CONDITIONAL DOMINANCE IN GAMES WITH UNAWARENESS*

Martin Meier† Burkhard C. Schipper‡

Preliminary: February 5, 2012

Abstract

Heifetz, Meier and Schipper (2011a) introduced generalized extensive-form games that allow for mutual unawareness. Here, we define the normal form of a generalized extensive-form game. The generalized normal-form game associated to a generalized extensive-form game may now consist of a collection of normal-form games. We use it to characterize extensive-form rationalizability (resp. prudent rationalizability) in generalized extensive-form games by iterative conditional strict (resp. weak) dominance in the associated generalized normal-form. We also show that the analogue to iterated admissibility for generalized normal-form games is not independent of extensive-form structure. This is because a player's information set not only determines which nodes he considers possible but also of which game tree(s) he is aware of.

Keywords: Unawareness, extensive-form games, normal-form games, extensive-form rationalizability, prudent rationalizability, iterated conditional dominance, iterated admissibility.

JEL-Classifications: C70, C72, D80, D82.

---

*Some of the material was previously part of the paper "Dynamic Unawareness and Rationalizable Behavior". We thank Aviad Heifetz for his substantial input. Moreover, we are grateful to Pierpaolo Battigalli and Amanda Friedenberg for helpful discussions. Martin was supported by the Spanish Ministerio de Educación y Ciencia via a Ramon y Cajal Fellowship (IAE-CSIC) and a Research Grant (SEJ 2006-02079). Burkhard is grateful for financial support from the NSF SES-0647811.

†Institut für Höhere Studien, Wien, and Instituto de Análisis Económico - CSIC, Barcelona. Email: martin.meier@ihs.ac.at

‡Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu

1 Introduction

We introduce the normal form associated to generalized extensive-form games with unawareness and study to what extent solution concepts to generalized extensive-form games can be characterized in the associated normal-form game. Intuitively, the extensive form is a more complete description of the strategic situation than the normal form since it specifies precisely the time dimension of the rules of the game. When does which player move? What and when does which player know about those moves etc.? Nevertheless it has been argued (e.g. Kohlberg and Mertens, 1986) that the normal form contains all strategically relevant information. Shimoji and Watson (1998) were able to characterize Pearce (1984) extensive-form rationalizability, a solution concept that embodies sophisticated extensive-form reasoning, with iterated conditional strict dominance in the associated normal-form game. Conceptually, extensive-form rationalizability is preferable over iterated conditional strict dominance since former is closer to actual reasoning that players may do in a game. It is conceivable that players' form beliefs about other's strategies and use them to rationalize their own strategies. Players then assume that other players think similarly, and this is used to form refined beliefs about the other's rational strategies and so on. In contrast, iterated conditional strict dominance is a mechanistic algorithm of eliminating strategies whose epistemic interpretation is less obvious. Yet, as Shimoji and Watson (1998) show both solution concepts select the same strategies at every level of iteration.[1] Moreover, Chen and Micali (2012) showed that iterated conditional strict dominance is order-independent. Thus, iterated conditional strict dominance is a "practical" algorithm for finding extensive-form rationalizable strategies.

In dynamic games with unawareness, this time dimension seems to be even more important since it specifies when a player's entire perception of the game may change, in which case she will have to reconsider what players did and knew, what they are able to do in future, and when they will know what. That is, generalized extensive-form games with unawareness formalize the changes in the awareness of players as well. The question is whether a sophisticated solution concept like extensive-form rationalizability adapted to generalized extensive-form games can still be characterized by iterated conditional strict dominance in a generalized associated normal-form. In this paper, we show that this is still the case. But in order to answer the question we have define the appropriate notion of normal form and extend the notion of iterated conditional strict dominance.

To provide some intuition, consider first the following standard "battle-of-the-sexes"

---

[1] That's probably why some authors like Brandenburger and Friedenberg (2011) don't even distinguish between these two concepts.

game where Bach and Stravinsky concerts are the two available choices for each player

		II	
		B	S
I	B	3, 1	0, 0
	S	0, 0	1, 3

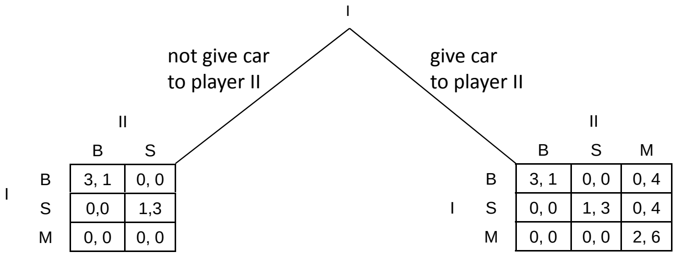
augmented by a dominant Mozart concert for player II:

			II	
		B	S	M
	B	3, 1	0, 0	0, 4
I	S	0, 0	1, 3	0, 4
	M	0, 0	0, 0	2, 6

The new game is dominance solvable, and (M,M) is the unique Nash equilibrium.

Suppose that the Mozart concert is in a distant town, and II can go there only if player I gives him her car in the first place: Here, if player I doesn't give the car to player II, player II may conclude by forward induction that player I would go to the Bach concert with the hope of getting the payoff 3 (because by giving the car to II, player I could have achieved the payoff 2). The best reply of player II is to follow suit and attend the Bach concert as well. Hence, in the unique rationalizable outcome, player I is not to give the car to player II and to go to the Bach concert.[2]

Figure 1:

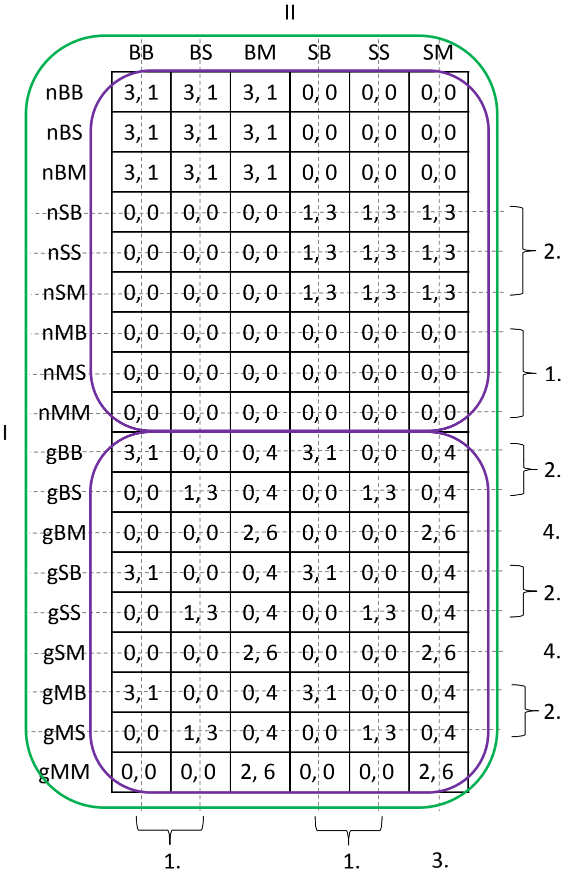


How is this solution captured by iterated conditional strict dominance? Figure 2 shows the associated normal-form game. Each action in the normal form is a strategy in

---

[2] See Heifetz, Meier, and Schipper (2011a) for further discussion of this example. For a discussion of forward induction in battle-of-the-sexes games see van Damme (1989).

Figure 2:



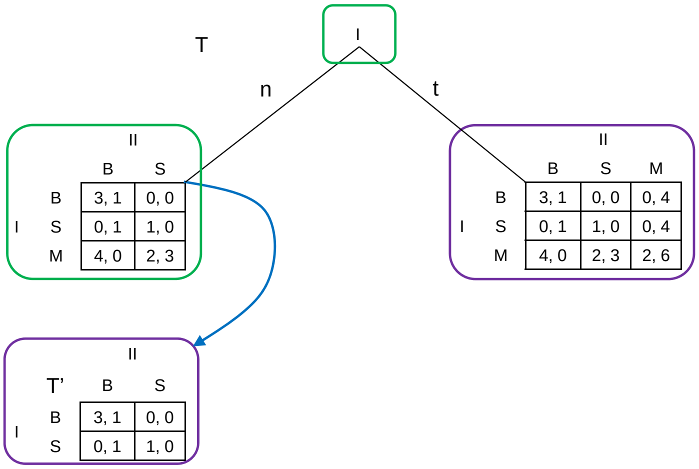
	BB	BS	BM	SB	SS	SM
nBB	3, 1	3, 1	3, 1	0, 0	0, 0	0, 0
nBS	3, 1	3, 1	3, 1	0, 0	0, 0	0, 0
nBM	3, 1	3, 1	3, 1	0, 0	0, 0	0, 0
nSB	0, 0	0, 0	0, 0	1, 3	1, 3	1, 3
nSS	0, 0	0, 0	0, 0	1, 3	1, 3	1, 3
nSM	0, 0	0, 0	0, 0	1, 3	1, 3	1, 3
nMB	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
nMS	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
nMM	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
gBB	3, 1	0, 0	0, 4	3, 1	0, 0	0, 4
gBS	0, 0	1, 3	0, 4	0, 0	1, 3	0, 4
gBM	0, 0	0, 0	2, 6	0, 0	0, 0	2, 6
gSB	3, 1	0, 0	0, 4	3, 1	0, 0	0, 4
gSS	0, 0	1, 3	0, 4	0, 0	1, 3	0, 4
gSM	0, 0	0, 0	2, 6	0, 0	0, 0	2, 6
gMB	3, 1	0, 0	0, 4	3, 1	0, 0	0, 4
gMS	0, 0	1, 3	0, 4	0, 0	1, 3	0, 4
gMM	0, 0	0, 0	2, 6	0, 0	0, 0	2, 6

the extensive form. E.g., nBB assigns the action "not give the car to player II" to player I's first information set (i.e. the root of the tree in Figure 1), B to the left game and B to the right game. In Figure 2 we also indicated the three normal-form information sets with rectangles. A normal-form information set is a subspace of strategy profiles that correspond to an information set in the extensive-form. The outer green rectangle corresponds to the root of the tree, an information of player I. The smaller purple rectangles correspond to the left and right subgame respectively. They are information sets of both players.

With the dashed lines and the numbers beside them, we indicated the order of iterative elimination of conditional strictly dominated strategies. For instance, at the first level, for player II strategy BS is strictly dominated by strategy BM *conditional* on being in the lower normal-form information set. Notice that it is not strictly dominated per se. Thus, conditioning on the lower normal-form information set is crucial. As is shown in Figure 2, iterative conditional strict dominance singles out the forward induction outcome of extensive-form rationalizability. This is just an example of the general result by Shimoji and Watson (1998).

How to apply such a procedure to dynamic games with unawareness? An example of such a game is shown in Figure 3. There are now two trees, an upper tree T and the lower game T' (i.e., the tree T').[3] This lower game T' is a partial description of the upper tree T. Any information set that belongs to both players is purple. The information sets that belong only to player I are green. Finally, when player I chooses n, then player II's information set is the purple information set in three T'. We indicate this with the blue arrow.

Figure 3: Example of a Generalized Extensive-form Game



By taking action n at the root of the tree T, player I can keep player II unaware of his action M since the information set of player II is at the lower tree T', a 2×2 game. If instead player I chooses t at the root of the upper tree T, then he makes player II aware of action M and the information set of player II is now at the right side of the upper tree T. At this information set player II knows that player I told him about action M but he also realizes if player I would not tell him about M, then he would be unaware of it and view the game as in the lower tree T'.

What is the associated normal form? For standard extensive-form games, the as-

---

[3] This game is similar to the Battle of the Sexes with unawareness in Heifetz, Meier, and Schipper (2011a). However, we changed the payoffs in order to demonstrate a particular feature of iterated conditional strict dominance in games with unawareness.

sociated normal-form game is the strategic game in which the players' actions are the strategies in the extensive-form game. A strategy assigns to each of the player's information set an action. Since in a standard extensive-form game, every player is aware of all actions, in principle he can "control" his entire strategy ex ante. In generalized extensive-form games, for each information set of a player her strategy specifies – from the point of view of the modeler – what the player would do if and when that information set of hers is ever reached. In this sense, a player does not necessarily 'own' her full strategy at the beginning of the game, because she might not be initially aware of all of her information sets. For instance, in Figure 3 a strategy of player II assigns an action to the right game in the upper tree T and an action to the lower game T'. A strategy for player I assigns an action to the root of the upper tree T, the left upper game, the right upper game and the lower tree T'. When players face the game in the lower tree T', then they choose partial strategies, i.e. strategies restricted to information sets in T'. For instance, the set of T'-partial strategies of player I is $\{B, S\}$. With this notion of strategy, we can define the associated generalized normal-form as usual. But notice that we get two normal-form games, one for the strategies and one for the T'-partial strategies. This is shown in Figure 4. Strategy $nMSB$ of player I means that he chooses n at the root of tree T, M in the left game of three T, S in the right game of tree T , and B in the lower tree T'. Strategy BS of player II means that he chooses B in the upper right game of tree T and S in the lower tree T'.

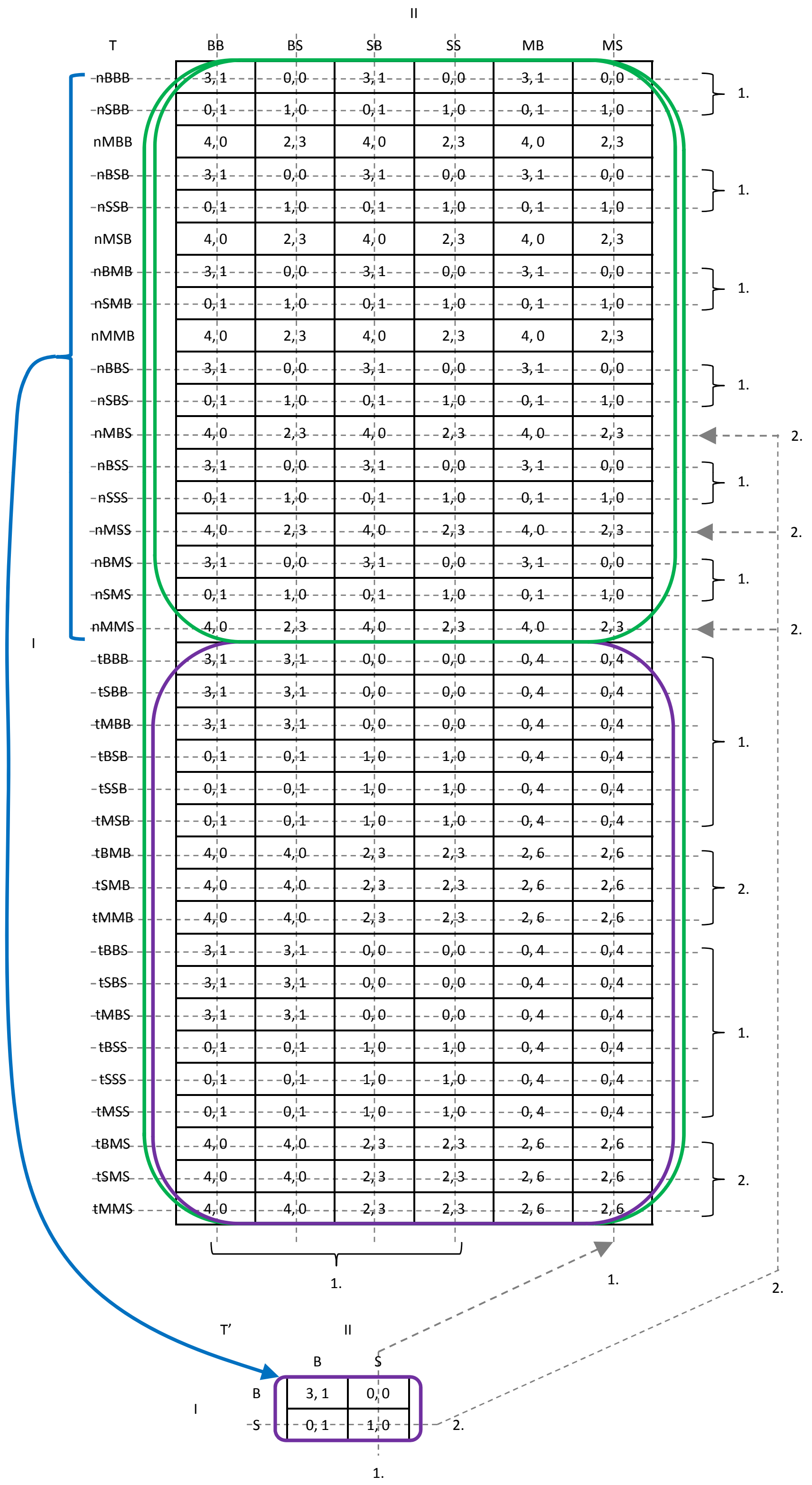
We also indicated the normal-form information sets. As before, green belongs to player I only, purple to both players, and the blue arrow indicates that when player I chooses any strategy with action n, then the player II is unaware of M and his information set is in the lower tree T'.

With the dashed lines and the numbers beside them, we indicate the order of iterative elimination of conditional strictly dominated strategies. But this algorithm is now more subtle since conditional dominance of a T' partial strategy may imply that all strategies in the game T with the same components are deleted as well. In the first round, this is the case for player II where deletion of S in the game T' implies that all strategies with S as the second component in the game T are eliminated as well. In particular this applies to strategy MS that is not otherwise conditionally strictly dominated in the upper normal form associated to T.[4] We indicate this with dashed arrows. A similar case arises for player II in the second round. After two rounds, the process stops. The

---

[4] This is the reason for presenting this example similar to the one in Heifetz, Meier, and Schipper (2011a) but with different payoffs. These different payoffs allow us to illustrate the elimination of a strategy in the upper normal form because it is conditionally dominated in the lower normal form.

Figure 4: Associated Generalized Normal-form Game of the Example



T	BB	BS	SB	SS	MB	MS
nBBB	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSBB	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMBB	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBSB	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSSB	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMSB	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBMB	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSMB	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMMB	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBBS	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSBS	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMBS	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBSS	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSSS	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMSS	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
nBMS	3, 1	0, 0	3, 1	0, 0	3, 1	0, 0
nSMS	0, 1	1, 0	0, 1	1, 0	0, 1	1, 0
nMMS	4, 0	2, 3	4, 0	2, 3	4, 0	2, 3
tBBB	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tSBB	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tMBB	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tBSB	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tSSB	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tMSB	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tBMB	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tSMB	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tMMB	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tBBS	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tSBS	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tMBS	3, 1	3, 1	0, 0	0, 0	0, 4	0, 4
tBSS	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tSSS	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tMSS	0, 1	0, 1	1, 0	1, 0	0, 4	0, 4
tBMS	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tSMS	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6
tMMS	4, 0	4, 0	2, 3	2, 3	2, 6	2, 6

T'	B	S
B	3, 1	0, 0
S	0, 1	1, 0

remaining strategies are the extensive-form rationalizabile strategies. More precisely, each round of elimination of conditionally strictly dominated strategies corresponds to the corresponding level of extensive-form rationalization. In Section 3, we show that this is generally the case.

The notion of extensive-form rationalizability was introduced by Pearce (1984). Since then the literature has slightly modified Pearce's original notion. Pearce's original definition is quite difficult to access and no text book treatment has been available. Battigalli (1997) presented an alternative definition that differs from Pearce's (1984) in at least two respects. First, Battigalli allows for correlated beliefs over opponents' strategies while Pearce required independence. Second, Battigalli's defined a procedure of iteratively eliminating beliefs. In contrast, Pearce defined a procedure of iteratively eliminating strategies. Battigalli shows that when one allows for correlation in Pearce's original definition, both his and Pearce's notion are equivalent. However, we believe that an elimination procedure of beliefs has the conceptual advantage of being closer to a reasoning procedure.

In Heifetz, Meier, and Schipper (2011b) we introduced a further modification. Besides generalizing extensive-form rationalizability to generalized extensive-form games, they also slightly enhanced the definition of rationality. In Pearce (1984) as well as in Battigalli (1997), the definition of rationality of a strategy is mute regarding information sets that the strategy itself rules out. Yet, if an information set is interpreted as a player's state of mind, then when evaluating a strategy the player should take any information set serious. Our notion of rationality, which we call here "would-be rationality" only in order to distinguish it from Pearce's notion (although we simply called it "rationality" in Heifetz, Meier, and Schipper, 2011a, b), requires the strategy to be rational at all information sets. We will show in Section 2 that this leads to a refinement of extensive-form rationalizability in terms of strategies but induces the same set of plans of actions.

Motivated by dynamic unawareness, Heifetz, Meier, and Schipper (2011b) introduced a refinement of extensive-form rationalizability. In generalized extensive-form games with unawareness, extensive-form rationalizability may involve imprudent behavior. It may be rationalizable for a player to make an opponent aware of one of the opponent's actions that is extremely bad for the player because the player believes that the opponent won't take this action. Using the idea of prudence, we introduced a version of extensive-form rationalizability that rules out such behavior. This solution concept, called prudent rationalizability, could be viewed as an extensive-form analogue of iterated admissibility in dynamic games with unawareness. While iterated admissibility is known to be a refinement of (normal-form) rationalizability in normal-form games, prudent rationalizability is surprisingly *not* always a refinement of extensive-form rationalizability in terms of strategies (not even for standard extensive-form games). However, Heifetz, Meier, and Schipper (2011b) proved that prudent rationalizable strategies do refine the set of *outcomes* obtainable by extensive-form rationalizable strategies. In Section 5, we characterize prudent rationalizability in generalized extensive-form games by iterated conditional elimination of weakly dominated strategies in the associated generalized normal-form game.

One could argue that the characterization of extensive-form rationalizability (resp. prudent rationalizability) by iterative conditional strict (resp. weak) dominance falls short of showing that one can capture sophisticated extensive-form reasoning in the normal-form. This is because we condition on normal-form information sets corresponding to information sets in the extensive-form game. Thus, we implicitly make use of extensive-form structures in the definition of iterative conditional strict (resp. weak) dominance. In standard game, the remedy would be to use iterative admissibility. Indeed, Brandenburger and Friedenberg (2011) show that for standard extensive-form games with perfect recall, iterated admissibility conincides with iterative conditional weak dominance at every level. Thus, in those games prudent rationalizability and iterated admissibility in the associated normal-form game coincides at every level. However, as we demonstrate, for generalized extensive-form games with unawareness the appropriate definition of iterative admissibility must make use of information sets as well. This is due to the fact that information sets in generalized extensive-form games do not only model a player's information in the standard sense but also his awareness. The player's awareness of strategies is crucial for admissibility since implicitly a player considers possible any of the opponents' strategies that he is aware of. Thus, in games with unawareness iterated admissibility is conceptually closer to iterated conditional weak dominance because it is *not* independent of extensive-form structure.

There is an increasing literature on extensive-form games with unawareness.[5] Halpern and Rêgo (2006), Rêgo and Halpern (2012), Feinberg (2009), and Heifetz, Meier, and Schipper (2011a) introduce general frameworks for extensive-form games with unawareness, while Li (2006) focuses on extensive-form games with unawareness but perfect information. All of the literature focuses on equilibrium concepts except for Heifetz, Meier, and Schipper (2011a, b) and Schipper and Woo (2012) who use versions of extensive-form rationalizability. Equilibrium notion are problematic under unawareness since they may lack a learning foundation in games with unawareness. An equilibrium is ideally inter-

---

[5] See http://www.econ.ucdavis.edu/faculty/schipper/unaw.htm for a comprehensive bibliography on unawareness including games with unawareness.

preted as a rest-point of some dynamic learning or adaptation process, or alternatively as a pre-meditated agreement or expectation. This interpretation is difficult to carry over to settings in which every increase of awareness is by definition a shock or a surprise. Once a player's view of the game itself is challenged in the course of play, it is hard to justify the idea that a convention or an agreement for the continuation of the game are readily available. Alternative non-equilibrium concepts whose interpretations are sensible even under unawareness and which - contrary to what is sometimes believed in the profession - yield sharp predictions are readily available in versions of extensive-form rationalizability. Ozbay (2007) and Filiz-Ozbay (2007) analyze games with one-sided unawareness and an equilibrium notion that entails forward-induction. Filiz-Ozbay (2007) studies an application to incomplete contracts in a context of insurance. Schipper and Woo (2012) apply prudent rationalizability introduced in Heifetz, Meier, and Schipper (2011b) (and also studied in this paper) to electoral campaigning. Finally, we like to mention that Meier and Schipper (2012) introduce (static) Bayesian games with unawareness and study the robustness of equilibrium to uncertainty about awareness of actions.

2 Generalized Extensive-Form Games

In this section we outline generalized extensive-form games with unawareness as introduced in Heifetz, Meier, and Schipper (2011a). To define a generalized extensive-form game Γ, consider first, as a building block, a finite perfect information game with a set of players I, a set of decision nodes N_0, active players I_n at node n with finite action sets A_n^i of player $i \in I_n$ (for $n \in N_0$), chance nodes C_0, and terminal nodes Z_0 with a payoff vector $(p_i^z)_{i \in I} \in \mathbb{R}^I$ for the players for every $z \in Z_0$. The nodes $\bar{N}_0 = N_0 \cup C_0 \cup Z_0$ constitute a tree.

Consider now a family $\mathbf{T}$ of subtrees of $\bar{N}_0$, partially ordered ($\preceq$) by inclusion. One of the trees $T_1 \in \mathbf{T}$ is meant to represent the modeler's view of the paths of play that are *objectively* feasible; each other tree represents the feasible paths of play as *subjectively* viewed by some player at some node at one of the trees.

In each tree $T \in \mathbf{T}$ denote by n_T the copy in T of the node $n \in \bar{N}_0$ whenever the copy of n is part of the tree T. However, in what follows we will typically avoid the subscript T when no confusion may arise.

Denote by N_i^T the set of nodes in which player $i \in I$ is active in the tree $T \in \mathbf{T}$. We require that all the terminal nodes in each tree $T \in \mathbf{T}$ are copies of nodes in Z_0. Moreover, if for two decision nodes n, $n' \in N_i^T$ (i.e. $i \in I_n \cap I_{n'}$) it is the case that

$A_n^i \cap A_{n'}^i \neq \emptyset$, then $A_n^i = A_{n'}^i$.

Denote by N the union of all decision nodes in all trees $T \in \mathbf{T}$, by C the union of all chance nodes, by Z the union of terminal nodes, and by $\bar{N} = N \cup C \cup Z$ (copies n_T of a given node n in different subtrees T are distinct from one another, so that $\bar{N}$ is a disjoint union of sets of nodes). For a node $n \in \bar{N}$ we denote by T_n the tree containing n.

For each decision node $n \in N$ and each active player $i \in I_n$ the information set is denoted by $\pi_i(n)$. It is the set of nodes that the player i considers as possible at n. $\pi_i(n)$ will be in a different tree than in the tree T_n if at n the player is unaware of some of the paths in T_n, and rather envisages the dynamic interaction as taking place in the tree containing $\pi_i(n)$. We impose properties analogous to standard extensive-form games with perfect recall (see Heifetz, Meier, and Schipper, 2011a, Properties I0 to I6).

We denote by H_i the set of i's information sets in all trees. For an information set $h_i \in H_i$, we denote by T_{h_i} the tree containing h_i. For two information sets h_i, h_i' in a given tree T, we say that h_i precedes h_i' (or that h_i' succeeds h_i) if for every $n' \in h_i'$ there is a path $n, ..., n'$ such that $n \in h_i$. We denote the precedence relation by $h_i \rightsquigarrow h_i'$.

Standard properties imply that if $n', n'' \in h_i$ where $h_i = \pi_i(n)$ is an information set, then $A_{n'}^i = A_{n''}^i$ (see Heifetz, Meier, and Schipper, 2011a, Remark 1, for details). Thus, if $n \in h_i$ we write also A_{h_i} for A_n^i.

Perfect recall property guarantees that with the precedence relation $\rightsquigarrow$ player i's information sets H_i form an *arborescence*: For every information set $h_i' \in H_i$, the information sets preceding it $\{h_i \in H_i : h_i \rightsquigarrow h_i'\}$ are totally ordered by $\rightsquigarrow$.

For trees $T, T' \in \mathbf{T}$ we denote $T \rightarrowtail T'$ whenever for some node $n \in T$ and some player $i \in I_n$ it is the case that $\pi_i(n) \subseteq T'$. Denote by $\hookrightarrow$ the transitive closure of $\rightarrowtail$. That is, $T \hookrightarrow T''$ if and only if there is a sequence of trees $T, T', \ldots, T'' \in \mathbf{T}$ satisfying $T \rightarrowtail T' \rightarrowtail \cdots \rightarrowtail T''$.

A generalized extensive-form game Γ consists of a partially ordered set $\mathbf{T}$ of subtrees of a tree $\bar{N}_0$ along with information sets $\pi_i(n)$ for every $n \in T$, $T \in \mathbf{T}$ and $i \in I_n$, satisfying all properties imposed in Heifetz, Meier, and Schipper (2011a).

For every tree $T \in \mathbf{T}$, the *T-partial game* is the partially ordered set of trees including T and all trees T' in Γ satisfying $T \hookrightarrow T'$, with information sets as defined in Γ. A T-partial game is a generalized game, i.e. it satisfies the same properties.

We denote by H_i^T the set of i's information sets in the T-partial game.

A (pure) strategy

$$s_i \in S_i \equiv \prod_{h_i \in H_i} A_{h_i}$$

for player i specifies an action of player i at each of her information sets $h_i \in H_i$. Denote by

$$S = \prod_{j \in I} S_j$$

the set of strategy profiles in the generalized extensive-form game.

If $s_i = (a_{h_i})_{h_i \in H_i} \in S_i$, we denote by

$$s_i (h_i) = a_{h_i}$$

the player's action at the information set h_i. If player i is active at node n, we say that at node n the strategy prescribes to her the action $s_i (\pi_i (n))$.

In generalized extensive-form games, a strategy cannot be conceived as an ex ante plan of action. If $h_i \subseteq T$ but $T \not\rightsquigarrow T'$, then at h_i player i may be interpreted as being unaware of her information sets in $H_i^{T'} \setminus H_i^T$.

Thus, a strategy of player i should rather be viewed as a list of answers to the hypothetical questions "what would the player do if h_i were the set of nodes she considered as possible?", for $h_i \in H_i$. However, there is no guarantee that such a question about the information set $h'_i \in H_i^{T'}$ would even be meaningful to the player if it were asked at a different information set $h_i \in H_i^T$ when $T \not\rightsquigarrow T'$. The answer should therefore be interpreted as given by the modeler, as part of the description of the situation.

For a strategy $s_i \in S_i$ and a tree $T \in \mathbf{T}$, we denote by s_i^T the strategy in the T-partial game induced by s_i. If $R_i \subseteq S_i$ is a set of strategies of player i, denote by R_i^T the set of strategies induced by R_i in the T-partial game, The set of i's strategies in the T-partial game is thus denoted by S_i^T. Denote by $S^T = \prod_{j \in I} S_j^T$ the set of strategy profiles in the T-partial game.

We say that a strategy profile $s \in S$ *reaches* the information set $h_i \in H_i$ if the players' actions and nature's moves (if there are any) in T_{h_i} lead to h_i with a positive probability. (Notice that unlike in standard games, an information set $\pi_i (n)$ may be contained in tree $T' \neq T_n$. In such a case, by definition $s_i (\pi_i (n))$ induces an action to player i also in n and not only in the nodes of $\pi_i (n)$.)

We say that the strategy $s_i \in S_i$ *reaches* the information set h_i if there is a strategy profile $s_{-i} \in S_{-i}$ of the other players such that the strategy profile (s_i, s_{-i}) reaches h_i. Otherwise, we say that the information set h_i is excluded by the strategy s_i.

Similarly, we say that the strategy profile $s_{-i} \in S_{-i}$ *reaches* the information set h_i if there exists a strategy $s_i \in S_i$ such that the strategy profile (s_i, s_{-i}) reaches h_i.

A strategy profile $(s_j)_{j\in I}$ *reaches a node* $n \in T$ if the players' actions $s_j\left(\pi_j\left(n'\right)\right)_{j\in I}$ and nature's moves in the nodes $n' \in T$ lead to n with a positive probability. Since we consider only finite trees, $(s_j)_{j\in I}$ reaches an information set $h_i \in H_i$ if and if there is a node $n \in h_i$ such that $(s_j)_{j\in I}$ reaches n.

As is the case also in standard games, for every given node, a given strategy profile of the players induces a distribution over terminal nodes in each tree, and hence an expected payoff for each player in the tree.

For an information set h_i, let $s_i/\tilde{s}_i^{h_i}$ denote the strategy that is obtained by replacing actions prescribed by s_i at the information set h_i and its successors by actions prescribed by $\tilde{s}_i$. The strategy $s_i/\tilde{s}_i^{h_i}$ is called an h_i-replacement of s_i.

The set of *behavioral* strategies is

$$\prod_{h_i\in H_i} \Delta\left(A_{h_i}\right).$$

3 Extensive-Form Rationalizability

Extensive-form rationalizability is an iterative procedure that for each player at each level rationalizes strategies with appropriate beliefs on the previous level rationalizable strategies of opponents. First, we need to define belief systems.

A *belief system* of player i

$$b_i = \left(b_i\left(h_i\right)\right)_{h_i\in H_i} \in \prod_{h_i\in H_i} \Delta\left(S_{-i}^{T_{h_i}}\right)$$

is a profile of beliefs - a belief $b_i\left(h_i\right) \in \Delta\left(S_{-i}^{T_{h_i}}\right)$ about the other players' strategies in the T_{h_i}-partial game, for each information set $h_i \in H_i$, with the following properties

- $b_i\left(h_i\right)$ reaches h_i, i.e. $b_i\left(h_i\right)$ assigns probability 1 to the set of strategy profiles of the other players that reach h_i.
- If h_i precedes h_i' $(h_i \rightsquigarrow h_i')$ then $b_i\left(h_i'\right)$ is derived from $b_i\left(h_i\right)$ by Bayes rule whenever possible.

Denote by B_i the set of player i's belief systems.

3.1 Rationality versus Would-Be Rationality

Given belief systems, we next have to define a notion of rationality. In this section we discuss two notions, rationality as originally defined in Pearce (1984) and would-be rationality, the notion actually used in Heifetz, Meier, and Schipper (2011a). To mention it upfront, both notions lead to the same plans of actions but would-be rationality gives rise to a refinement in terms of strategies. Moreover, we think that would-be rationality is arguably more natural in extensive-form games because players choose actions at information sets rather than entire strategies. The comparison of these two notions is relevant here because iterative conditional dominance a la Shimoji and Watson (1998) characterizes extensive-form rationalizability à la Pearce (1984) and Battigalli (1997).

For a belief system $b_i \in B_i$, a strategy $s_i \in S_i$ and an information set $h_i \in H_i$, define player i's expected payoff **at** h_i to be the expected payoff for player i in T_{h_i} given $b_i(h_i)$, the actions prescribed by s_i at h_i and its successors, and conditional on the fact that h_i has been reached.[6]

We say that with the belief system b_i and the strategy s_i player i is *rational* at the information set $h_i \in H_i$ if either s_i doesn't reach h_i in the tree T_{h_i}, or if s_i does reach h_i in the tree T_{h_i} then there exists no h_i-replacement of s_i which yields player i a higher expected payoff in T_{h_i} given the belief $b_i(h_i)$ on the other players' strategies $S_{-i}^{T_{h_i}}$.

We say that with the belief system b_i and the strategy s_i player i *would be rational* at the information set $h_i \in H_i$ if there exists no action $a'_{h_i} \in A_{h_i}$ such that only replacing the action $s_i(h_i)$ by a'_{h_i} results in a new strategy s'_i which yields player i a higher expected payoff at h_i given the belief $b_i(h_i)$ on the other players' strategies $S_{-i}^{T_{h_i}}$.

The difference between these two definitions is as follows. The definition of *rationality* of a strategy s_i at an information set h_i takes a global perspective. It is mute regarding information sets which the strategy s_i itself rules out. Also, at an information set h_i which s_i *does* reach, it considers h_i-replacements, which may alter s_i not only at h_i, but also simultaneously at h_i and/or at some of the succeeding information sets of player i.

In contrast, the second definition takes a local perspective. It takes seriously the reasoning about rationality *assuming* that h_i has been reached, whether this assumption

---

[6] Even if this condition is counterfactual due to the fact that the strategy s_i does not reach h_i. The conditioning is thus on the event that nature's moves, if there are any, have led to the information set h_i, and assuming that player i's past actions (in the information sets preceding h_i) have led to h_i *even if these actions are distinct than those prescribed by* s_i.

is *realistic* (when h_i can in fact be reached with a positive probability given the actions prescribed by s_i at preceding information sets) or *counterfactual* (when h_i is ruled out by i's own actions with the strategy s_i at preceding information sets). Moreover, it considers alternative actions a'_{h_i} only at h_i itself. This is motivated by the implicit assumption that at h_i, player i is certain that at future information sets she will be acting according to the strategy s_i, but at the same time she also realizes that at each such future information set she will have the opportunity to re-consider her action, and that at h_i she has no way to commit herself to the action she will be taking at such a future information set.

We find the second definition more appealing in the context of unawareness. With unawareness, a player does not necessarily conceive of her entire strategy. Rather, she might be aware only of a subset of her information sets. She may plan what to do if and when such an information set is reached. However, once her level of awareness gets increased along the path of play, she may suspect that a similar revelation can happen again. She may then realize that whatever she plans to do, with her current level of awareness, is in fact subject to reconsideration. That's why with unawareness, what a strategy specifies for future information sets should better be conceptualized as expressing current beliefs about one's future actions rather than as a rigid plan to which the player is bound to conform.

The following lemma describes the close connection between the two definitions when all of the information sets h_i are considered. The lemma follows from the principle of optimality in dynamic programming. The explicit proof appears in the appendix.

Lemma 1 *With a belief system b_i of player i,*

(i) if a strategy s_i of player i would be rational *at* all *information sets $h_i \in H_i$ then it is* rational *at all information sets $h_i \in H_i$; and*

(ii) if a strategy s_i of player i is rational *at* all *information sets $h_i \in H_i$, then there exists a strategy $\hat{s}_i$ which coincides with s_i at all information sets reached by s_i, such that $\hat{s}_i$* would be rational *at all information sets $h_i \in H_i$.*

The connection between the two definitions described in Lemma 1 is related to the notion of a *plan of action* (Rubinstein 1991, Reny 1992). A plan of player i specifies her action when she is called to play, and does not specify what she would do at information sets which are ruled out by that plan. Formally, a plan of action for player i is an equivalence class of strategies $\mathcal{P}_i \subset S_i$ such that two strategies $s_i, \hat{s}_i$ are in $\mathcal{P}_i$ if and only if for every strategy profile s_{-i} of the other players, (s_i, s_{-i}) and $(\hat{s}_i, s_{-i})$ induce the same distribution over terminal nodes in each of the trees of the game Γ. If $s_i \in \mathcal{P}_i$ we say that the strategy s_i *induces* the plan of action $\mathcal{P}_i$.

With this terminology, Lemma 1 implies:

Lemma 2 *For a given belief system* b_i *of player* i*, there exists a strategy* s_i *which* is rational *at all information sets* $h_i \in H_i$ *and induces the plan of action* $\mathcal{P}_i$ *if and only if there exists a strategy* $\hat{s}_i$ *which* would be rational *at all information sets* $h_i \in H_i$ *and induces the plan of action* $\mathcal{P}_i$*.*

We now turn to define rationalizability in generalized extensive-form games.

Definition 1 (Would-be rationalizable strategies) *Define, inductively, the following sequence of belief systems and strategies of player* i*.*

$B_i^1 = B_i$

$S_i^1 = \{s_i \in S_i$*: there exists a belief system* $b_i \in B_i^1$
with which for every information set $h_i \in H_i$ *player* i *is* ***would-be rational*** *at* $h_i\}$

$\vdots$

$B_i^k = \{b_i \in B_i^{k-1}$: *for every information set* h_i*, if there exists some profile of the other players' strategies* $s_{-i} \in S_{-i}^{k-1} = \prod_{j \neq i} S_j^{k-1}$ *such that* s_{-i} *reaches* h_i *in the tree* T_{h_i}*, then* $b_i\left(h_i\right)$ *assigns probability 1 to* $S_{-i}^{k-1,T_{h_i}}\}$

$S_i^k = \{s_i \in S_i$*: there exists a belief system* $b_i \in B_i^k$ *with which for every information set* $h_i \in H_i$ *player* i ***would be rational*** *at* $h_i\}$

The set of player i*'s* ***would-be rationalizable strategies*** *is*

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k.$$

Remark 1 $S_i^k \subseteq S_i^{k-1}$ *for every* $k > 1$*.*

Proof. Consider $s_i \in S_i^k$. By definition, s_i would-be rational at each of player i's information sets given some belief system $b_i \in B_i^k$. Since $B_i^k \subseteq B_i^{k-1}$, s_i would also be rational at each of player i's information sets given a belief system in B_i^{k-1}, namely given b_i. Hence $s_i \in S_i^{k-1}$. $\square$

We like to mention that Definition 1 is the notion of extensive-form rationalizability used in Heifetz, Meier, and Schipper (2011a, b) although they called it there simply "extensive-form rationalizability".

The generalization of Pearce's (1984) notion of extensive-form correlated rationalizable strategies using replacements and the Pearce's notion of rationality is introduced next. The inductive definition below generalizes Definition 2 in Battigalli (1997), which he proved to be equivalent to Pearce's original definition when correlation is allowed.

Definition 2 (Extensive-form rationalizable strategies) *Define, inductively, the following sequence of belief systems and strategies of player i.*

$\hat{B}_i^1 = B_i$

$\hat{S}_i^1 = \{s_i \in S_i$*: there exists a belief system* $b_i \in \hat{B}_i^1$
with which for every information set $h_i \in H_i$ *player* i *is* ***rational*** *at* $h_i\}$

$\vdots$

$\hat{B}_i^k = \{b_i \in \hat{B}_i^{k-1}$: *for every information set* h_i*, if there exists some profile of the other players' strategies* $s_{-i} \in \hat{S}_{-i}^{k-1} = \prod_{j \neq i} \hat{S}_j^{k-1}$ *such that* s_{-i} *reaches* h_i *in the tree* T_{h_i}*, then* $b_i(h_i)$ *assigns probability 1 to* $\hat{S}_{-i}^{k-1,T_{h_i}}\}$

$\hat{S}_i^k = \{s_i \in S_i$*: there exists a belief system* $b_i \in \hat{B}_i^k$ *with which for every information set* $h_i \in H_i$ *player* i *is* ***rational*** *at* $h_i\}$

The set of player i's ***extensive-form rationalizable strategies*** *is*

$$\hat{S}_i^\infty = \bigcap_{k=1}^{\infty} \hat{S}_i^k.$$

Remark 2 $\hat{S}_i^k \subseteq \hat{S}_i^{k-1}$ *for every* $k > 1$.

Proof. Analogous to the proof of Remark 1 above. □

Lemma 2 above implies the following proposition.

Proposition 1 *For every finite generalized extensive-form game, the set of strategies* S_i^k *is contained in* $\hat{S}_i^k$*, but* S_i^k *induces a set of plans of action identical to the set of plans of action induced by* $\hat{S}_i^k$*. Consequently, the set of would-be rationalizable strategies is contained in the set of extensive-form correlated rationalizable strategies,*

$$S_i^\infty = \bigcap_{k=1}^{\infty} S_i^k \subseteq \hat{S}_i^\infty = \bigcap_{k=1}^{\infty} \hat{S}_i^k$$

but both sets induce the same set of plans of actions.

The inclusion mentioned in the proposition may be strict. For instance, in our first example in the introduction (Figure 1), it is rationalizable for player 1 not to give the car to player 2 and to subsequently go to the Bach concert, but to have gone to the Stravinsky concert (or to the Bach concert, or to the Mozart concert) *had he given the car to 2*. In contrast, the only *would-be* rationalizable strategy of player 1 is not to give the car to player 2 and subsequently attend the Bach concert, but to have gone to the Mozart concert had he given the car to player 2. As the proposition asserts, no difference arises between rationality and would-be rationality along the uniquely realized path.

Proposition 2 *For every finite generalized extensive-form game, the set of would-be rationalizable strategies is non-empty. Consequently, the set of extensive-form correlated rationalizable strategies is non-empty.*

The proof of the first statement appears in Heifetz, Meier, and Schipper (2011a). The second sentence follows from the first sentence and the previous lemma.

Heifetz, Meier, and Schipper (2011a) show that in a Bach-Stravinsky-Mozart example with unawareness similar to the one in the introduction that no player may have a unique would-be rationalizable strategy. When we compare this example to the analogous game in which both players are aware of the Mozart concert but player I has the option of not providing her car for going to this concert (Figure 1), we note that the strategic implications of unawareness of actions are distinct from a situation in which both players are aware of the actions but some action may not always be available. The reason is that if player I keeps player II unaware of the Mozart concert, then player II can not infer the intention of player II to go to the Bach concert. In other words, awareness of an available action (providing the car for going to the Mozart concert) and certainty that it hasn't been taken has stronger strategic implications than unawareness of the very same action. One may conjecture now that in games with unawareness of actions, the would-be rationalizable outcome is not unique while in the analogous game with unavailability of the very same actions there may be a unique would-be rationalizable outcome. However, this is not generally the case. There exist games where with unavailability of actions there are more would-be rationalizable outcomes than with unawareness of the same actions. The game in Figure 3 is such an example.[7] We summarize this discussion:

---

[7] That is, replace the payoffs in Figure 1 with the payoffs from Figure 3. The resulting standard game is analogous to Figure 3 but with unavailability of actions rather than unawareness of actions. One can

Remark 3 *Would-be rationalizability and extensive-form rationalizability do not necessarily yield sharper predictions under unavailability of actions than under unawareness of the same actions.*

4 Characterization of Extensive-Form Rationalizability by Conditional Dominance

4.1 Associated Normal-Form Games

Consider a generalized extensive-form game Γ with a partially ordered set of trees $\mathbf{T}$. The *associated normal-form game* G is defined by $\langle I, \langle (S_i^T)_{i \in I}, (u_i^T)_{i \in I} \rangle_{T \in \mathbf{T}} \rangle$, where I is the set of players in Γ and S_i^T is player i's set of T-partial strategies. If player i is not active in trees $T' \in \mathbf{T}$ with $T \hookrightarrow T'$, then $S_i^T = \emptyset$. Recall that if player i is active at node $n \in T$, then at node n the strategy $s_i \in S_i^T$ prescribes to her the action $s_i(\pi_i(n))$. Hence, each profile of strategies in S^T induces a distribution over terminal nodes in T (even if there is a player active in T with no information set in T). $u_i^T(s)$ is the expected value of the payoffs associated with the terminal nodes in T reached by $s \in S^T$ weighted by the probabilities associated to the moves of nature. (Note that while strategy profiles in S^T reach terminal nodes also in trees $T' \in \mathbf{T}$, $T \hookrightarrow T'$, u_i^T concerns payoffs in the tree T only.)

Recall that H_i^T denotes player i's set of extensive-form information sets in the T-partial game. For each $h_i \in H_i^T$, let $S^T(h_i) \subseteq S^T$ be the subset of the T-partial strategy space containing T-partial strategy profiles that reach the information set h_i. Define also $S_i^T(h_i) \subseteq S_i^T$ and $S_{-i}^T(h_i) \subseteq S_{-i}^T$ to be the set of player i's T-partial strategies reaching h_i and the set of profiles of the other players' T-partial strategies reaching h_i respectively. For the entire game denote by $S(h_i) \subseteq S$ the set of strategy profiles that reach h_i. Similarly, $S_i(h_i) \subseteq S_i$ and $S_{-i}(h_i) \subseteq S_{-i}$ are the set of player i's strategies reaching h_i and the set of profiles of the other players' strategies reaching h_i respectively.

Given Γ and its associated normal-form game G, define player i's set of *normal-form*

---

show that in this game with unavailability of actions the extensive-form rationalizable outcome is not unique.

information sets[8] by

$$\mathcal{X}_i = \{S^{T_{h_i}}(h_i) : h_i \in H_i\}.$$

These are the "normal-form versions" of information sets in the generalized extensive-form game.

For $T \in \mathbf{T}$, any set $Y \subseteq S^T$ is called a *restriction* for player i (or an i-product set) of T-partial strategies if $Y = Y_i \times Y_{-i}$ for some $Y_i \subseteq S_i^T$ and $Y_{-i} \subseteq S_{-i}^T$. Clearly, a player's normal-form information set is a restriction. I.e., if $S^{T_{h_i}}(h_i)$ is a normal-form information set of player i, then it is a restriction for player i of T_{h_i}-partial strategy profiles.

4.2 Iterated Conditional Strict Dominance and Extensive-Form Rationalizability

We say that $s_i \in S_i^T$ is *strictly dominated* in a restriction $Y \subseteq S^T$ if $s_i \in Y_i$, $Y_{-i} \neq \emptyset$, and there exists a mixed strategy $\sigma_i \in \Delta(Y_i)$ such that $u_i^T(\sigma_i, s_{-i}) > u_i^T(s_i, s_{-i})$ for all $s_{-i} \in Y_{-i}$.

Denote by $\mathbf{S} = \bigcup_{T \in \mathbf{T}} S^T$ and $\mathbf{S}_i = \bigcup_{T \in \mathbf{T}} S_i^T$.

For $T \hookrightarrow T'$ and a T-partial strategy $s_i \in S_i^T$, we denote the T'-partial strategy $s_i^{T'} \in S_i^{T'}$ induced by s_i. For $\tilde{s}_i \in S_i^{T'}$, define

$$[\tilde{s}_i] := \bigcup_{T \hookrightarrow T'} \{s_i \in S_i^T : s_i^{T'} = \tilde{s}_i\}.$$

That is, $[\tilde{s}_i]$ is the set of strategies in $\mathbf{S}_i$ which at information sets $h_i \in H_i^{T'}$ prescribe the same actions as strategy $\tilde{s}_i$.

Let $(Y^T)_{T \in \mathbf{T}}$ be a collection of i-product sets, one for each $T \in \mathbf{T}$. Define $\mathbf{Y} = \bigcup_{T \in \mathbf{T}} Y^T$. Given such a $\mathbf{Y}$, we say that $s_i \in S_i^T$ is *conditionally strictly dominated on* $(\mathcal{X}_i, \mathbf{Y})$ if for some $\tilde{s}_i \in S_i^{T'}$, $T \hookrightarrow T'$, $s_i \in [\tilde{s}_i]$, we have that $\tilde{s}_i$ is strictly dominated in $X \cap Y^{T'}$ for some normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^{T'}$.

Note that this definition implies as a special case that $s_i \in S_i^T$ is conditionally strictly dominated on $(\mathcal{X}_i, \mathbf{Y})$ if there exists a normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^T$

---

[8] We abuse here slightly existing terminology. In the literature on standard games, normal-form information sets refer more generally to subsets of the strategy space of a pure strategy reduced normal-form game for which there exists an extensive-form game with corresponding information sets (see Mailath, Samuelson, and Swinkels, 1993). For our characterization, we are just interested in the normal-form versions of information sets of a given generalized extensive-form game.

such that s_i is strictly dominated in $X \cap Y^T$. Yet, the domination "across" normal-forms makes the definition a non-trivial generalization of conditional strict dominance in standard games.

For $\mathbf{Y}$ define

$$U_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not conditionally strictly dominated on } (\mathcal{X}_i, \mathbf{Y})\},$$

$$U(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} \left(U_i(\mathbf{Y}) \cap S_i^T\right),$$

and

$$U_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} \left(U_j(\mathbf{Y}) \cap S_j^T\right).$$

Define inductively

$U^0(\mathbf{S}) = \mathbf{S}$,

$U^{k+1}(\mathbf{S}) = U(U^k(\mathbf{S}))$ for $k \geq 0$,

$U^{\infty}(\mathbf{S}) = \bigcap_{k=0}^{\infty} U^k(\mathbf{S})$,

and similarly for $U_i^k(\mathbf{S})$ and $U_{-i}^k(\mathbf{S})$.

Example. We will illustrate the definitions with the introductory example in Figure 3. In this strategic situation, player I may deceive player II by hiding player II's dominant action M. Recall that player I is the row player, while player II is the column player. For the row player in the upper normal form, the first component of her strategy refers to actions at the root of the upper tree, the second to her action in the upper left subgame, the third to the upper right subgame, and the last component to the action in the lower game. For the column player, the first component of his strategy refers to the action taken in the upper information set while the second is the action taken in the lower information set.

The entire upper normal form is the normal-form information set (marked green) of player I (but not player II) associated with player I's information set at the beginning of the T-partial game (but not in the T'-partial game). We denote this information set by $X_I(\emptyset^T)$. The upper green rectangle in the upper normal form is the normal-form information set of player I (but not of player II) corresponding to her extensive-form information set after the history n in the T-partial game (but not in the T'-partial game). We denote it by $X_I(n)$. The lower purple rectangle in the upper normal-form game is the normal-form information set for both player I and II corresponding to the information sets after history t in the T-partial game (but not in the T'-partial game). We denote it by $X_i(t)$.

Finally, the lower normal-form game is a normal-form information set (marked purple) for both player I and II both for corresponding information sets in the T'-partial normal form and in the T-partial normal-form game. It is also the normal-form information set for player II corresponding to his information set $\pi_{II}(n)$ in the T-partial game. We indicate this with the blue arrow. We denote it by $X_i(\emptyset^{T'}) = X_{II}(n)$.

The definition of $\mathbf{S}_i$ is illustrated by the example $\mathbf{S}_{II} = \{BB, BS, SB, SS, MB, MS, B, S\}$, while the definition $[\tilde{s}_i]$ can be illustrated by [”S”] $= \{BS, SS, MS, S\}$. These are all the strategies of player II that prescribe the action “S” (“Stravinsky”) at the information set $\pi_{II}(n)$.

Iterated elimination of conditionally strictly dominated strategies proceeds as follows:

$$\begin{aligned}
U_i^0(\mathbf{S}) &= \mathbf{S}_i, i = I, II, \\
U_I^1(\mathbf{S}) &= \{nMBB, nMSB, nMMB, nMBS, nMSS, nMMS, \\
&\quad tBMB, tSMB, tMMB, tBMS, tSMS, tMMS, \\
&\quad B, S\}, \\
U_{II}^1(\mathbf{S}) &= \{MB, B\}.
\end{aligned}$$

For instance, strategy $nSBB$ is conditionally strictly dominated by $nMBB$ in the normal-form information set $X_I(\emptyset^T)$ or $X_I(n)$. More interestingly, MS is conditionally strictly dominated on $(\mathcal{X}_{II}, \mathbf{S})$ because $MS \in$ [”S”] and S is strictly dominated by B in $X_{II}(n)$. So this example demonstrates that *an action in the upper normal form may be deleted because of strict dominance in the lower normal form.* This is one reason why we chose this game to demonstrate iterated conditional strict dominance rather than the introductory example.

Applying the definitions iteratively yields

$$\begin{aligned}
U_I^2(\mathbf{S}) &= \{nMBB, nMSB, nMMB, B\}, \\
&= U_I^k(\mathbf{S}) \text{ for } k \geq 2, \\
U_{II}^2(\mathbf{S}) &= U_{II}^2(\mathbf{S}) = \{MB, B\}, \\
&= U_{II}^k(\mathbf{S}) \text{ for } k \geq 1.
\end{aligned}$$

Note that $U_i^\infty(\mathbf{S}) \cap S_i = \hat{S}_i^\infty$. That is, the set of strategies remaining after iterated elimination of conditionally strictly dominated strategies coincides with the set of extensive-form correlated rationalizable strategies. Both solution concepts predict that player I will not give the car to player II and attend the Mozart concert, while player II will attend the Bach concert.

Our main result is that iterated conditional strict dominance characterizes extensive-form rationalizability in generalized extensive-form games.

Proposition 3 *For every finite generalized extensive-form game,* $U_i^k(\mathbf{S}) \cap S_i = \hat{S}_i^k$, $k \geq 1$. *Consequently,* $U_i^\infty(\mathbf{S}) \cap S_i = \hat{S}_i^\infty$.

The proof is contained in the appendix.

5 Prudent Rationalizability and Iterated Conditional Weak Dominance

Heifetz, Meier, and Schipper (2011b) defined an outcome refinement of would-be rationalizability called prudent rationalizability. Here, we will call it prudent would-be rationalizability to make transparent that the notion of would-be rationality is used, and to distinguish it from an analogous notion of prudent rationalizability that uses the notion of rationality à la Pearce (1984). The definition is as follows:

Definition 3 (Prudent would-be rationalizability) *Let*

$$\bar{S}_i^0 = S_i$$

For $k \geq 1$ define inductively

$$\bar{B}_i^k = \left\{ b_i \in B_i : \begin{array}{l} \textit{for every information set } h_i,\ \textit{if there exists some profile} \\ s_{-i} \in \bar{S}_{-i}^{k-1} = \prod_{j \neq i} \bar{S}_j^{k-1} \textit{ of the other players' strategies} \\ \textit{such that } s_{-i} \textit{ reaches } h_i \textit{ in the tree } T_{h_i},\ \textit{then the support} \\ \textit{of } b_i(h_i) \textit{ is the set of strategy profiles } s_{-i} \in \bar{S}_{-i}^{k-1, T_{h_i}} \textit{ that reach } h_i \end{array} \right\}$$

$$\bar{S}_i^k = \left\{ s_i \in \bar{S}_i^{k-1} : \begin{array}{l} \textit{there exists } b_i \in \bar{B}_i^k \textit{ such that for all } h_i \in H_i \textit{ player } i \\ \boldsymbol{would\ be\ rational} \textit{ at } h_i \end{array} \right\}$$

The set of prudent would-be rationalizable strategies of player i is

$$\bar{S}_i^\infty = \bigcap_{k=1}^\infty \bar{S}_i^k$$

Analogous to extensive-form rationalizability, we can define a notion similar to prudent rationalizability but using Pearce's (1984) notion of rationality.

Definition 4 (Prudent rationalizability) *Let*

$$\breve{S}_i^0 = S_i$$

For $k \geq 1$ define inductively

$$\breve{B}_i^k = \left\{ b_i \in B_i : \begin{array}{l} \textit{for every information set } h_i, \textit{ if there exists some profile} \\ s_{-i} \in \breve{S}_{-i}^{k-1} = \prod_{j \neq i} \breve{S}_j^{k-1} \textit{ of the other players' strategies} \\ \textit{such that } s_{-i} \textit{ reaches } h_i \textit{ in the tree } T_{h_i}, \textit{ then the support} \\ \textit{of } b_i \left(h_i \right) \textit{ is the set of strategy profiles } s_{-i} \in \breve{S}_{-i}^{k-1,T_{h_i}} \textit{ that reach } h_i \end{array} \right\}$$

$$\breve{S}_i^k = \left\{ s_i \in \breve{S}_i^{k-1} : \begin{array}{l} \textit{there exists } b_i \in \breve{B}_i^k \textit{ such that for all } h_i \in H_i \textit{ player } i \\ \textit{is } \textbf{\textit{rational}} \textit{ at } h_i \end{array} \right\}$$

The set of prudent rationalizable strategies of player i is

$$\breve{S}_i^{\infty} = \bigcap_{k=1}^{\infty} \breve{S}_i^k$$

Lemma 2 implies the following proposition.

Proposition 4 *For every finite generalized extensive-form game, the set of strategies $\bar{S}_i^k$ is contained in $\breve{S}_i^k$, but $\bar{S}_i^k$ induces a set of plans of action identical to the set of plans of action induced by $\breve{S}_i^k$. Consequently, the set of prudent would-be rationalizable strategies is contained in the set of prudent rationalizable strategies,*

$$\bar{S}_i^{\infty} = \bigcap_{k=1}^{\infty} \bar{S}_i^k \subseteq \breve{S}_i^{\infty} = \bigcap_{k=1}^{\infty} \breve{S}_i^k$$

but both sets induce the same set of plans of actions.

Proposition 5 *For every finite generalized extensive-form game, the set of prudent would-be rationalizable strategies is non-empty. Consequently, the set of prudent rationalizable strategies is non-empty.*

The proof of the first statement appears in Heifetz, Meier, and Schipper (2011b). The second sentence follows from the first sentence and the previous lemma.

5.1 Iterated Conditional Weak Dominance

We say that $s_i \in S_i^T$ is *weakly dominated* in a restriction $Y \subseteq S^T$ if $s_i \in Y_i$, $Y_{-i} \neq \emptyset$, and there exists a mixed strategy $\sigma_i \in \Delta(Y_i)$ such that $u_i^T(\sigma_i, s_{-i}) \geq u_i^T(s_i, s_{-i})$ for all $s_{-i} \in Y_{-i}$ and $u_i^T(\sigma_i, s_{-i}) \geq u_i^T(s_i, s_{-i})$ for some $s_{-i} \in Y_{-i}$

Given a $\mathbf{Y}$, we say that $s_i \in S_i^T$ is *conditionally weakly dominated on* $(\mathcal{X}_i, \mathbf{Y})$ if for some $\tilde{s}_i \in S_i^{T'}$, $T \hookrightarrow T'$, $s_i \in [\tilde{s}_i]$, we have that $\tilde{s}_i$ is weakly dominated in $X \cap Y^{T'}$ for some normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^{T'}$.

Note that this definition implies as a special case that $s_i \in S_i^T$ is conditionally weakly dominated on $(\mathcal{X}_i, \mathbf{Y})$ if there exists a normal-form information set $X \in \mathcal{X}_i$, $X \subseteq S^T$ such that s_i is weakly dominated in $X \cap Y^T$. Yet, the weak domination "across" normal forms makes this definition a non-trivial generalization of conditional weak dominance.

For $\mathbf{Y}$ define

$$W_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not conditionally weakly dominated on } (\mathcal{X}_i, \mathbf{Y})\},$$

$$W(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} \left(W_i(\mathbf{Y}) \cap S_i^T\right),$$

and

$$W_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} \left(W_j(\mathbf{Y}) \cap S_j^T\right).$$

Define inductively

$W^0(\mathbf{S}) = \mathbf{S}$,

$W^{k+1}(\mathbf{S}) = W(W^k(\mathbf{S}))$ for $k \geq 0$,

$W^\infty(\mathbf{S}) = \bigcap_{k=0}^\infty W^k(\mathbf{S})$,

and similarly for $W_i^k(\mathbf{S})$ and $W_{-i}^k(\mathbf{S})$.

Proposition 6 *For every finite generalized extensive-form game,* $W_i^k(\mathbf{S}) \cap S_i = \breve{S}_i^k$, $k \geq 1$. *Consequently,* $W_i^\infty(\mathbf{S}) \cap S_i = \breve{S}_i^\infty$.

Proof. The proof is analogous to Proposition 3. Instead of using Lemma 3 in Pearce (1984), we would now use Lemma 4 in Pearce (1984). □

5.2 Iterated Admissibility

For standard extensive-form games with perfect recall, Brandenburger and Friedenberg (2011, Proposition 3.1) showed that iterated elimination of conditionally weakly dominated strategies coincides with iterated admissibility at each level of iteration in the associated normal-form game.

How to define iterated admissibility in generalized normal-form games with unawareness? Consider the example in Figure 4. A first straightforward approach could be to apply iterated admissibility to each of the normal-forms games separately. At the first level, the set of admissible strategies coincides with all the set of strategies that are not conditionally strict dominated except for MS of player II. (Recall that under conditional strict dominance, we were able to delete strategy MS because S was strictly dominated in the lower game T'.) Notice that at the second level, strategy MB becomes now weakly dominated by MS. But MB is the only extensive-form rationalizable strategy and prudent rationalizable strategy of player II in the T-partial game. What is wrong with this straightforward approach is that it does not eliminate a strategy in the T-partial game when it is weakly dominated by another strategy in a lower game. Strategies MB and MS differ in the second component only, the action of player II in the lower game T'. It is in this lower game T' that S is dominated by B, and hence any strategy prescribing S at the lower game T' should be eliminated. This motivates to define iterated admissibility as procedure that conditions more coarsely on a normal-form instead of an information set. But not any normal-form game will do. We also need to insure that for each information set, we condition on the "correct" normal-form game, namely the normal-form game that represents the player's awareness at this information set.

More formally, let $\mathcal{S}_i = \{S^{T_{h_i}} : h_i \in H_i\}$. Given a $\mathbf{Y}$, we say that $s_i \in S_i^T$ is *NF-conditionally weakly dominated on* $(\mathcal{S}_i, \mathbf{Y})$ if for some $\tilde{s}_i \in S_i^{T'}$, $T \hookrightarrow T'$, $s_i \in [\tilde{s}_i]$, we have that $\tilde{s}_i$ is weakly dominated in $S^{T'} \cap Y^{T'}$ for some normal form $S^{T'} \in \mathcal{S}_i$.

Note that this definition implies as a special case that $s_i \in S_i^T$ is NF-conditionally weakly dominated on $(\mathcal{S}_i, \mathbf{Y})$ if there exists a normal form $S^T \in \mathcal{S}_i$, such that s_i is weakly dominated in $S^T \cap Y^T$. Again, the weak domination "across" normal forms makes this definition a non-trivial generalization of weak dominance.

For $\mathbf{Y}$ define

$$\tilde{W}_i(\mathbf{Y}) = \{s_i \in \mathbf{S}_i : s_i \text{ is not NF-conditionally weakly dominated on } (\mathcal{S}_i, \mathbf{Y})\},$$

$$\tilde{W}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{i \in I} \left(\tilde{W}_i(\mathbf{Y}) \cap S_i^T \right),$$

and

$$\tilde{W}_{-i}(\mathbf{Y}) = \bigcup_{T \in \mathbf{T}} \prod_{j \in I \setminus \{i\}} \left(W_j(\mathbf{Y}) \cap S_j^T \right).$$

Define inductively

$\tilde{W}^0(\mathbf{S}) = \mathbf{S}$,

$\tilde{W}^{k+1}(\mathbf{S}) = \tilde{W}(\tilde{W}^k(\mathbf{S}))$ for $k \geq 0$,

$\tilde{W}^\infty(\mathbf{S}) = \bigcap_{k=0}^\infty \tilde{W}^k(\mathbf{S})$,

and similarly for $\tilde{W}_i^k(\mathbf{S})$ and $\tilde{W}_{-i}^k(\mathbf{S})$.

We could call any strategy of player i in $\tilde{W}_i(\mathbf{Y})$ admissible on $(\mathcal{S}_i, \mathbf{Y})$, and any strategy of player i in $\tilde{W}_i^\infty(\mathbf{S})$ as iterative admissible. But this terminology is arguably somewhat misleading. While similar to standard games, iterated admissibility can be understood as a special version of iterated elimination of conditionally weakly dominated strategies, in games with unawareness iterated admissibility is conceptually closer to iterated conditional weak dominance since it makes explicit use of information sets (for selecting appropriately among the normal-form games). This is due to the fact that information sets in generalized extensive-form games do not only model a player's information but also his awareness. The player's awareness of strategies is crucial for admissibility since it does not rule out any opponents' strategy from being played among the opponents' strategies a player is aware of.

In the previous section, we have seen that prudent rationalizability is characterized by iterated elimination of conditional weakly dominated strategies. How to characterize iterated admissibility by iterative elimination procedure using beliefs analogous to prudent rationalizability?

A *relaxed belief system* of player i

$$b_i = (b_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta\left(S_{-i}^{T_{h_i}}\right)$$

is a profile of beliefs - a belief $b_i(h_i) \in \Delta\left(S_{-i}^{T_{h_i}}\right)$ about the other players' strategies in the T_{h_i}-partial game, for each information set $h_i \in H_i$. Compared to belief systems, for relaxed belief systems we neither require that $b_i(h_i)$ reaches h_i nor Bayesian updating.

Denote by $\ddot{B}_i$ the set of player i's relaxed belief systems.

We say that with the relaxed belief system b_i and the strategy s_i player i is *relaxed rational* at the information set $h_i \in H_i$ if there does not exist an alternative strategy s_i' which yields player i a higher expected in T_{h_i} given the belief $b_i(h_i)$ on the other players' strategies $S_{-i}^{T_{h_i}}$.

Definition 5 (Prudent relaxed rationalizability) *Let*

$$\ddot{S}_i^0 = S_i$$

For $k \geq 1$ define inductively

$$\ddot{B}_i^k = \left\{ b_i \in \ddot{B}_i : \begin{array}{c} \textit{for every information set } h_i, \textit{ if there exists some profile} \\ s_{-i} \in \ddot{S}_{-i}^{k-1} = \prod_{j \neq i} \ddot{S}_j^{k-1} \textit{ of the other players' strategies} \\ \textit{such that } s_{-i} \textit{ reaches } h_i \textit{ in the tree } T_{h_i}, \textit{ then the support} \\ \textit{of } b_i\left(h_i\right) \textit{ is the set of strategy profiles } s_{-i} \in \ddot{S}_{-i}^{k-1,T_{h_i}} \end{array} \right\}$$

$$\ddot{S}_i^k = \left\{ s_i \in \ddot{S}_i^{k-1} : \begin{array}{l} \textit{there exists } b_i \in \ddot{B}_i^k \textit{ such that for all } h_i \in H_i \textit{ player } i \\ \textit{is } \textbf{\textit{relaxed rational}} \textit{ at } h_i \end{array} \right\}$$

The set of prudent relaxed rationalizable strategies of player i is

$$\ddot{S}_i^\infty = \bigcap_{k=1}^{\infty} \ddot{S}_i^k$$

We show that prudent relaxed rationalizability is characterized by iterated admissibility in generalized extensive-form games.

Proposition 7 *For every finite generalized extensive-form game, $\tilde{W}_i^k\left(\mathbf{S}\right) \cap S_i = \ddot{S}_i^k$, $k \geq 1$. Consequently, $\tilde{W}_i^\infty\left(\mathbf{S}\right) \cap S_i = \ddot{S}_i^\infty$.*

The proof is contained in the appendix.

It turns out though that prudent relaxed rationalizability coincides with prudent rationalizability.

Proposition 8 *For every finite generalized extensive-form game and every $i \in I$, $\breve{S}_i^k = \ddot{S}_i^k$, $k \geq 0$. Consequently, $\breve{S}_i^\infty = \ddot{S}_i^\infty$.*

The proof is contained in the appendix.

The following corollary shows that iterated admissibility is equivalent to iterated conditional weak dominance at every level. This result generalizes Brandenburger and Friedenberg (2011, Proposition 3.1) to generalized extensive-form games.

Corollary 1 *For every finite generalized extensive-form game, $\tilde{W}_i^k\left(\mathbf{S}\right) = W_i^k\left(\mathbf{S}\right)$, $k \geq 1$. Consequently, $\tilde{W}_i^\infty\left(\mathbf{S}\right) = W_i^\infty\left(\mathbf{S}\right)$.*

A Proofs

A.1 Proof of Lemma 1

Recall property I3 of information sets of generalized extensive-form games from Heifetz, Meier, and Schipper (2011a). For each player $i \in I$, the information sets satisfy

I3 No divining of currently unimaginable paths, no expectation to forget currently conceivable paths: If $n' \in \pi_i(n) \subseteq T'$ (where $T' \in \mathbf{T}$ is a tree) and there is a path $n', \ldots, n'' \in T'$ such that $i \in I_{n'} \cap I_{n''}$ then $\pi_i(n'') \subseteq T'$.

(i) By (I3), all information sets of player i along a path starting in h_i and ending at a terminal node are contained in T_{h_i}. Therefore, it is enough to show the claim for every subtree T in the generalized extensive-form game. Since player i's belief system b_i satisfies updating consistency as defined by Perea (2002), the proof of Theorem 3.1 of Perea (2002) implies the claim.[9]

(ii) If a strategy $s_i$of player i is rational at all information sets $h_i \in H_i$, then in particular s_i would be rational in all information sets $h_i \in H_i$ reached by s_i. Denote by $H_i^{-s_i}$ the set of information sets not reached by s_i. By (I3), the expected payoff for player i (given the belief system b_i) from choosing an action in $h_i \in H_i^{-s_i}$ does not depend on her choices at information sets outside T_{h_i}.

Furthermore, $H_i^{-s_i}$ is an arborescence with respect to the precedence relation $\rightsquigarrow$. Hence, a standard backward-induction procedure on $H_i^{-s_i}$ yields an optimal action $a^*_{h_i} \in A_{h_i}$ for player i (given b_i) at h_i for every information set $h_i \in H_i^{-s_i}$. Replacing by $a^*_{h_i}$ the action prescribed by s_i at h_i for every $h_i \in H_i^{-s_i}$ yields a new strategy $\hat{s}_i$ which would be rational at all information sets $h_i \in H_i$. $\square$

A.2 Proof of Proposition 3

A *general belief system* of player i

$$\tilde{b}_i = (\tilde{b}_i(h_i))_{h_i \in H_i} \in \prod_{h_i \in H_i} \Delta(S_{-i}^{T_{h_i}})$$

---

[9] Formally, Theorem 3.1 in Perea (2002) refers to two-player games, but as he remarks at the top of p. 325, the argument can be extended in a straightforward manner to games with more than two players and correlated beliefs about other players' strategies.

is a profile of beliefs – a belief $\tilde{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}})$ about the other players' strategies in the T_{h_i}-partial extensive-form game, for each information set $h_i \in H_i$, such that $\tilde{b}_i(h_i)$ reaches h_i, i.e., $\tilde{b}_i(h_i)$ assigns probability 1 to the set of strategy profiles of the other players that reach h_i. The difference between a belief system and a general belief system is that in the latter *we do not impose Bayes rule.*

For $k \geq 1$ let $\tilde{B}_i^k$ and $\tilde{S}_i^k$ be defined inductively like $\hat{B}_i^k$ and $\hat{S}_i^k$ in Definition 2, respectively, the only change being that belief systems are replaced by generalized belief systems.

Lemma 3 *$U_i^k(\mathbf{S}) \cap S_i = \tilde{S}_i^k$ for $k \geq 1$. Consequently, $U_i^\infty(\mathbf{S}) \cap S_i = \tilde{S}_i^\infty$.*

Proof of the Lemma. We proceed by induction. The case $k = 0$ is straightforward since $U_i^0(\mathbf{S}) \cap S_i = S_i = \tilde{S}_i^0$ for all $i \in I$.

Suppose now that we have shown $U_i^k(\mathbf{S}) \cap S_i = \tilde{S}_i^k$ for all $i \in I$. We want to show that $U_i^{k+1}(\mathbf{S}) \cap S_i = \tilde{S}_i^{k+1}$ for all $i \in I$.

"$\subseteq$": First we show, if $s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$ then $s_i \in \tilde{S}_i^{k+1}$.

$s_i \in U_i^{k+1}(\mathbf{S}) \cap S_i$ if $s_i \in S_i$ is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$.

$s_i \in S_i$ is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$ if for all $T' \in \mathbf{T}$ with $T_1 \hookrightarrow T'$ and all $\tilde{s}_i \in S_i^{T'}$ such that $s_i \in [\tilde{s}_i]$, we have that there does not exist a normal-form information set $X \in \mathcal{X}_i$ with $X \subseteq S^{T'}$ such that $\tilde{s}_i$ is strictly dominated in $X \cap U^k(\mathbf{S})$.

For any information set $h_i \in H_i$, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is not strictly dominated in $S^{T_{h_i}}(h_i) \cap U^k(\mathbf{S})$, then

(i) either $\tilde{s}_i$ does not reach h_i, in which case $\tilde{s}_i$ is trivially rational at h_i; or

(ii) by Lemma 3 in Pearce (1984) there exists a belief $\tilde{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}}(h_i) \cap U_{-i}^k(\mathbf{S}))$ for which $\tilde{s}_i$ is rational at h_i. Since by the induction hypothesis $U^k(\mathbf{S}) \cap S = \tilde{S}^k$, we have in this case that there exists a belief at h_i with $\tilde{b}_i(h_i)(\tilde{S}_{-i}^{k,T_{h_i}}) = 1$ for which $\tilde{s}_i$ is rational at h_i.

By the definitions of $[\tilde{s}_i]$ and "reach", if $\tilde{s}_i$ reaches h_i in the tree T_{h_i} and $s_i \in [\tilde{s}_i]$, then s_i reaches h_i in the tree T_{h_i}. Hence, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is rational at h_i given $\tilde{b}_i(h_i)$, then $s_i \in [\tilde{s}_i]$ is rational at h_i given $\tilde{b}_i(h_i)$.

We need to show that beliefs in (ii) define a generalized belief system in $\tilde{B}_i^{k+1}$. Consider any $\tilde{b}_i' = (\tilde{b}_i'(h_i))_{h_i \in H_i} \in \tilde{B}_i^{k+1}$. For all $h_i \in H_i$ for which there exists a profile of player i's opponents' strategies $s_{-i} \in \tilde{S}^k_{-i}$ that reach h_i, replace $\tilde{b}'_i(h_i)$ by $\tilde{b}_i(h_i)$ as defined in (ii). Call the new belief system $\tilde{b}_i$. Then this is a generalized belief system. Moreover, $\tilde{b}_i \in \tilde{B}^{k+1}_i$.

Hence, if s_i is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$ then there exists a generalized belief system $\tilde{b}_i \in \tilde{B}^{k+1}_i$ for which s_i is rational at every $h_i \in H_i$. Thus $s_i \in \tilde{S}^{k+1}_i$.

"$\supseteq$": We show next, if $s_i \in \tilde{S}^{k+1}_i$ then $s_i \in U^{k+1}_i(\mathbf{S}) \cap S_i$.

If $s_i \in \tilde{S}^{k+1}_i$ then there exists a generalized belief system $\tilde{b}_i \in \tilde{B}^{k+1}_i$ such that for all $h_i \in H_i$ the strategy s_i is rational given $\tilde{b}_i(h_i)$. That is, either

(I) s_i does not reach h_i, or

(II) s_i reaches h_i and there does not exist an h_i-replacement of s_i which yields a higher expected payoff in T_{h_i} given $\tilde{b}_i(h_i)$ that assigns probability 1 to T_{h_i}-partial strategies of player i's opponents in $\tilde{S}^{k,T_{h_i}}_{-i}$ that reach h_i in T_{h_i}. By the induction hypothesis, $\tilde{S}^k_{-i} = U^k_{-i}(\mathbf{S}) \cap S^{T_{h_i}}_{-i}$. Hence $\tilde{b}_i(h_i) \in \Delta(U^k_{-i}(\mathbf{S}) \cap S^{T_{h_i}}_{-i}(h_i))$.

If $s_i \in [\tilde{s}_i]$ with $\tilde{s}_i \in S^{T_{h_i}}_i$ and s_i reaches h_i in the tree T_{h_i}, then $\tilde{s}_i$ reaches h_i in the tree T_{h_i}. Hence, if $s_i \in [\tilde{s}_i]$ with $\tilde{s}_i \in S^{T_{h_i}}_i$ is rational at h_i given $\tilde{b}_i(h_i)$, then $\tilde{s}_i$ is rational at h_i given $\tilde{b}_i(h_i)$.

Thus, if s_i is rational at h_i given $\tilde{b}_i(h_i)$, then $\tilde{s}_i \in S^{T_{h_i}}_i$ with $s_i \in [\tilde{s}_i]$ is not strictly dominated in $U^k_{-i}(\mathbf{S}) \cap S^{T_{h_i}}_{-i}(h_i)$ either because s_i does not reach h_i (case (I)), or because of Lemma 3 in Pearce (1984) (in case (II)).

It then follows that if the strategy s_i is rational at all $h_i \in H_i$ given $\tilde{b}_i$ then s_i is not conditionally strictly dominated on $(\mathcal{X}_i, U^k(\mathbf{S}))$. Hence $s_i \in U^{k+1}_i(\mathbf{S}) \cap S_i$. □

Lemma 4 $\tilde{S}^k_i = \hat{S}^k_i$ *for* $k \geq 1$. *Consequently,* $\tilde{S}^\infty_i = \hat{S}^\infty_i$.

Proof of the Lemma. $\hat{S}^k_i \subseteq \tilde{S}^k_i$ for $k \geq 1$ since if s_i is rational at each information set $h_i \in H_i$ given the belief system $b_i \in B_i$ then there exists a generalized belief system $\tilde{b}_i \in \tilde{B}^k_i$, namely $\tilde{b}_i = b_i$, such that s_i is rational at each information set $h_i \in H_i$ given $\tilde{b}_i$.

We need to show the reverse inclusion, $\tilde{S}^k_i \subseteq \hat{S}^k_i$ for $k \geq 1$. The first step is to show how to construct a (consistent) belief system from a generalized belief system. Let s_i be rational given $\tilde{b}_i \in \tilde{B}^1_i$, i.e., $s_i \in \tilde{S}^1_i$. Consider an information set $h^0_i \in H_i$ such that in T_{h_i} there does not exist an information set h_i that precedes h^0_i. Define $b_i(h^0_i) \equiv \tilde{b}_i(h^0_i)$.

Assume, inductively, that we have already defined b_i for a subset of information sets $H_i' \subseteq H_i$ such that for each $h_i' \in H_i'$ all the predecessors of h_i' are also in H_i'. For each successor information set h_i'' of each information set $h_i' \in H_i'$ such that $h_i'' \notin H_i'$ define $b_i\left(h_i''\right)$ as follows:

- If $b_i\left(h_i'\right)$ reaches h_i'' define $b_i\left(h_i''\right)$ by using Bayes rule, i.e. if $s_{-i}^{T_{h_i'}} \in S_{-i}(h_i'')$

  $$b_i\left(h_i''\right)(s_{-i}^{T_{h_i'}}) = \frac{b_i\left(h_i'\right)(s_{-i}^{T_{h_i'}})}{\sum_{\tilde{s}_{-i}^{T_{h_i'}} \in S_{-i}(h_i'')} b_i(h_i')(\tilde{s}_{-i}^{T_{h_i'}})}$$

  and $b_i\left(h_i''\right)(s_{-i}^{T_{h_i'}}) = 0$ else.

- If $b_i\left(h_i'\right)$ does not reach h_i'' let $b_i(h_i'') \equiv \tilde{b}_i(h_i'')$.

Since there are finitely many information sets in H_i, this inductive definition will be concluded in a finite number of steps.

Next, assuming that s_i is rational at each information set $h_i \in H_i$ with the generalized belief system $\tilde{b}_i$, we will show that s_i is also rational at each information set $h_i \in H_i$ according to the belief system b_i.

Consider again $h_i^0 \in H_i$ with no predecessors in $T_{h_i^0}$. Since $b_i(h_i^0) = \tilde{b}_i(h_i^0)$ and s_i is rational at h_i^0 given $\tilde{b}_i\left(h_i^0\right)$, s_i is also rational at h_i^0 given $b_i\left(h_i^0\right)$.

Assume, inductively, that we have already shown the claim for a subset of information sets $H_i' \subseteq H_i$ such that for each $h_i' \in H_i'$ all the predecessors of h_i' are also in H_i'. Consider a successor information set h_i'' of an information set $h_i' \in H_i'$ such that $h_i'' \notin H_i'$. Notice that each h_i''-replacement is also an h_i'-replacement. Therefore,

- If $b_i\left(h_i'\right)$ reaches h_i'', $b_i\left(h_i''\right)$ is derived from $b_i(h_i')$ by Bayes rule, and hence any h_i''-replacement improving player i's expected payoff according to $b_i\left(h_i''\right)$ would improve player i's payoff also according to $b_i(h_i')$, contradicting the induction hypothesis. Hence s_i is rational at h_i'' given $b_i\left(h_i''\right)$.

- If $b_i\left(h_i'\right)$ does not reach h_i'', then $b_i\left(h_i''\right) = \tilde{b}_i(h_i'')$. Hence, s_i is rational at h_i'' also given $b_i\left(h_i''\right)$.

Applying the same argument inductively yields $\tilde{S}_i^k = \hat{S}_i^k \; \forall k \geq 1$. This concludes the proof of the lemma. □

Lemmata 3 and 4 together yield $U_i^k(\mathbf{S}) \cap S_i = \hat{S}_i^k$ for $k \geq 1$. Since it applies for all $k \geq 1$ and $i \in I$, this completes the proof of the proposition. $\square$

A.3 Proof of Proposition 7

We proceed by induction. The case $k = 0$ is straightforward since $\tilde{W}_i^0(\mathbf{S}) \cap S_i = S_i = \ddot{S}_i^0$ for all $i \in I$.

Suppose now that we have shown $\tilde{W}_i^k(\mathbf{S}) \cap S_i = \ddot{S}_i^k$ for all $i \in I$. We want to show that $\tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i = \ddot{S}_i^{k+1}$ for all $i \in I$.

"$\subseteq$": First we show, if $s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$ then $s_i \in \ddot{S}_i^{k+1}$.

$s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$ if $s_i \in S_i$ is not NF-conditionally weakly dominated on $(\mathcal{S}_i, \tilde{W}^k(\mathbf{S}))$.

$s_i \in S_i$ is not NF-conditionally weakly dominated on $(\mathcal{S}_i, \tilde{W}^k(\mathbf{S}))$ if for all $T' \in \mathbf{T}$ with $T_1 \hookrightarrow T'$ and all $\tilde{s}_i \in S_i^{T'}$ such that $s_i \in [\tilde{s}_i]$, we have that $\tilde{s}_i$ is not weakly dominated in $S_i^{T'} \cap \tilde{W}^k(\mathbf{S})$.

For any information set $h_i \in H_i$, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is not weakly dominated in $S^{T_{h_i}} \cap \tilde{W}^k(\mathbf{S})$, then by Lemma 4 in Pearce (1984) there exists a full support belief $\ddot{b}_i(h_i) \in \Delta(S_{-i}^{T_{h_i}} \cap \tilde{W}_{-i}^k(\mathbf{S}))$ for which $\tilde{s}_i$ is relaxed rational at h_i. Since by the induction hypothesis $\tilde{W}^k(\mathbf{S}) \cap S = \ddot{S}^k$, we have in this case that there exists a full support belief at h_i with $\ddot{b}_i(h_i) \in \Delta(\ddot{S}_{-i}^{k,T_{h_i}})$ for which $\tilde{s}_i$ is relaxed rational at h_i.

By definition of $[\tilde{s}_i]$, if $\tilde{s}_i \in S_i^{T_{h_i}}$ is relaxed rational at h_i given $\ddot{b}_i(h_i)$, then $s_i \in [\tilde{s}_i]$ is relaxed rational at h_i given $\ddot{b}_i(h_i)$.

Note that the profile of beliefs $(\ddot{b}_i(h_i))_{h_i \in H_i}$ define a relaxed belief system in $\ddot{B}_i^{k+1}$.

Hence, if s_i is not NF-conditionally weakly dominated on $(\mathcal{S}_i, \tilde{W}^k(\mathbf{S}))$ then there exists a relaxed belief system $\ddot{b}_i \in \ddot{B}_i^{k+1}$ for which s_i is relaxed rational at every $h_i \in H_i$. Thus $s_i \in \ddot{S}_i^{k+1}$.

"$\supseteq$": We show next, if $s_i \in \ddot{S}_i^{k+1}$ then $s_i \in \tilde{W}_i^{k+1}(\mathbf{S}) \cap S_i$.

If $s_i \in \ddot{S}_i^{k+1}$ then there exists a relaxed belief system $\ddot{b}_i \in \ddot{B}_i^{k+1}$ such that for all $h_i \in H_i$ the strategy s_i is relaxed rational given $\ddot{b}_i(h_i)$. That is, there does not exist an alternative strategy s_i' which yields a strictly higher expected payoff in T_{h_i} given $\ddot{b}_i(h_i)$ that has full support on $\ddot{S}_{-i}^{k,T_{h_i}}$.

By the induction hypothesis, $\ddot{S}_{-i}^k = \tilde{W}_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$. Hence $\ddot{b}_i(h_i)$ has full support on $\tilde{W}_{-i}^k(\mathbf{S}) \cap S_{-i}^{T_{h_i}}$.

If s_i is relaxed rational at h_i with $\ddot{b}_i(h_i)$, then for any $\tilde{s}_i \in S_i^{T_{h_i}}$ with $s_i \in [\tilde{s}_i]$, $\tilde{s}_i$ is relaxed rational at h_i with $\ddot{b}_i(h_i)$. By Lemma 4 in Pearce (1984), $\tilde{s}_i$ is not weakly dominated in $\tilde{W}^k_{-i}(\mathbf{S}) \cap S^{T_{h_i}}_{-i}$

It then follows that if the strategy s_i is rational at all $h_i \in H_i$ given $\tilde{b}_i$ then s_i is not NF-conditionally weakly dominated on $(\mathcal{S}_i, \tilde{W}^k(\mathbf{S}))$. Hence $s_i \in \tilde{W}^{k+1}_i(\mathbf{S}) \cap S_i$. □

A.4 Proof of Proposition 8

The proof proceeds by induction. By definition $\breve{S}^0_i = \ddot{S}^0_i$.

Assume $\breve{S}^k_i = \ddot{S}^k_i$. We show that $\breve{S}^{k+1}_i = \ddot{S}^{k+1}_i$.

"$\supseteq$": Fix $s_i \in \ddot{S}^{k+1}_i$ and $h_i \in H_i$ such that s_i reaches h_i (otherwise s_i is trivially rational at h_i). There exists a relaxed belief system $\ddot{b}_i \in \ddot{B}^{k+1}_i$ such that s_i is relaxed rational at h_i. By definition of $\ddot{B}^{k+1}_i$, $\ddot{b}_i(h_i)$ has full support on $\ddot{S}^{k,T_{h_i}}_{-i}$. Assume $\ddot{S}^{k,T_{h_i}}_{-i} \cap S^{T_{h_i}}_{-i}(h_i) \neq \emptyset$. Then $\ddot{b}_i(h_i)\left(\ddot{S}^{k,T_{h_i}}_{-i} \cap S^{T_{h_i}}_{-i}(h_i)\right) > 0$. Thus, we can consider $\ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)$. Note that $\ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right) \in \breve{B}^{k+1}_i$. (Otherwise, if $\ddot{S}^{k,T_{h_i}}_{-i} \cap S^{T_{h_i}}_{-i}(h_i) = \emptyset$, then s_i is trivially rational at h_i.)

Suppose by contradiction that s_i is not rational at h_i with $\ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)$. There exists a h_i-replacement of s_i, call it $s_i/\tilde{s}^{h_i}_i$, such that $s_i/\tilde{s}^{h_i}_i$ yields a strictly higher expected payoff in T_{h_i} given the belief $\ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)$. That is, $u^{T_{h_i}}_i\left(s_i/\tilde{s}^{h_i}_i, \ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)\right) > u^{T_{h_i}}_i\left(s_i, \ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)\right)$. Since $u^{T_{h_i}}_i\left(s_i, \ddot{b}_i(h_i)\right) \geq u^{T_{h_i}}_i\left(s_i/\tilde{s}^{h_i}_i, \ddot{b}_i(h_i)\right)$, we must have $\ddot{b}_i(h_i)\left(S^{T_{h_i}}_{-i}(h_i)\right) < 1$. Now

$$\begin{aligned}
u^{T_{h_i}}_i\left(s_i, \ddot{b}_i(h_i)\right) &\geq u^{T_{h_i}}_i\left(s_i/\tilde{s}^{h_i}_i, \ddot{b}_i(h_i)\right)\\
&= \ddot{b}_i(h_i)\left(S^{T_{h_i}}_{-i}(h_i)\right) u^{T_{h_i}}_i\left(s_i/\tilde{s}^{h_i}_i, \ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)\right)\\
&\quad + \left(1 - \ddot{b}_i(h_i)\left(S^{T_{h_i}}_{-i}(h_i)\right)\right) u^{T_{h_i}}_i\left(s_i/\tilde{s}^{h_i}_i, \ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)\right)\\
&> \ddot{b}_i(h_i)\left(S^{T_{h_i}}_{-i}(h_i)\right) u^{T_{h_i}}_i\left(s_i, \ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)\right)\\
&\quad + \left(1 - \ddot{b}_i(h_i)\left(S^{T_{h_i}}_{-i}(h_i)\right)\right) u^{T_{h_i}}_i\left(s_i, \ddot{b}_i(h_i)\left(\cdot \middle| S^{T_{h_i}}_{-i}(h_i)\right)\right)\\
&= u^{T_{h_i}}_i\left(s_i, \ddot{b}_i(h_i)\right)
\end{aligned}$$

a contradiction.

"$\subseteq$": Let $\breve{S}^{k,T_{h_i}}_{-i}(h_i)$ denote the set of opponents' level k-prudent rationalizable T_{h_i}-partial strategy profiles that reach h_i. For each $T \in \mathbf{T}$ for which H^T_i is nonempty, let

$\breve{S}_{-i}^{k,T}(H_i^T) = \bigcup_{h_i \in H_i^T} \breve{S}_{-i}^{k,T}(h_i)$. Note that if $\breve{S}_{-i}^{k,T}(H_i^T) \subsetneqq \breve{S}_{-i}^{k,T}$ then there is a terminal history in T that is not reached by any information set $h_i \in H_i^T$. (In general, $\breve{S}_{-i}^{k,T}(H_i^T)$ may not be a cross-product set).

We claim that there exists a nonempty subset of information sets $G_i^T \subseteq H_i^T$ such that $\left\{\breve{S}_{-i}^{k,T}(h_i)\right\}_{h_i \in G_i^T}$ forms a partition of $\breve{S}_{-i}^{k,T}(H_i^T)$. To define G_i^T, we first define the rank of an information set $h_i \in H_i^T$ as the number of information sets in H_i^T required to pass in order to reach a terminal node in T. Note that if $h_i \in H_i^T$ with $n \in h_i$ there is path from n to n' in T with $i \in I_{n'}$, then property I3 (No divining of currently unimaginable paths, no expectation to forget currently conceivable paths) insures that there exists an information set $h_i' \in H_i^T$ with $n' \in h_i'$. Note further that there may be terminal nodes in T that are not reached by any information set in H_i^T.

Using the definition of rank of an information set, we construct a subset of information sets G_i^T as follows: For any information set $h_i \in H_i^T$ of lowest rank, let $h_i \in G_i^T$ if there is no information set $h_i' \in H_i^T$ that precedes h_i. Otherwise, let $h_i' \in G_i^T$ if there is no information set $h_i'' \in H_i^T$ that precedes h_i', etc. Since T is finite, H_i^T is finite, and the procedure terminates after finite steps. (In particular, if player i moves at the root of T, G_i^T is a singleton whose only information set contains the root.)

By construction, $\bigcup_{h_i \in G_i^T} \breve{S}_{-i}^{k,T}(h_i)$ covers $\breve{S}_{-i}^{k,T}(H_i^T)$. Moreover, from property I6 (Perfect recall) follows that for any two $h_i, h_i' \in G_i^T$ with $h_i \neq h_i'$ there is no profile of opponents' T-partial strategies $s_{-i}^T \in \breve{S}_{-i}^{k,T}(h_i) \cap \breve{S}_{-i}^{k,T}(h_i')$.

Fix $s_i \in \breve{S}_i^{k+1}$ and $h_i \in H_i$ such that s_i reaches h_i (otherwise s_i is trivially rational at h_i). There exists a belief system $\breve{b}_i \in \breve{B}_i^{k+1}$ such that s_i is rational at h_i. By definition of $\breve{B}_i^{k+1}$, the support of $\breve{b}_i(h_i)$ is the set $\breve{S}_{-i}^{k,T_{h_i}}(h_i)$.

Construct a belief system $\ddot{b}_i$ by setting for arbitrary $\varepsilon \in (0,1)$,

$$\ddot{b}_i(h_i)(s_{-i}) \quad := \quad \begin{cases} \frac{1-\varepsilon}{\left|G_i^{T_{h_i}}\right|}\breve{b}_i(h_i)(s_{-i}) & \text{if } s_{-i}^T \in \breve{S}_{-i}^{k,T_{h_i}}(h_i), \text{ and} \\ \frac{\varepsilon}{\left|\breve{S}_{-i}^{k,T_{h_i}} \setminus \breve{S}_{-i}^{k,T_{h_i}}(H_i^{T_{h_i}})\right|} & \text{if } s_i^T \in \breve{S}_{-i}^{k,T_{h_i}} \setminus \breve{S}_{-i}^{k,T_{h_i}}(H_i^{T_{h_i}}). \end{cases}$$

This defines a full support probability measure $\ddot{b}_i(h_i) \in \Delta\left(\breve{S}_{-i}^{k,T_{h_i}}\right)$ for each information set $h_i \in H_i$.

$\ddot{B}_i^{k+1}$ is the set of all belief systems defined as above from any $\breve{b}_i \in \breve{B}_i^{k+1}$ and $\varepsilon \in (0,1)$.

Finally, note that any strategy $s_i \in \breve{S}_i^{k+1}$ that with $\breve{b}_i \in \breve{B}_i^{k+1}$ is rational at every information set $h_i \in H_i$ continues to be rational with a belief system $\ddot{b}_i \in \ddot{B}_i^{k+1}$ at every information set $h_i \in H_i$. Thus $s_i \in \ddot{S}_i^{k+1}$. □

References

[1] Battigalli, P. (1997). On rationalizability in extensive games, *Journal of Economic Theory* **74**, 40-61.

[2] Brandenburger, A. and A. Friedenberg (2011). The relationship between rationality on the matrix and the tree, mimeo.

[3] Chen, J. and S. Micali (2012). The order independence of iterated dominance in extensive games, mimeo.

[4] Feinberg, Y. (2009). Games with unawareness, mimeo.

[5] Filiz-Ozbay, E. (2007). Incorporating unawareness into contract theory, in: Samet, D. (ed.), Proceedings of the 11th conference on Theoretical Aspects of Rationality and Knowledge, Presses Universitaires de Louvain, 135-144.

[6] Halpern, J. and L. Rêgo (2006). Extensive games with possibly unaware players, in: *Proc. Fifth International Joint Conference on Autonomous Agents and Multiagent Systems*, 744-751.

[7] Heifetz, A., Meier, M., and B. C. Schipper (2011a). Dynamic unawareness and rationalizable behavior, mimeo.

[8] Heifetz, A., Meier, M., and B. C. Schipper (2011b). Prudent rationalizability in generalized extensive-form games, mimeo.

[9] Kohlberg, E. and J.-F. Mertens (1986). On the strategic stability of equilibrium, *Econometrica* **54**, 1003-1037.

[10] Li, J. (2006). Dynamic games with perfect awareness information, mimeo.

[11] Mailath, G., Samuelson, L., and J. Swinkels (1993). Extensive form reasoning in normal form games, *Econometrica* **61**, 273-302.

[12] Meier, M. and B.C. Schipper (2012). Bayesian games with unawareness and unawareness perfection, mimeo.

[13] Ozbay, E. (2007). Unawareness and strategic announcements in games with uncertainty, in: Samet, D. (ed.), Proceedings of the 11th conference on Theoretical Aspects of Rationality and Knowledge, Presses Universitaires de Louvain, pp. 231-238.

[14] Pearce, D.G. (1984). Rationalizable strategic behavior and the problem of perfection, *Econometrica* **52**, 1029-1050.

[15] Perea, A. (2002). A note on the one-deviation property in extensive form games, *Games and Economic Behavior* **40**, 322-338.

[16] Rêgo, L. and J. Halpern (2012). Generalized Solution Concepts in Games with Possibly Unaware Players, *International Journal of Game Theory* **41**, 131–155.

[17] Reny, P. (1992). Backward induction, normal form perfection and explicable equilibria, *Econometrica* **60**, 627-649.

[18] Rubinstein, A. (1991), Comments on the Interpretation of Game Theory, *Econometrica* **59**, 909-924.

[19] Schipper, B.C. and H.Y. Woo (2012). Political awareness and microtargeting of voters in electoral competition, mimeo.

[20] Shimoji, M. and J. Watson (1998). Conditional dominance, rationalizability, and game forms, *Journal of Economic Theory* **83**, 161-195.