu(s_+)\) and \(\mu(s_-)\) must be of order \(O(\sigma_c)\), where \(\sigma_c\) is the standard deviation of consumption growth.

**Proof:**

Consider first that assumption 4 implies a time varying wealth-consumption ratio. Since the wealth consumption ratio is at most a function of the switching states, \(s_\pm\), one can express its recursive equation as:

\[
 f(s_t) = 1 + \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) \frac{C(a_{t+1})}{C(a_t)} f(s_{t+1}) \\
 = 1 + \sum_{s_{t+1} \in \{s^+, s^-\}} \left( \sum_{a_{t+1} \subseteq s_{t+1}} \phi(a_{t+1}|a_t) \frac{C(a_{t+1})}{C(a_t)} \right) f(s_{t+1}) \\
 = 1 + \pi(s_+|s_t) \frac{e^{\mu_c(s_+)} f(s_+)}{R_f(1 + \lambda(s_+|s_t))} + \pi(s_-|s_t) \frac{e^{\mu_c(s_-)} f(s_-)}{R_f(1 + \lambda(s_-|s_t))} \tag{27}
\]

where \(\lambda(s_{t+1}|s_t)\) is the risk premium associated with a portfolio that mimics aggregate consumption growth only in the state \(s_{t+1}\) conditional on the current state being \(s_t\) (otherwise, the portfolio pays nothing). After solving these linear equations for \(f(s_\pm)\), log of \(f(s_\pm)\) can be expressed as

\[
 \ln f(s_\pm) \approx A + \pi(s_\mp|s_\pm) \left( \mu_c(s_\pm) - \lambda(s_\mp|s_\pm) \right) \\
 - \pi(s_\mp|s_\mp) \left( \mu_c(s_\mp) - \lambda(s_\mp|s_\pm) \right)
\]

\[31\text{This is also in the spirit of Campbell and Cochrane (1999).}\]
and thus
\[
\ln f(s_+ - \ln f(s_-) = \left( \mu_c(s_+) - \mu_c(s_-) \right) + \pi(s_-|s_-) \lambda(s_-|s_-) + \pi(s_+|s_-) \lambda(s_+|s_-) \\
- \pi(s_-|s_+) \lambda(s_+|s_-) - \pi(s_+|s_+) \lambda(s_-|s_-)
\]

As argued in Section 4.1, Assumption 4 iv requires \( \lambda(s_i|s_j) \ll \mu_c(s_i) \) (the risk premium for aggregate consumption shocks is insignificant). Now, the standard deviation of the log consumption ratio is proportional to \( |\ln f(s_+ - \ln f(s_-)|. \) Thus the difference between \( \mu(s_+) \) and \( \mu(s_-) \) must be of the same order of magnitude as standard deviation of the wealth-consumption ratio; and, since the latter is of the same order of magnitude as \( \sigma_c \) (Assumption 4 v), the Theorem is established.

The primary economic implication of the Theorem is that time variation in the wealth-consumption ratio must be associated with time variation in the expected consumption growth rate - this is a general property that does not depend on the two-state nature of the model. We elaborate on this and other issues below:

1. \( X(s_t) \) and consumption growth must be unconditionally correlated (although shocks to either are contemporaneously uncorrelated). Knowledge about the current expected growth rate of consumption can help to identify the current state of \( X(a_t) \). Conversely, the current state of \( X(a_t) \) can be used to make a more precise forecast of the current growth rate of consumption, but gives no other information about future consumption states. Perhaps the most important implication is that whatever the nature of the factor driving the pricing kernel (i.e., \( X(a_t) \)), it cannot be independent of the state of aggregate consumption. Examination of the proof of the Theorem reveals that this is a direct consequence of (i) the assumption of a low equity premium for shocks to aggregate consumption, and (ii) the assumption of low (or even zero) correlation between the risk free rate and wealth consumption ratio. The Theorem, in fact, can be easily generalized. Regardless of the number of states spanned by \( s_t \), one can write an equation analogous to (27) which effectively says that if \( \lambda(s_{t+1}|s_t) \) and \( R_f(s_t) \) do not vary much with the wealth-consumption ratio, then variations in \( f(s_t) \) can only arise from variations in \( \mu(s_t) \). In the two state model, since shocks to \( X(a_t) \) are perfectly correlated with shocks to expected consumption growth, \( f(a_t) \) is a great proxy for \( X(a_t) \).  

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32 The proportionality constant is less than or equal to \( \frac{1}{2} \).
33 More precisely, one can always find constants, \( A \) and \( \epsilon \) so that \( X(s_t) = Af(s_t)^\epsilon \).
2. The difference between $\mu(s_+) \text{ and } \mu(s_-)$ is sufficiently sizeable to imply that $s_t$ corresponds to the state of the economy within a business cycle. In other words, the model is consistent with Hamilton’s (1989) description of GNP growth both in form as well as in substance. Identifying $s_+$ (resp. $s_-$) with economic expansion (resp. recession), one obtains through Assumption 4(i) the state transition probability matrix. Since in a binary switching model state $j$ has average duration of $\frac{1}{1-P(s_j|s_j)}$, $P(s_+|s_+)$ (continued economic expansion) is between 0.89-0.96, while $P(s_-|s_-)$ (continued economic recession) is between 0.75-0.83.

Although we have not committed to explaining the source of aggregate shifts in risk aversion, assuming that consumption risk has little pricing implication, that the risk-free rate is uncorrelated with the wealth-consumption ratio, and that the log of the wealth-consumption ratio varies more than consumption growth, leads to an association of $f(s_t)$, and therefore $X(s_t)$ with the business cycle. A priori, there is nothing seemingly obvious in these assumptions to link the changes in the wealth consumption ratio to the business cycle. Oddly enough, we motivated our model by arguing that aggregate consumption or its derivatives cannot explain stylized facts, and after deriving a model in which consumption need not play a dominant role we find that, after all, the additional factors cannot be independent of consumption altogether.\textsuperscript{34}

We cannot be totally divorced from a consumption based story; what differentiates our model is that it is not shocks to aggregate consumption that are important, but shocks to variables that have forecasting power with respect to consumption - in particular, business cycle variables.

3. Why has such a relationship not shown up in other models? The answer is that in all other consumption-based macro asset pricing models, shocks to consumption growth (as opposed to shocks to expected growth) drive the risk premium. This is true of the Lucas (1978) model, the Epstein-Zin (1989) model, and the habit formation models (Abel(1990), Constantinides (1990), Sudaresan (1989), and Campbell and Cochrane (1999, 2000)). The latter, in particular, rely on shocks to variables that forecast the past (i.e., consumption histories) instead of variables that forecast the future (i.e., consumption growth).

4. Why is it that shocks to expected consumption are so important to agents in the economy while consumption shocks are not? Our framework does not permit a direct answer

\textsuperscript{34}In fact, in the two state model, the filtration of macro states coincides with that of aggregate consumption: shocks to aggregate consumption and shocks to expected aggregate consumption.
to this. According to our model, the interpretation is that changes in the forecast of consumption growth are somehow connected to changes in the degree of commonality of unforeseen contingencies across agents in the economy. We concede that this is not a completely satisfactory answer, but note that the point of our model is to avoid conjecture and assume that agents in the economy are just as ignorant as we are.

Our next result concerns the form of $e^{-\delta(a_{t+1})}$ in the pricing kernel. Recall that

$$e^{-\delta(a_{t+1})} \equiv \frac{G(a_{t+1})}{F(a_{t+1})G(a_t)} \frac{X(s_{t+1})}{X(s_t)} \frac{C(a_t)}{C(a_{t+1})} \frac{1}{F(a_{t+1})}$$

where $\frac{1}{F(a_{t+1})}$ is the ‘expected’ change in each agent’s risk aversion parameter under the (possibly non-linear) weighting operator $\mathbb{E}_t^i$. The next result establishes that $e^{-\delta(a_{t+1})}$ necessarily does not equal one. As discussed in Section 3, this has very important economic implications: (i) the market is necessarily incomplete, and (ii) investors do not have ‘objective’ expected utility preferences with respect to microstates. States that cannot be hedged and are not priced must exist, and these will have an economically significant impact on states that can be hedged (the economic impact arises due to the subjective or non-linear weighting scheme for microstates in investors’ preferences). It is mainly the following result that justifies our elaborate approach as opposed to one in which a representative agent with random risk aversion is assumed.

**Theorem 4 (Calibration).** $e^{-\delta(a_{t+1})} = 1$ is not consistent with Assumptions 1-4.

**Proof:**

Assume $\delta(a_{t+1}) = 0$ and define

$$P(s_{t+1}|s_t)K_{s_t,s_{t+1}} \equiv \sum_{a_{t+1} \in s_{t+1}} \phi(a_{t+1}|a_t \in s_t)$$

For instance, $P(s_-|s_+)K_{s_+,s_-}$ is the price of a claim that pays $1$ if the economy moves from expansion to recession. Now, since $\delta(a_{t+1}) = 0$, one can use Assumption 3 to write

$$K_{s_+,s_-} = P(s_-|s_+)K_{s_+,s_+} \left( \frac{R_R(s_-)}{R_R(s_+)} \right)^\gamma$$

$$K_{s_-,s_+} = P(s_+|s_-)K_{s_-,s_-} \left( \frac{R_R(s_-)}{R_R(s_+)} \right)^{-\gamma}$$

where $R_R(s_t) \equiv \gamma \frac{X(s_t)}{X(s_t)+1}$. 
The risk-free rate in state $s_t$ is simply $R_f(s_t) \equiv P(s_+|s_t)K_{st,s_+} + P(s_-|s_t)K_{st,s_-}$. The maximum Sharpe Ratio in state $s_t$ is given by the risk-free rate times the standard deviation of the state-price deflator, conditional on $s_t$. Assumption 4(iv) implies that only a small portion of that standard deviation is due to conditional shocks to aggregate consumption; equivalently, it states that in fitting to the large observed Sharpe Ratio, one can make the following approximation (which ignores the contribution due to aggregate consumption shocks):

$$SR(s_t) \approx R_f(s_t) \sqrt{P(s_t|s_t)(1 - P(s_t|s_t))} \bigg| K_{st,+} - K_{st,-}$$

The risk-free rate and Sharpe Ratio equations can be used to derive

$$\frac{SR(s_-)}{SR(s_+)} = \frac{\sqrt{P(s_-|s_-)(1 - P(s_-|s_-))}}{SR(s_+)(2P(s_+ - 1) + \sqrt{P(s_+|s_+)(1 - P(s_+|s_+))})}$$

The unconditional maximum Sharpe Ratio, $SR$, is related to $SR(s_+)$ and $SR(s_-)$ via the unconditional transition probability, $P(+) = \frac{1 - P(s_-|s_-)}{1 - P(s_+|s_-)P(s_-|s_-)}$; i.e., $SR \equiv P(+)SR(s_+) + (1 - P(+)SR(s_-)$. The latter relationship can be used with Eqn. (28) to write $\frac{SR(s_-)}{SR(s_+)}$ as an algebraic function of $SR, P(s_-|s_-)$ and $P(s_+|s_+)$. 

For $P(s_+|s_+) > P(s_-|s_-) > \frac{1}{2}$, this function is decreasing in $SR$ and $P(s_-|s_-)$, and increasing in $P(s_+|s_+)$. Thus within the range of parameters implied by Assumptions 4(i-ii), $\frac{SR(s_-)}{SR(s_+)}$ can be no larger than the value it attains at $P(s_+|s_+) = 16/17, P(s_-|s_-) = 2/3$ and $SR = 0.16$; explicitly calculating this value we conclude that $\frac{SR(s_-)}{SR(s_+)} \leq 1.27$. This is not consistent with Assumption 4(ii). □

We also calculated model parameters constrained by Assumptions 1-4 and constant interest rates. Assuming the intervals for observed macro economic variabled specified in the Assumptions is uniform around the true values, in less than 5% of cases was $\beta < 1$ and $\frac{SR(s_-)}{SR(s_+)} > 1.1$. This gives reassurance that the calibration theorem is robust to a great deal of error in measuring the degree of predicatibility in expected returns (i.e., errors in measuring $\frac{SR(s_-)}{SR(s_+)}$).

**Remark 4.** What is wrong with the simplified model of Section 4.2? The answer is two fold: as mentioned in that section, the implied variation in the risk free rate are too large, and

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35 The resulting algebraic equation is too tedious to write, but can be easily derived using standard symbolic algebra software.

36 The uniform distribution assumption made with respect to the confidence intervals of the observed macroeconomic moments is conservative. For a point estimate, (i.e., the center of the confidence intervals) $\delta(x_{t+1}) = 0$ is soundly rejected; thus, assuming the confidence intervals conformed to that given by a multivariate normal distribution would tend to greatly reduce the frequency at which $\beta < 1$ and $\frac{SR(s_-)}{SR(s_+)} > 1.1$. 

34
the magnitude of its correlation with the wealth-consumption ratio is too high. Essentially, one cannot have a high equity risk premium with substantial time variation, low premium for aggregate consumption shocks, low correlation between the wealth-consumption and risk free rate, and set $\delta(a_{t+1}) = 0$.

Our model gives little guidance as to the structural form of $e^{-\delta(a_{t+1})}$ when it is not equal to one. This is not to say that any guess will do the trick. We now provide an example of a pricing kernel consistent with Assumptions 1-4 and constant interest rates. We assume that

$$z(s_t) \equiv X(s_t)f(s_t)$$

and look for model parameters that rationalize this assumption and the stylized facts. The pricing kernel simplifies to

$$\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t)\beta\left(\frac{C(a_{t+1})}{C(a_t)} \frac{X(a_{t+1})}{X(a_t)} + 1 \frac{X(a_t)}{X(a_{t+1})} \left(1 + e^{-\delta(s_{t+1}|s_t)}\right)\right)^{-\gamma}$$

There are four possible values that $\delta(s_{t+1}|s_t)$ may take on, two values for $X(s_{t+1})$, and two remaining parameters in $\gamma$ and $\beta$. Although this may seem to be a great number of degrees of freedom, the system is actually over-constrained by the stylized facts. First note that, since $C(a_{t+1})/C(a_t)$ is conditionally independent of $s_{t+1}$, the risk premium for consumption shocks is entirely fixed by the standard deviation of $\left(\frac{C(a_{t+1})}{C(a_t)}\right)^{-\gamma}$. In principle, this pins down $\gamma$. That leaves seven parameters that may be fit to nine derivative variables: the maximal Sharpe Ratios, $SR(s_{t+1})$, the risk free return, $R_f(s_{t+1}) \equiv R_f$, the constraint from Eqn. (16) (i.e., $z(s_{t+1}) \equiv X(s_{t+1})f(s_{t+1})$), and finally, the stylized facts in Assumption 4($v - vi$); the latter relate the unconditional standard deviation of wealth to that of consumption, fix the variance decomposition of changes in aggregate wealth, and pin down the correlation between aggregate consumption and aggregate wealth.

Table 1 summarizes our parameter assumptions and the consequent values for the macro-economic variables in different macro states. The transition probabilities are Hamilton’s (1989) estimates for his GNP model. From these, we calculate the average numbers of recession (expansion) quarters ($\tau(s_{t+1})$ in the table) to be 4.1 (10.5). The consumption growth parameters can, in principle, be estimated as in Hamilton (1989), although here we simply ‘guess’ at their values: during recessions consumption typically shrinks by 0.5% quarterly while during expansionary times the economy grows by an average of 1% quarterly.\footnote{These growth rates are consistent with GNP estimates for Hamilton’s (1989) data.}
fix the conditional quarterly standard deviation for log-consumption growth at $\sigma_c = 0.8\%$ (i.e., the volatility of the random walk), so that together with our other assumptions, the quarterly unconditional standard deviation for consumption growth is $\sigma^{uncond}_c = 1.0\%$ (consistent with stylized facts); likewise, the quarterly unconditional growth rate of consumption is $\mu^{uncond}_c = 1\%$ (consistent with stylized facts); likewise, the quarterly unconditional growth rate of consumption is $\mu^{uncond}_c = 1\%$. The relative risk aversion parameter, $\gamma$ is set to 1. This ensures a low risk premium associated with consumption shocks ($\lambda_c = 1$ basis point), effectively guaranteeing the absence of a hedging market for aggregate consumption.

The remaining parameters of the model ($X(s_t), \delta(s_t|s_t)$ and $\beta$) are chosen to (roughly) match the countercyclical maximal Sharpe Ratios in the two states of the business cycle (i.e., $SR(s_t)$), the constant risk free rate (set at $R_f = 1.005$ per quarter), the constraints $z(s_t) = X(s_t)f(s_t), \beta < 1$, and the remaining stylized facts from Assumption 4 ($v-vi$). $X(s_t)$, the other component of relative risk aversion, varies between 5 (expansory phases), and 7.23 (recessions). Alternatively, $R_R \equiv \gamma \frac{X(s_t)}{1+X(s_t)}$ varies between 0.83 and 0.88. The standard deviation of $\ln(X(s_t))$ (i.e., $\sigma_z$) is tabulated in the two states: $X(s_t)$ is higher and more volatile during recessions with overall unconditional quarterly standard deviation of 16.6%. Note that $\ln R_R(s_t)$, by contrast, has a substantially lower standard deviation.

The Sharpe ratio is set well within bounds set by empirical observations. The wealth-consumption ratio is also calculated. Its level is high ($\sim 100$) due to the small difference between the risk-free rate and unconditional mean of consumption growth. We also calculate the standard deviation (conditional and unconditional) of $f(s_t)$ and $w(s_t)$. The unconditional standard deviation of wealth is much higher than the conditional moments due to the fact that conditionally, consumption is uncorrelated with the wealth-consumption ratio, but unconditionally, there is an important relationship. Unconditionally, the standard deviation of log-wealth is calculated to be roughly thrice that of aggregate consumption - this is different from the factor of two implied by Assumption 4 ($v$), but given the small magnitudes in question, not sufficiently different to reject the model outright.

Empirically, the wealth-consumption ratio cannot be observed directly; assuming wealth is co-integrated with consumption, Lettau and Ludvigson (1999, 2001) approximately measure the standard deviation of log aggregate wealth and the log wealth-consumption ratio, 38 Again, recall that relative risk aversion for a representative agent with $\delta(a_t) = 0$ is $R_R(s_t) = \gamma \frac{X(s_t)}{1+X(s_t)}$. Since $\delta(a_t) \neq 0$, we cannot truly speak of a representative agent (markets are incomplete), but one may still draw intuition from the case $\delta(a_t) = 0$.

39 Recall that in the standard model the wealth consumption ratio is $\frac{dr}{r_f + \lambda_c - \mu_e}$, where $dt$ is the unit of time over which the variables are measured (a quarter, in this case).
as well as its correlation between these variables and log consumption growth. Consistent with their findings, we calculate that most (i.e., $1 - 12.4\% = 87.6\%$) of the contribution to the variance of log wealth comes from the wealth-consumption ratio whose variation is transitory due to its stationarity. Both log-wealth and the log wealth-consumption ratio vary more in recessions, during which the wealth-consumption ratio is below average. The conditional correlation between the wealth-consumption ratio and consumption growth is, by construction, zero; the unconditional correlation, on the other hand, is calculated to be $69.1\%$ – above the approximate value of $\sim 53\%$ derived from Lettau and Ludvigson (2001). Overall, the model roughly fits all the stylized facts of Assumption 4.

5 Concluding Remarks: An Empirically testable formulation of the pricing kernel

Our main goal in this paper is to argue that a sensible asset pricing theory can be constructed from decision-theoretic primitives that involves factors other than pure aggregate consumption, and yet does not commit to the identification of such factors. The stylized models considered in Section 4 derive from a general equilibrium in which heterogeneous agents experience aggregate consumption risk as well as unforeseen contingencies. A large component of macro risk arises from commonalities across unforeseen contingencies faces by agents. These shocks need not be conditionally related to innovations in aggregate consumption, but must be related to innovations in the expected growth rate of aggregate consumption. Thus, at least according to our model, changes in expected consumption growth rates play an important role in asset pricing. In addition, the wealth-consumption ratio directly proxies for variations in expected growth rates and therefore can be used as a proxy for non-consumption factors in the pricing kernel.

In the previous section, we assumed that aside from shocks to consumption, the only relevant pricing factor corresponds to the aggregate relative risk aversion factor, $X(a_t)$. Moreover, in the two state model, the wealth-consumption ratio $f(a_t)$ can be tautologically set to equal $AX(a_t)\epsilon$. The calibrated model is, of course, stylized, and cannot be used but to illustrate that many empirical facts can be parsimoniously captured. In particular, $\delta(a_{t+1})$ is not observable, so it is impossible to conduct further cross-sectional tests of the two-state model. However, one thing the models in Section 4.2 and 4.3 do suggest, is that, at least at

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40 The difference is only marginally significant.
a crude level, one is not entirely in error by insisting on the identification \( f(a_t) = AX(a_t)^\epsilon \). If, in addition, one postulates that \( F(a_t) = 1 \), then \( z(a_t) \) is identified as the true price of a console bond. In both cases the pricing kernel is completely parameterized in terms of observed (or approximately observed) macro economic variables: aggregate consumption, aggregate wealth, and the price of a console bond. These assumptions directly lead to a non-trivial yet completely identified pricing kernel:

\[
\phi(a_{t+1}|a_t) \approx \pi(a_{t+1}|a_t) \beta \left( \frac{Af_t^\epsilon + 1}{Af_t^\epsilon + 1} \frac{Ay_{t+1}f_{t+1} \frac{C_{t+1}}{C_t} f_t^\epsilon + 1}{Ay_{t+1}f_{t+1}^{1+\epsilon} + 1} \right)^{-\gamma}
\]

\( y_{t+1} \equiv \frac{1}{z_{t+1}} \) is the yield to maturity on a real console bond. This expression involves two more parameters to estimate than the usual CRRA representative agent expression and can be used to construct the usual Euler equation tests. Note that, everything else being equal, high yield to maturity tends to increase the effective relative risk aversion with respect to changes in the wealth consumption ratio and aggregate consumption. Moreover, the derivative of the kernel with respect to \( y_{t+1} \) is proportional to \( 1 - \frac{C_{t+1}}{C_t} f_{t+1}^\epsilon \). The latter can also be interpreted as being proportional to the factor loading of \( y_{t+1} \) on the state price deflator. In particular, the sign of the market price of risk for \( y_{t+1} \) is roughly the sign of \( \frac{C_{t+1}}{C_t} \frac{f_t^\epsilon}{f_{t+1}^\epsilon} - 1 \). Assuming the wealth-consumption ratio is stationary, the latter is counter-cyclical. In other words, excess returns on long maturity bonds are highest during ‘good’ times and may even be negative during ‘bad’ times.
Appendix:

Proof of Proposition 1:

The argument is inductive. At macro state \( a_{t+1} \in \mathcal{F}_{t+1} \) and microstate \( s^i_{t+1} \) we conjecture the form of the indirect utility function in Eqn. (15), and assume that at date \( t \) the investor maximizes a derived utility as in Eq. (7). Equations (9)-(11), lead to utility in event \( a_{t+1} \in \mathcal{F}_{t+1} \) of

\[
U^s_t(c_t^i, w_{t+1}^i, a_{t+1}) = \frac{(s^i_t + 1)^{1-\gamma}}{s^i_t (1 - \gamma)} \Delta + \beta \rho(a_{t+1}) z(a_{t+1}) \frac{(s^i_{t+1} F(a_{t+1}) + 1)^{1-\gamma}}{s^i_t (1 - \gamma) F(a_{t+1})^{1-\gamma}} \tag{29}
\]

where \( w_{t+1}^i \) is the total wealth with which agent \( i \) begins the period in state \( a_{t+1} \):

\[
w_{t+1}^i \equiv c^i_t(a_{t+1}) \Delta + \sum_{a_s \subseteq a_{t+1} \text{ s.t. } s > t} c^i_s(a_s) \phi(a_s|a_{t+1}) \Delta \tag{30}
\]

Under the model assumptions it can be readily verified that Eqn. (29) holds for date \( T - 1 \) by making the identification, \( \rho(a_T) = \Delta, z(a_T) = \Delta \) and \( w_T^i = c_T^i \).

The agent’s optimization program is as follows:

\[
V^*_t s^i_t (w_t^i, a_t) = \max_{\lambda, c_t^i, \{c_s^i(a_s)\}} \left\{ \sum_{a_{t+1} \subseteq a_t} \pi(a_{t+1}|a_t) U^s_t \left( c_t^i, w_{t+1}^i, a_{t+1} \right) - \lambda \left( c_t^i \Delta + \sum_{a_s \subseteq a_t \text{ s.t. } s > t} c^i_s(a_s) \phi(a_s|a_t) \Delta - w_t^i \right) \right\} \tag{31}
\]

where \( \lambda \) is a Lagrange multiplier. The Lagrange multiplier can be eliminated from the first order conditions for \( c_t^i(a_{t+1}) \) and \( c_t^i \) to give

\[
\phi(a_{t+1}|a_t) = \pi(a_{t+1}|a_t) \frac{\partial}{\partial c} U^s_t \left( c_t^i, w_{t+1}^i, a_{t+1} \right) \tag{32}
\]

Eq. (32) can be written as

\[
\phi(a_{t+1}|a_t) = \beta \pi(a_{t+1}|a_t) \rho(a_{t+1}) F(a_{t+1})^\gamma \left( \frac{s^i_{t+1} F(a_{t+1}) + 1}{s^i_t c_t^i + 1} \right) - \gamma \tag{33}
\]
By solving for \( (s^i_t F(a_{t+1}) \frac{w^i_{t+1}}{z(a_{t+1})} + 1)^{1-\gamma} \) one can use (29) to write the optimized utility from Eq. (31) as,

\[
V_t^* s^i_t(w^i_t, a_t) = \left( \frac{s^i_t c^i_t + 1}{s^i_t(1-\gamma)} \right) (\Delta + \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{\gamma}})
\] (34)

The left hand side is not an explicit function of \( w^i_t \). To fix this, note that Eq. (33) can be manipulated to give

\[
F(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{\gamma}} (s^i_t c^i_t + 1) = s^i_t F(a_{t+1}) \frac{w^i_{t+1}}{z(a_{t+1})} + 1
\] (35)

Multiplying by \( \frac{\phi(a_{t+1}|a_t)z(a_{t+1})}{F(a_{t+1})} \) and then summing over state prices gives (with the help of the budget constraint):

\[
(c^i_t s^i_t + 1) \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{\gamma}} = \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) + s^i_t (w^i_t - c^i_t \Delta)
\] (36)

The above can be solved for \( c^i_t \) in terms of \( w^i_t \) and the results used in (34). By defining \( z(a_t) \) and \( \rho(a_t) \) as in (16)-(17), and after some manipulation, one derives (14) which leads directly to the conjectured form in (15).

**Proof of Proposition 2:**

Using the fact that the events, \( \{a_s\} \) for \( s > t \) partition the state space, the first order condition for \( c^i_t(a_s) \) for \( s > t + 1 \) in (31) can be written as

\[
\phi(a_{s}|a_t) = \pi(a_{t+1}|a_t) \phi(a_{s}|a_{t+1}) \frac{\partial}{\partial w^i_t} U^a_t(c^i_t(a_t), w^i_{t+1})
\] (37)

Substituting from Eq. (32) and rearranging, the last equation becomes

\[
\phi(a_{s}|a_{t+1}) = \frac{\phi(a_{s}|a_t)}{\phi(a_{t+1}|a_t)}
\] (38)

Setting \( \phi(a_{s}) \equiv \phi(a_{s}|a_1) \), these equations can be used to inductively derive Eq. (1).

It is now possible to return to directly analyze the state prices. Specifically, Eqn. (36) can be written as

\[
(c^i_t + \frac{1}{s^i_t}) \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{\gamma}} = \frac{1}{s^i_t} \sum_{a_{t+1} \subseteq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \frac{w^i_{t+1}}{F(a_{t+1})} + (w^i_t - c^i_t \Delta)
\] (39)
Proof of Theorem 2:

From Equations (40)-(41), one can derive, respectively:

\[ \frac{1}{G(t)} \sum_{a_{t+1} \leq a_t} \phi(a_{t+1}|a_t) z(a_{t+1}) \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{\gamma}} = \frac{1}{G(t)} \sum_{a_{t+1} \leq a_t} \frac{\phi(a_{t+1}|a_t) z(a_{t+1})}{F(a_{t+1})} + \left( w(a_t) - C(a_t) \Delta \right) \]

where \( w(a_t) \) is aggregate per-capita wealth and \( \frac{1}{G(t)} \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{1}{s_i} \). In the same way, one can also manipulate Eqn. (35) to give:

\[ F(a_{t+1}) (C(a_t) + \frac{1}{G(t)} \left( \frac{\phi(a_{t+1}|a_t)}{\beta \pi(a_{t+1}|a_t) \rho(a_{t+1})} \right)^{-\frac{1}{\gamma}} = \frac{1}{G(t)} \frac{w(a_{t+1})}{z(a_{t+1})} F(a_{t+1}) \]

From Equations (40)-(41), one can derive, respectively:

\[ \rho(a_t)^{-\frac{1}{\gamma}} = G(t)C(a_t) + 1 \]

\[ \phi(a_{t+1}|a_t) = \beta \pi(a_{t+1}|a_t) \rho(a_{t+1}) \left( \frac{G(t) \frac{w(a_{t+1})}{z(a_{t+1})} + \frac{1}{F(a_{t+1})}}{G(a_t) C(a_t) + 1} \right)^{-\gamma} \]

Proof of Theorem 2:

Taking Eqn. (42) plugging it into (43), state prices are given by:

\[ \phi(a_{t+1}|a_t) = \beta \pi(a_{t+1}|a_t) \left( \frac{G(a_{t+1}) C(a_{t+1}) + 1}{G(a_t) C(a_t) + 1} \right) \left( \frac{G(a_t) \frac{w(a_{t+1})}{z(a_{t+1})} + \frac{1}{F(a_{t+1})}}{G(a_{t+1}) \frac{w(a_{t+1})}{z(a_{t+1})} + 1} \right)^{-\gamma} \]

Setting \( \frac{F(a_{t+1}) G(a_{t+1})}{G(a_t)} \equiv e^{-\delta(a_{t+1})} \), the last expression gives Eq. (22):

\[ \phi(a_{t+1}|a_t) = \beta \pi(a_{t+1}|a_t) \left( \frac{G(a_{t+1}) C(a_{t+1}) + 1}{G(a_t) C(a_t) + 1} \right) \left( \frac{G(a_t) \frac{w(a_{t+1})}{z(a_{t+1})} + \frac{1}{F(a_{t+1})}}{G(a_{t+1}) \frac{w(a_{t+1})}{z(a_{t+1})} + 1} \right)^{-\gamma} \]
References


