Unawareness, Beliefs, and Speculative Trade*

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Abstract

We define a generalized state-space model with interactive unawareness and probabilistic beliefs. Such models are desirable for potential applications of asymmetric unawareness. Applying our unawareness belief structures, we show that the common prior assumption is too weak to rule out speculative trade in all states. Yet, we prove a generalized "No-speculative-trade" theorem according to which there cannot be common certainty of strict preference to trade. Moreover, we prove a generalization of the "No-agreeing-to-disagree" theorem. Finally, we show the existence of a universal unawareness belief type space.

Keywords: Unawareness, awareness, common prior, agreement, speculative trade, universal type-space, interactive epistemology, inattention.

JEL-Classifications: C70, C72, D53, D80, D82.

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1 Introduction

Unawareness is probably the most common and most important kind of ignorance. Business people invest most of their time not in updating prior beliefs and crossing out states of the world that they previously assumed to be possible. Rather, their efforts are mostly aimed at exploring unmapped terrain, trying to figure out business opportunities that they could not even have spelled out before. More broadly, every book we read, every new acquaintance we make, expands our horizon and our language, by fusing it with the horizons of those we encounter, turning the world more intelligible and more meaningful to us than it was before (Gadamer, 1960).

With this in mind, we should not be surprised that the standard state-spaces aimed at modeling knowledge or certainty are not adequate for capturing unawareness (Dekel, Lipman, and Rustichini, 1998). Indeed, more elaborate models are needed (Fagin and Halpern, 1988, Modica and Rustichini, 1994, 1999, Halpern, 2001). In all of these models, the horizon of propositions the individual has in her disposition to talk about the world is always a genuine part of the description of the state of affairs.

Things become even more intricate when several players are involved. Each player may not only have different languages, but may also form a belief on the extent to which other players are aware of the issues that she herself has in mind. Even more complex, the player may be uncertain as to the sub-language that each other player attributes to her or to others; and so on.

Heifetz, Meier, and Schipper (2006) showed how an unawareness structure consisting of a *lattice of spaces* is adequate for modeling mutual unawareness. Every space in the lattice captures one particular horizon of meanings or propositions. Higher spaces capture wider horizons, in which states correspond to situations described by a richer vocabulary. The join of several spaces – the lowest space at least as high as every one of them – corresponds to the fusion of the horizons of meanings expressible in these spaces.

In a companion work (Heifetz, Meier, and Schipper, 2008), we showed the precise sense in which such unawareness structures are adequate and general enough for modeling mutual unawareness. We put forward an axiom system, which extends to the multi-player case a variant of the axiom system of Modica and Rustichini (1999). We then showed how the collections of all maximally-consistent sets of formulas in our system form a canonical unawareness structure.[1] In a parallel work, Halpern and Rêgo (2008) devised

[1]Each space in the lattice of this canonical unawareness structure consists of the maximally consistent sets of formulas in a sub-language generated by a subset of the atomic propositions.

another sound and complete axiomatization for our class of unawareness structures.[2]

In this paper we extend unawareness structures so as to encompass probabilistic beliefs (Section 2) rather than knowledge or ignorance. The definition of types (Definition 1), and the way beliefs relate across different spaces of the lattice, is a non-trivial modification of the coherence conditions for knowledge operators in unawareness structures, as formulated in Heifetz, Meier, and Schipper (2006). We show that we obtain all properties of unawareness suggested in the literature.

Having structures with both unawareness and probabilistic beliefs raises the question about the differences between probability zero events and events that an agent is unaware of. In Appendix B, we show how to "flatten" an unawareness belief structure by taking the union of all spaces and assigning zero probability to all states of which the individual is unaware. Since the "flattened" type space is a standard type space, the Dekel-Lipman-Rustichini (1998) critique applies and the epistemic notion of unawareness becomes trivial. At an epistemic level, unawareness has very different properties than probability zero belief. For instance, one property that is satisfied by unawareness is symmetry (see Proposition 5). An agent is unaware of an event if and only if she is unaware of its negation. Clearly, such a property cannot be satisfied by probability zero belief because if an agent assigns probability zero to an event, then she must assign probability one to its complement. Schipper (2012) shows that this feature captures also behavioral differences between unawareness and probability zero belief. Let's say a decision maker chooses among different contracts for buying a firm. The seconds contract may differ from a first contract only in a consequence for an event E that is disadvantageous to the buyer. If the decision maker is indifferent between both contracts, then this is consistent with E being Savage null. Yet, if the decision maker is also indifferent between the first and a third contract that differs from the first only in assigning this disadvantageous consequence to the negation of the event E instead the event E itself, then this behavior is inconsistent with the negation of the event E or the event E itself being Savage null. The decision maker behaves as if both the event E and its negation are Savage null, which is impossible but consistent with unawareness of the E and of its negation. Thus, when the primitives of a decision model are fixed, unawareness has behavioral implications distinct

[2]The precise connection between Fagin and Halpern (1988), Modica and Rustichini (1999), Halpern (2001), and Heifetz, Meier, and Schipper (2006) is understood from Halpern and Rêgo (2008) and Heifetz, Meier, and Schipper (2008). The connection between Heifetz, Meier, and Schipper (2006, 2008) and Galanis (2011a) is explored in Galanis (2011b). The connection between Li (2009) and Fagin and Halpern (1988) is explored in Heinsalu (2011a). The connections with the models of Ewerhart (2001) and Feinberg (2009) are yet to be explored.

from zero probability.

In Section 3, we present as an economic application of unawareness belief structures an analysis of speculative trade under unawareness. We start by defining the notion of a common prior in unawareness belief structures. Conceptually, a prior of a player is a convex combination of (the beliefs of) her types (see e.g. Samet, 1998). If the priors of the different players coincide, we have a common prior. A prior of a player induces a prior on each particular space in the lattice, and if the prior is common to the players, the induced prior on each particular space is common as well.

What are the implications of the existence of a common prior? First, we extend an example from Heifetz, Meier, and Schipper (2006) and show that *speculative trade* is compatible with the existence of a common prior (Section 1.1). This need not be surprising if one views unawareness as a particular kind of "delusion", since we know that with deluded beliefs, speculative trade is possible even with a common prior (Geanakoplos, 1989). Nevertheless, we show that a positive common prior is *not compatible* with common certainty of *strict preference* to carry out speculative trade. That is, even though types with limited awareness are, in a particular sense, deluded, a common prior precludes the possibility of common certainty of the event that based on private information players are willing to engage in a zero-sum bet with *strictly positive* subjective gains to everybody. This is so because unaware types are "deluded" only concerning aspects of the world outside their vocabulary, while a common prior captures a prior agreement on the likelihood of whatever the players do have a common vocabulary. An implication of this generalized "No-speculative-trade" theorem is that arbitrary small transaction fees (like a Tobin tax) rule out speculative trade under unawareness. We complement this result by generalizing Aumann's (1976) "No-Agreeing-to-disagree" result to unawareness belief structures.

In Section 4 we return to the foundations of unawareness belief structures. Unawareness belief structures capture unawareness and beliefs, beliefs about beliefs (including beliefs about unawareness), beliefs about that etc. in a parsimonious way familiar from standard type spaces. That is, hierarchies of beliefs are captured implicitly by states and type mappings. A construction of unawareness belief structures from explicit hierarchies of beliefs is complicated by the multiple awareness levels involved. A player with a certain awareness level may believe that another player has a lower awareness level and believes that the first player has yet a lower awareness level etc. In Section 4, we present such a hierarchical construction and show the existence of a universal unawareness type space that contains all belief hierarchies. Heinsalu (2011b) independently proves the existence of a universal unawareness type space for the measurable case. Our approach differs from his in that we present an explicit construction of hierarchies of beliefs and thus a proof that is constructive.

In Section 2 we present our interactive unawareness belief structure. In Section 3 we apply unawareness belief structures to study speculative trade under unawareness, prove a "No-speculative-trade" theorem, and discuss the common prior assumption. In Section 4, we present an explicit construction of hierarchies of beliefs and show the existence of a universal unawareness type space. Finally, in Section 5 we conclude with a discussion of the related literature. Some further properties of our unawareness belief structures are relegated to an appendix. Proofs are relegated to an appendix as well.

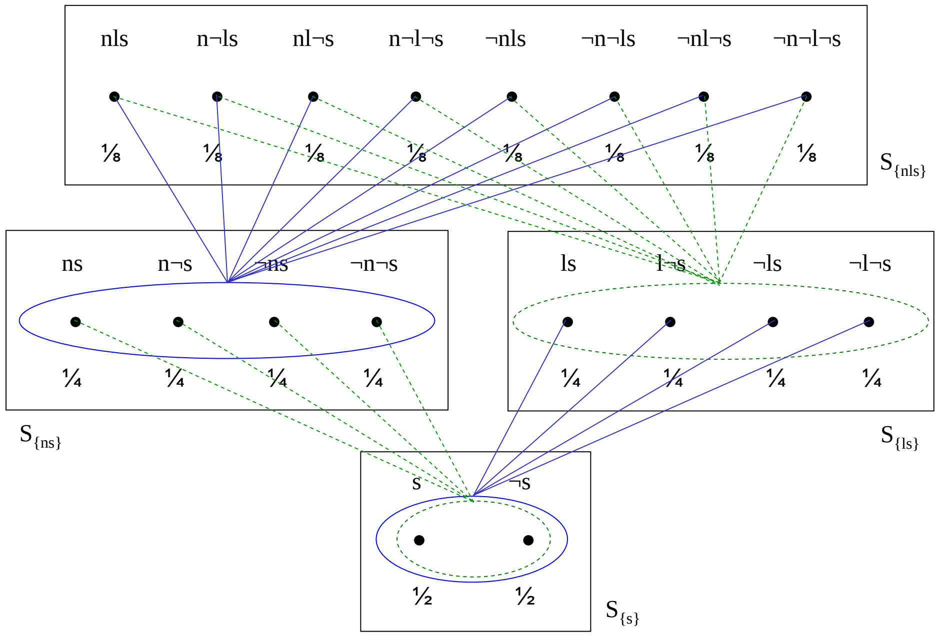
1.1 Introductory Example - Speculation under Unawareness

The purpose of the following example is threefold: First, it shall motivate the study of unawareness and speculation under unawareness. Second, it should illustrate informally some features of our model. Third, it is a counter example to the standard "No-speculative-trade" theorems in the context of unawareness.

Consider a probabilistic version of the speculative trade example of Heifetz, Meier, and Schipper (2006). There is an owner, o, of a firm and a potential buyer, b, whose awareness differ. The owner is aware that there may be a costly lawsuit $[l]$ involving the firm, but she is unaware of a potential novelty $[n]$ enhancing the value of the firm. In contrast, the buyer is aware that there might be an innovation, but he is unaware of the lawsuit. Both are aware that the firm may face high sales $[s]$ or not in future.

Both agents can only reason and form beliefs about contingencies of which they are aware of respectively. The information structure is given in Figure 1. There are four state-spaces of different expressive power. The description of each state is printed above the state. While the upmost space, $S_{\{nls\}}$, contains all contingencies, the space $S_{\{ls\}}$ misses the novelty, $S_{\{ns\}}$ misses the law suit, and $S_{\{s\}}$ is capable of expressing only events pertaining to the sales. At any state in the upmost space $S_{\{nls\}}$, the buyer's belief has full support on the lower space $S_{\{ns\}}$ (as given by the solid ellipse and lines) and the seller's belief has full support on $S_{\{ls\}}$ (dashed ellipse and lines). Thus the buyer forms beliefs about sales and the novelty but is unaware of the law suit, and the seller forms beliefs about sales and the law suit but is unaware of the novelty. At any state in $S_{\{ns\}}$ the seller's belief has full support on the lower space $S_{\{s\}}$. That is, the buyer is certain that the seller is unaware of the novelty. Analogously, the seller is certain that the buyer is

Figure 1: Information Structure in the Speculative Trade Example



unaware of the law suit since at any state in $S_{\{ls\}}$ the belief of the buyer has full support on the space $S_{\{s\}}$. Figure 1 provides an example of an unawareness structure developed in this paper. The probability distribution given in each space illustrates an example of a common prior in unawareness structures, that is, a projective system of probability measures whose posteriors are the players' beliefs. I.e., the prior on a lower space is the marginal of the prior in the upmost space. The beliefs of both agents are consistent with the common prior.

Suppose that the status quo value of the firm with high sales (s) is 100 dollars, but only 80 dollars with low sales ($\neg s$). If the potential innovation (n) obtains, this would add 20 dollars to the value of the firm, whereas the potential lawsuit (l) would cost the firm 20 dollars. According to the beliefs at state (nls) (and any other state in the upmost state-space), the buyer's expected value of the firm is 100, whereas the seller's expected value of the firm is 80 dollars. However, the buyer (resp. seller) is certain that the seller's (resp. buyer's) expected value is 90 dollars.

We assume that both players are rational in the sense of maximizing their respective payoff given their belief and awareness. The buyer (resp. seller) prefers to buy (resp. sell) at price x if his (resp. her) expected value of the firm is at least (resp. at most) x. The buyer (resp. seller) strictly prefers to buy (resp. sell) at price x if his (resp. her) expected value of the firm is strictly above (resp. strictly below) x.

Note that despite the fact that both agents' beliefs are consistent with the common prior, at state (nls) and at the price 90 dollars, there is common certainty of willingness to trade, but each player *strictly* prefers to trade. This is impossible in standard state-space structures with a common prior. In standard "No-speculative-trade" theorems, if there is common certainty of willingness to trade, then agents are necessarily indifferent to trade (Milgrom and Stokey, 1982).

Despite this counterexample to the "No-speculative-trade" theorems, we can prove in Section 3 a generalized "No-speculative-trade" theorem according to which, if there is a common prior, then there cannot be *common certainty of strict preference to trade*. In the above example we have common certainty of willingness to trade and strict preference to trade but there is no common certainty of *strict* preference to trade.

2 Model

2.1 State-Spaces

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be a complete lattice of disjoint *state-spaces*, with the partial order $\succeq$ on $\mathcal{S}$. A complete lattice is a lattice such that each subset has a least upper bound (i.e., supremum) and a greatest lower bound (i.e., infimum). If S_α and S_β are such that $S_\alpha \succeq S_\beta$ we say that "S_α is more expressive than S_β – states of S_α describe situations with a richer vocabulary than states of S_β".[3] Denote by $\Omega = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ the union of these spaces. Each $S \in \mathcal{S}$ is a measurable space, with a σ-field $\mathcal{F}_S$.

Spaces in the lattice can be more or less "rich" in terms of facts that may or may not obtain in them. The partial order relates to the "richness" of spaces. The upmost space of the lattice may be interpreted as the "objective" state-space. Its states encompass full descriptions.

2.2 Projections

For every S and S' such that $S' \succeq S$, there is a measurable surjective projection $r_S^{S'}$: $S' \longrightarrow S$, where r_S^S is the identity. ("$r_S^{S'}(\omega)$ is the restriction of the description ω to the

[3] Here and in what follows, phrases within quotation marks hint at intended interpretations, but we emphasize that these interpretations are not part of the definition of the set-theoretic structure.

more limited vocabulary of S.") Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: If $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

Projections "translate" states from "more expressive" spaces to states in "less expressive" spaces by "erasing" facts that can not be expressed in a lower space.

2.3 Events

Denote $g(S) = \{S' : S' \succeq S\}$. For $D \subseteq S$, denote $D^{\uparrow} = \bigcup_{S' \in g(S)} \left(r_S^{S'}\right)^{-1}(D)$. ("All the extensions of descriptions in D to at least as expressive vocabularies.")

An *event* is a pair (E, S), where $E = D^{\uparrow}$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Some fact may obtain in a subset of a space. Then this fact should be also "expressible" in "more expressive" spaces. Therefore the event contains not only the particular subset but also its inverse images in "more expressive" spaces.

To illustrate the definition of event, consider Figure 1. The event "high sales", $\{s\}^{\uparrow}$, contains the state s in space $S_{\{s\}}$, states ns and $\neg ns$ in space $S_{\{ns\}}$, states ls and $\neg ls$ in space $S_{\{ls\}}$ as well as states nls, $n\neg ls$, $\neg nls$, and $\neg n \neg ls$ in space $S_{\{nls\}}$.

Let Σ be the set of *measurable events* of Ω, i.e., $D^{\uparrow}$ such that $D \in \mathcal{F}_S$, for some state-space $S \in \mathcal{S}$. Note that unless $\mathcal{S}$ is a singleton, Σ is not an algebra because it contains distinct $\emptyset^S$ for all $S \in \mathcal{S}$. The event $\emptyset^S$ should be interpreted as a "logical contradiction phrased with the expressive power available in S". It is quite natural to have distinct vacuous events since contradictions can be phrased with differing expressive powers.

2.4 Negation

If $(D^{\uparrow}, S)$ is an event where $D \subseteq S$, the negation $\neg(D^{\uparrow}, S)$ of $(D^{\uparrow}, S)$ is defined by $\neg(D^{\uparrow}, S) := ((S \setminus D)^{\uparrow}, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^{\uparrow} := \neg(D^{\uparrow}, S)$. Note that by our notational convention, we have $\neg S^{\uparrow} = \emptyset^S$ and $\neg \emptyset^S = S^{\uparrow}$, for each space $S \in \mathcal{S}$. $\neg D^{\uparrow}$ is typically a proper subset of the complement $\Omega \setminus D^{\uparrow}$. That is, $(S \setminus D)^{\uparrow} \subsetneq \Omega \setminus D^{\uparrow}$.

Intuitively, there may be states in which the description of an event $D^{\uparrow}$ is both expressible and valid – these are the states in $D^{\uparrow}$; there may be states in which its description is expressible but invalid – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither its description nor its negation are expressible – these are the states in

$$\Omega \setminus \left(D^{\uparrow} \cup \neg D^{\uparrow}\right) = \Omega \setminus S\left(D^{\uparrow}\right)^{\uparrow}.$$

Thus our structure is not a standard state-space model in the sense of Dekel, Lipman, and Rustichini (1998).

2.5 Conjunction and Disjunction

If $\left\{\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)\right\}_{\lambda \in L}$ is a collection of events (with $D_{\lambda} \subseteq S_{\lambda}$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right)$ is defined by $\bigwedge_{\lambda \in L}\left(D_{\lambda}^{\uparrow}, S_{\lambda}\right) := \left(\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right), \sup_{\lambda \in L} S_{\lambda}\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda \in L} S_{\lambda}$ exists. If $S = \sup_{\lambda \in L} S_{\lambda}$, then we have $\left(\bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}\right) = \left(\bigcap_{\lambda \in L}\left(\left(r_{S_{\lambda}}^{S}\right)^{-1}\left(D_{\lambda}\right)\right)\right)^{\uparrow}$. Again, abusing notation, we write $\bigwedge_{\lambda \in L} D_{\lambda}^{\uparrow} := \bigcap_{\lambda \in L} D_{\lambda}^{\uparrow}$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

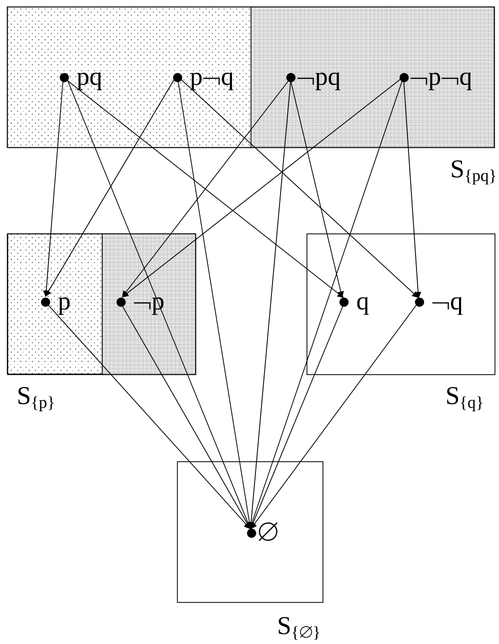
Intuitively, to take the intersection of events $(D_{\lambda}^{\uparrow}, S_{\lambda})_{\lambda \in L}$, we express them "most economically in the smallest language" in which they are all expressible $S = \sup_{\lambda \in L} S_{\lambda}$, take the intersection, and then the union of inverse images obtaining the event $\left(\bigcap_{\lambda \in L}((r_{S_{\lambda}}^{S})^{-1}(D_{\lambda}))\right)^{\uparrow}$ that is based in S.

We define the relation $\subseteq$ between events (E, S) and (F, S'), by $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets *and* $S' \preceq S$. If $E \neq \emptyset$, we have that $(E, S) \subseteq (F, S')$ if and only if $E \subseteq F$ as sets. Note however that for $E = \emptyset^{S}$ we have $(E, S) \subseteq (F, S')$ if and only if $S' \preceq S$. Hence we can write $E \subseteq F$ instead of $(E, S) \subseteq (F, S')$ as long as we keep in mind that in the case of $E = \emptyset^{S}$ we have $\emptyset^{S} \subseteq F$ if and only if $S \succeq S(F)$. It follows from these definitions that for events E and F, $E \subseteq F$ is equivalent to $\neg F \subseteq \neg E$ only when E and F have the same base, i.e., $S(E) = S(F)$.

Intuitively, to say "E implies F" we must be able to express F in the "language" used to express E. Hence, it must be that $S(F) \preceq S(E)$. The inclusion is then just $E \cap S(E) \subseteq F \cap S(E)$.

The disjunction of $\left\{D_{\lambda}^{\uparrow}\right\}_{\lambda \in L}$ is defined by the de Morgan law $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} = \neg\left(\bigwedge_{\lambda \in L} \neg\left(D_{\lambda}^{\uparrow}\right)\right)$. Typically $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} \subsetneqq \bigcup_{\lambda \in L} D_{\lambda}^{\uparrow}$, and if all D_{λ} are nonempty we have that $\bigvee_{\lambda \in L} D_{\lambda}^{\uparrow} = \bigcup_{\lambda \in L} D_{\lambda}^{\uparrow}$ holds if and only if all the $D_{\lambda}^{\uparrow}$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of (at most countably many measurable)

Figure 2: Event Structure



events is a (measurable) event.

Apart from the measurability conditions, the event-structure outlined so far is analogous to Heifetz, Meier, and Schipper (2006, 2008). An example is shown in Figure 2. It depicts a lattice with four spaces and projections. The event that p obtains is indicated by the dotted areas, whereas the grey areas illustrate the event that not p obtains. $S_p \cup S_q$ is for instance not an event in our structure.

2.6 Probability Measures

Here and in what follows, the term 'events' always refers to measurable events in Σ unless otherwise stated.

Let $\Delta(S)$ be the set of probability measures on $(S, \mathcal{F}_S)$. We consider this set itself as a measurable space endowed with the σ-field $\mathcal{F}_{\Delta(S)}$ generated by the sets $\{\mu \in \Delta(S) : \mu(D) \geq p\}$, where $D \in \mathcal{F}_S$ and $p \in [0, 1]$.

2.7 Marginals

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \npreceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.8 Types

I is the nonempty set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a type mapping $t_i : \Omega \longrightarrow \bigcup_{\alpha \in \mathcal{A}} \Delta(S_\alpha)$, which is measurable in the sense that for every $S \in \mathcal{S}$ and $Q \in \mathcal{F}_{\Delta(S)}$ we have $t_i^{-1}(Q) \cap S \in \mathcal{F}_S$. We require the type mapping t_i to satisfy the following properties:*[4]

(0) Confinement: If $\omega \in S'$ then $t_i(\omega) \in \triangle(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

$t_i(\omega)$ represents individual i's belief at state ω. Properties (0) to (3) guarantee the consistent fit of beliefs and awareness at different state-spaces. *Confinement* means that at any given state $\omega \in \Omega$ an individual's belief is concentrated on states that are all described with the same "vocabulary" - the "vocabulary" available to the individual at ω. This "vocabulary" may be less expressive than the "vocabulary" used to describe statements in the state ω."

[4] Recall that S_μ is the space on which μ is a probability measure. Thus, $S_{t_i(\omega)}$ is the space on which $t_i(\omega)$ is a probability measure.

Properties (1) to (3) compare the types of an individual in a state $\omega \in S'$ and its projection to ω_S, for some $S \preceq S'$. Property (1) and (2) mean that at the projected state ω_S the individual believes everything she believes at ω given that she is aware of it at ω_S. Property (3) means that at ω an individual cannot be unaware of an event that she is aware of at the projected state $\omega_{S'}$.

Remark 1 *Property (1) of the type mappings in Definition 1 is implied by the Properties (0),(2), and (3).*

Define[5]

$$Ben_i(\omega) := \left\{\omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)\right\}.$$

This is the set of states at which individual i's type or the marginal thereof coincides with her type at ω. Such sets are events in our structure:

Remark 2 *For any $\omega \in \Omega$, $Ben_i(\omega)$ is an $S_{t_i(\omega)}$-based event, which is not necessarily measurable.*[6] *We have $Ben_i(\omega) = \{\omega' \in S_{t_i(\omega)} : t_i(\omega') = t_i(\omega)\}^{\uparrow} = \{Ben_i(\omega) \cap S_{t_i(\omega)}\}^{\uparrow}$.*

Recall that by definition $t_i(\omega)(E) = t_i(\omega)(E \cap S_{t_i(\omega)})$. Moreover, recall that with event we mean measurable event in our event structure unless otherwise stated; both facts will be used throughout the paper.

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for an event E, then $t_i(\omega)(E) = 1$.*

This assumption implies introspection (Property (va)) in Proposition 4 in the appendix. Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 is equivalent to $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{S} := \left\langle \mathcal{S}, \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta \preceq S_\alpha}, (t_i)_{i \in I} \right\rangle$ an interactive unawareness belief structure.*

For some of our results, we will consider the finite case. A *finite* unawareness belief structure is an unawareness belief structure, where $\mathcal{S}$ is finite, each $S \in \mathcal{S}$ is finite, and for all $S \in \mathcal{S}$, $\mathcal{F}_S$ is the set of all subsets of S.

[5]The name "Ben" is chosen analogously to the "ken" in knowledge structures, see Samet (1990, p. 193).

[6]Even in a standard type-space, if the σ-algebra is not countably generated, then the set of states where a player is of a certain type might not be measurable.

2.9 Awareness and Unawareness

The definition of awareness is analogous to the definition in unawareness knowledge structures (see Remark 6 in Heifetz, Meier, and Schipper, 2008).

Definition 3 *For $i \in I$ and an event E, define the awareness operator*

$$A_i(E) := \{\omega \in \Omega : t_i(\omega) \in \Delta(S), S \succeq S(E)\}$$

if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.

An individual is aware of an event if and only if his type is concentrated on a space in which the event is "expressible." That is, individual i being aware of E means that he "understands what E is".

Proposition 1 *If E is an event then $A_i(E)$ is an $S(E)$-based event.*

This proposition shows that the set of states in which an individual is aware of an event is indeed an event in our structure. Moreover, in the nonempty case note that $A_i(E) = \{\omega \in S(E) : S_{t_i(\omega)} = S(E)\}^{\uparrow} = \{A_i(E) \cap S(E)\}^{\uparrow}$. The awareness operator is convenient to work with since the event $A_i(E)$ has the same base-space as the event E.

Unawareness is naturally defined as the negation of awareness:

Definition 4 *For $i \in I$ and an event E, the unawareness operator is defined by*

$$U_i(E) = \neg A_i(E).$$

Note that the definition of our negation and Proposition 1 imply that if E is an event, then $U_i(E)$ is an $S(E)$-based event.

Note further that Definition 3 and 4 apply also to events that are not necessarily measurable.

2.10 Belief

The p-belief-operator is defined as usual (see for instance Monderer and Samet, 1989):

Definition 5 *For $i \in I$, $p \in [0,1]$ and an event E, the p-belief operator is defined by*

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise.

Proposition 2 *If E is an event then $B_i^p(E)$ is an $S(E)$-based event.*

This proposition shows that the set of states in which an individual believes an event with probability at least p is an event in our structure that has the same base-space as the event E.

Note that $B_i^p(E) = \{\omega \in S(E) : t_i(\omega)(E) \geq p\}^{\uparrow}$. That is, for every operator on events, everything can be expressed in the base space and then the union of inverse images can be taken.

We make note of the particular case $p = 1$ that we call certainty.

The p-belief operator has the standard properties stated in Proposition 4 in Appendix A.

2.11 Properties of Awareness and Belief

Dekel, Lipman, and Rustichini (1998) showed that in a standard state-space unawareness must be trivial, even if the belief operator satisfies only very weak properties. In contrast, we show in Proposition 5 in the appendix that we obtain all properties of unawareness suggested in the literature. One noteworthy property is symmetry, $A_i(E) = A_i(\neg E)$. It means that an individual i is aware of an event E if and only if he is aware of the negation of E. This property makes clear that awareness is qualitatively very different from the notion of probabilistic belief.

Although we model awareness of events, symmetry suggests that we model a notion of awareness of issues or questions. Let an issue or question (E.g., "is the stock market crashing?") be such that it can be answered in the affirmative ("The stock market is crashing.") or the negative ("The stock market is not crashing."). By symmetry, an individual is aware of an event if and only if she is aware of its negation. Thus we model the awareness of questions and issues rather than just single events. In fact, another noteworthy property called weak necessitation, $A_i(E) = B_i^1(S(E)^{\uparrow})$, means that an individual is aware of an event E if and only if she is aware of any event that can be "expressed" in the base-space of E.

Interactive beliefs are defined as usual (e.g. Monderer and Samet, 1989). From now on, we assume that the set of individuals I is at most countable.

Definition 6 *The mutual p-belief operator on events is defined by*

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} \left(B^1\right)^n(E).$$

We say that an event E is common certainty *at $\omega \in \Omega$ if $\omega \in CB^1(E)$.*

That is, the mutual p-belief of an event E is the event in which everybody p-believes the event E. Common certainty of E is the event that everybody is certain of the event E, and everybody is certain that everybody is certain of the event E, everybody is certain of that, ... ad infinitum. Common certainty is the generalization of common knowledge to the probabilistic notion of certainty. Note that Proposition 2 and the definition of the conjunction of events imply that $B^p(E)$ and $CB^1(E)$ are $S(E)$-based events, for any measurable event E.

To illustrate beliefs about beliefs we return to the introductory example. What does it mean for instance that the buyer is certain that the seller's expected value for the firm is 90 dollars? Note that at any state $\omega \in S_{\{s\}}$ the seller's type mapping is $t_s(\omega)(\{s\}) = t_s(\omega)(\{\neg s\}) = \frac{1}{2}$. Since the value of the firm is 100 dollars in state s while it is just 80 dollars in state $\neg s$, the seller's expected value of the firm is 90 dollars at any state in $S_{\{s\}}$. This holds also for all states in $S_{\{ns\}}$ since at any of those states the seller's type coincides with his type at states in $S_{\{s\}}$. Since at any state $\omega \in S_{\{nls\}}$ the buyer's type $t_b(\omega)$ is a probability measure on $S_{\{ns\}}$, the buyer is certain at ω that the seller's expected value of the firm is 90 dollars.

Analogously to mutual belief and common belief, we define mutual awareness and common awareness:

Definition 7 *The mutual awareness operator on events is defined by*

$$A(E) = \bigcap_{i \in I} A_i(E),$$

and the common awareness operator on events is defined by

$$CA(E) = \bigcap_{n=1}^{\infty} (A)^n (E).$$

Mutual awareness of an event E is the event that everybody is aware of E. Common awareness of an event E is the event that everybody is aware of E, everybody is aware that everybody is aware of E, everybody is aware of that ... ad infinitum.

In Propositions 6 and 7 in the appendix, we state several properties of belief and awareness in the multiperson context. One noteworthy property is $A_i(E) = A_i A_j(E)$. If individual i is aware of an event E, then she can also conceive that some other individual j is aware of the event E. Another property is that mutual awareness coincides with common awareness, $A(E) = CA(E)$. That is, if everybody is aware of an event, then everybody can conceive that everybody is aware of the event, everybody is aware of that, etc. Finally, it is noteworthy that common certainty implies common awareness, $CB^1(E) \subseteq CA(E)$. This property will be used repeatedly in the next section.

3 Common Prior, Agreement, and Speculation

In this section, we define a common prior and explore the implications. In Section 1.1, we showed by example that the common prior assumption is too weak to rule out speculative trade under unawareness. With unawareness, it is possible to have common certainty of willingness to trade but everybody has a strict preference to trade. Yet, we are able to prove a "No-speculative-trade" theorem according to which there cannot be common certainty of strict preference to trade under unawareness. In the same vein, we prove a "No-Agreeing-to-Disagree" theorem.

3.1 Priors and Common Priors

In a standard type-space S, a *prior* P_i^S of player i is a convex combination of the beliefs of i's types in S (Samet, 1998). That is, for every event $E \in \mathcal{F}_S$,

$$P_i^S(E) = \int_S t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{1}$$

In particular, if S is finite or countable, and if $\mathcal{F}_S$ is the powerset of S, this equality holds if and only if

$$P_i^S(E) = \sum_{s \in S} t_i(s)(E)\, P_i^S(\{s\}). \tag{2}$$

In words, to find the probability $P_i^S(E)$ that the prior P_i^S assigns to an event E, one should check the beliefs $t_i(s)(E)$ ascribed by player i to the event E in each state $s \in S$, and then average these beliefs according to the weights $P_i^S(\{s\})$ assigned by the prior P_i^S to the different states $s \in S$.

P^S is a *common prior* on S if P^S is a prior for every player $i \in I$.

Here we generalize these definitions to unawareness structures, as follows.

Definition 8 (Prior) *A prior for player i is a system of probability measures $P_i = \left(P_i^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ such that*

1. *The system is projective: If $S' \preceq S$ then the marginal of P_i^S on S' is $P_i^{S'}$. (That is, if $E \in \Sigma$ is an event whose base-space $S(E)$ is lower or equal to S', then $P_i^S(E) = P_i^{S'}(E)$.)*

2. *Each probability measure P_i^S is a convex combination of i's beliefs in S: For every event $E \in \Sigma$ such that $S(E) \preceq S$,*

$$P_i^S(E \cap S \cap A_i(E)) = \int_{S \cap A_i(E)} t_i(\cdot)(E)\, dP_i^S(\cdot). \tag{1u}$$

$P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is a common prior if P is a prior for every player $i \in I$.

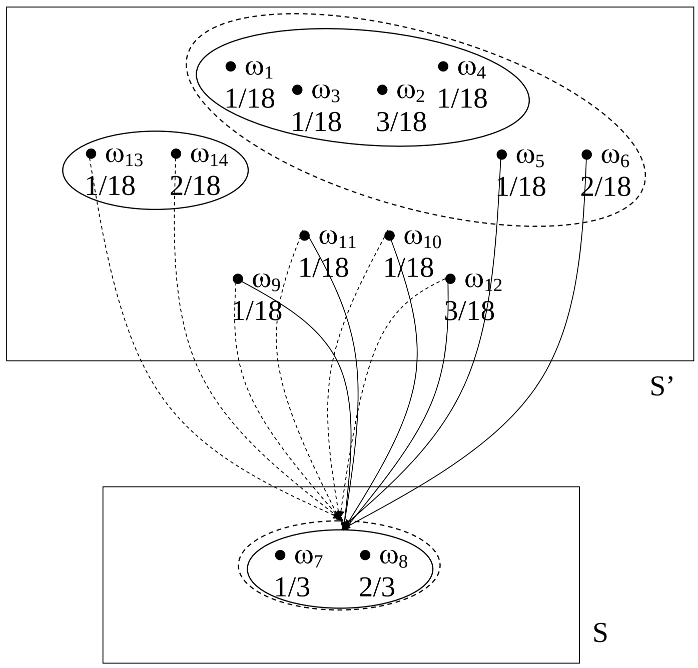
In particular, if S is finite or countable, and if $\mathcal{F}_S$ is the powerset of S, equality (1u) holds if and only if

$$P_i^S(E \cap S \cap A_i(E)) = \sum_{s \in S \cap A_i(E)} t_i(s)(E)\, P_i^S(\{s\}). \tag{2u}$$

What is the reason for the difference between (1) and (1u) (or similarly between (2) and (2u))? With unawareness, $t_i(s)(E)$ is well defined only for states $s \in S$ in which player i is aware of E, i.e., the states $s \in S \cap A_i(E)$. This is the cause for the difference in the definition of the domain of integration (or summation) on the right-hand side. Consequently, E (or equivalently $E \cap S$) on the left-hand side of (1) and (2) is replaced by $E \cap S \cap A_i(E)$ in (1u) and (2u).

The introductory example of speculative trade under unawareness has a common prior as indicated by the fractions below each state in Figure 1). To see Property 1., observe that the distribution on lower spaces coincides with the marginal of the distribution on the higher space. For Property 2., consider for instance the event of "high sales", $[s]$, and space $S_{\{s\}}$. On one hand, $P^{(S_{\{s\}})}([s] \cap S_{\{s\}} \cap A_i([s])) = \frac{1}{2}$. On the other hand, we have $t_i(s)([s]) \cdot P^{(S_{\{s\}})}(\{s\}) + t_i(\neg s)([s])P^{(S_{\{s\}})}(\{\neg s\}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$.

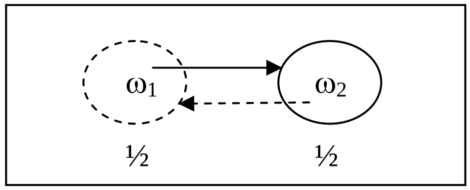
Figure 3: Illustration of a Common Prior



Another example of an unawareness structure with a common prior is given in Figure 3. Odd (resp. even) states in the upper space project to the odd (resp. even) state in the lower space. There are two individuals, one indicated by the solid lines and ellipses and another by dashed lines and ellipses. Note that the ratio of probabilities over odd and even states in each "information cell" coincides with the ratio in the "information cell" in the lower space.

A discussion of the interpretation of a common prior in unawareness structures is deferred to Section 3.4.

Figure 4: Speculative Trade with Delusion



3.2 Speculative Trade

In this section, we investigate whether the common prior assumption implies the absence of speculative trade (e.g. Milgrom and Stokey, 1982). The example in Section 1.1 shows that speculation is possible under unawareness even if we assume that there is a positive common prior. Despite this counter example to the "No-speculative-trade" theorems, we prove below a generalized "No-speculative-trade" theorem according to which, if there is a positive common prior, then there cannot be common certainty of *strict* preference to trade. That is, even with unawareness it is not the case that "everything goes". We find this surprising, because unawareness can be interpreted as a special form of "delusion": At a given state of a space, a player's belief may be concentrated in a very different lower state-space. It is known that speculative trade is possible in delusional standard state-space structures with a common prior. For instance, consider the information structure in Figure 4. The common prior and the information structure allows the dashed player to have a posterior of $t_{dashed}(\omega_1)(\{\omega_1\}) = t_{dashed}(\omega_2)(\{\omega_1\}) = 1$ and the solid player $t_{solid}(\omega_1)(\{\omega_2\}) = t_{solid}(\omega_2)(\{\omega_2\}) = 1$. So they may happily disagree on the expected value of a random variable defined on this standard state-space.

Denote by $[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\}$.

Definition 9 *A common prior* $P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ *is* positive *if and only if for all* $i \in I$ *and* $\omega \in \Omega$*: If* $t_i(\omega) \in \triangle(S')$*, then* $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$ *and* $P^S\left(([t_i(\omega)] \cap S')^{\uparrow} \cap S\right) > 0$ *for all* $S \succeq S'$.

For every type, a positive common prior puts a positive weight on the set of "stationary" states where the player has this type. It can be viewed as a technical condition that serves the same purpose as the assumption in Aumann (1976) that the prior puts strict positive weight on each partition cell in his finite partitional structure. This assumption is for instance satisfied in the introductory example in which we show the possibility of speculative trade under unawareness. The positivity condition ensures that the common prior indeed imposes consistency on the types. To see this, consider once again Figure 3. Replace the common prior by a prior that assigns $\frac{1}{6}$ to each state ω_9, ω_{10}, ω_{11} and $\frac{3}{6}$ to ω_{12}, and zero to all other states in S'. The prior probabilities for states in S remain unchanged. This prior is a common prior but it does not satisfy the positivity assumption of Definition 9. In particular, this common prior does not constrain any player's types with beliefs on S'. So, for unawareness belief structures the positivity assumption on the common prior ensures that the common prior constrains the beliefs of types not just locally on some space but across the lattice. Essentially, it is in the spirit of the common prior assumption according to which different beliefs are only due to differences in information. The positivity condition also implies that for each player there can be at most countably many types in each space. Moreover, in terms of awareness it implies that for every pair of players, i and j, and every event E, if i is certain that j is aware of the event E, then j is indeed aware of the event E with probability 1.

Next we define the set of states in which a player believes the expectation of a random variable to be above (resp. below) some real number x.

Definition 10 *Let x_1 and x_2 be real numbers and v a random variable on Ω. Define the sets* $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d(t_1(\omega))(\cdot) \leq x_1\right\}$ *and* $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d(t_2(\omega))(\cdot) \geq x_2\right\}$. *We say that at ω, conditional on his information, player 1 (resp. player 2) believes that the expectation of v is weakly below x_1 (resp. weakly above x_2) if and only if $\omega \in E_1^{\leq x_1}$ (resp. $\omega \in E_2^{\geq x_2}$).*

Note that the sets $E_1^{\leq x_1}$ or $E_2^{\geq x_2}$ may not be events in our unawareness belief structure, because $v(\omega) \neq v(\omega_S)$ is allowed, for $\omega \in S' \succ S$. Yet, we can define p-belief, mutual p-belief, and common certainty for measurable subsets of Ω, and show that the properties stated in Propositions 4 and 6 obtain as well.[7] The proofs are analogous and thus omitted.

We are now ready to state our "No-speculative-trade" result:

Theorem 1 *Let $\underline{S}$ be a finite unawareness belief structure and $P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ be a positive common prior. Then there is no state $\tilde{\omega} \in \Omega$ such that there are a random*

[7] A measurable subset of Ω is an $E \subseteq \Omega$ such that $E \cap S \in \mathcal{F}_S$, for all $S \in \mathcal{S}$. $\neg E$ is then understood to be the relative complement of E with respect to the union of state-spaces rather than our definition of the negation of an event. This plays a role in point (ii) of Proposition 4 applied to measurable subsets of Ω.

variable $v : \Omega \longrightarrow \mathbb{R}$ *and* $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, *with the following property: at* $\tilde{\omega}$ *it is common certainty that conditional on her information, player 1 believes that the expectation of* v *is weakly below* x_1 *and, conditional on his information, player 2 believes that the expectation of* v *is weakly above* x_2.

The theorem says that if there is a positive common prior, then there can not be common certainty of strict preference to trade.[8] Together with our example of speculative trade under unawareness we conclude that a common prior does not rule out speculation under unawareness but it can never be common certainty that both players expect to strictly gain from speculation. The theorem implies as a corollary that given a positive common prior, *arbitrary small transaction fees (e.g., a Tobin tax) rule out speculative trade under unawareness.*

We should note that the simple model leaves open what happens if in the introductory example the buyer offers more than 90 dollars to the seller. In this case the seller may suspect that he is unaware of *some* event that the buyer is aware of. It is not clear whether the seller would accept such an offer or not, and what the buyer would learn from it. Such kind of reasoning is outside the model. Grant and Quiggin (2011) discuss a heuristic for this case in our example.

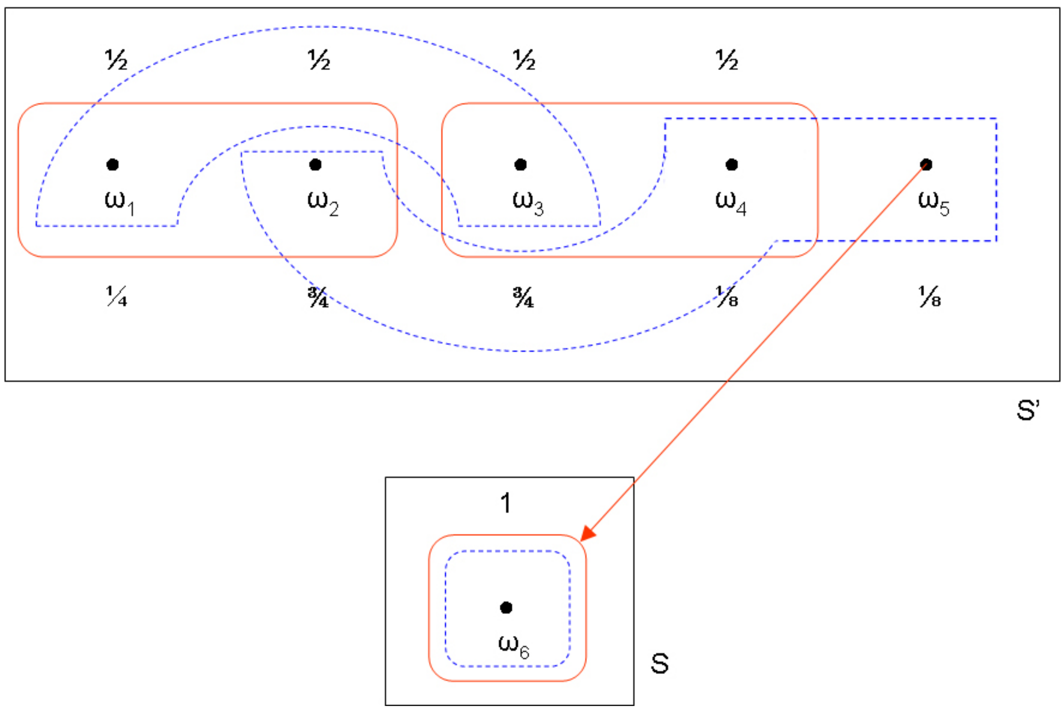
One may ask whether the absence of speculative trade implies a common prior under unawareness. The previous result suggests that heterogeneous unawareness with a common prior is "intermediate" between common awareness with heterogeneous priors on the one hand, and common awareness with a common prior on the other hand. With heterogeneous priors even in standard state-spaces, common certainty of strict preference to trade is possible. In standard state-spaces, the absence of speculative trade implies a common prior (see for instance Feinberg, 2000). This is the converse to the "No-speculative-trade" theorem. The following example shows that under unawareness the converse of our "No-speculative-trade" theorem does not hold.

Example 1 Consider the information structure with two spaces in Figure 5. There are two players: The information structure of the first (resp. second) player is given by the solid (resp. intermitted) objects. The belief of the first (resp. second) player is given above (resp. below) the states. Since the relative weights differ, there can not be a positive common prior. In fact, there is not even a common prior since equation (2u)

[8] In Meier and Schipper (2010), we extend the above "No-speculative-trade" theorem to infinite unawareness belief structures.

of Definition 8 imposed on the priors of both individuals would imply that the common prior assigns probability zero to all states in S'. Note that the only measurable sets that are common certainty among both players are $\Omega = S' \cup S$ and S. Yet, it is not true that in all states in Ω or S player 1's expectation of a random variable differs from player 2's expectation. E.g., at ω_6 both player's expectations of the random variable must agree. Thus, the absence of common certainty of strict preference to trade does not imply the existence of a (positive) common prior.

Figure 5: Information Structure of the Counter-Example



3.3 Agreement

For an event E and $p \in [0,1]$ define the set $[t_i(E) = p] := \{\omega \in \Omega : t_i(\omega)(E) = p\}$, if $\{\omega \in \Omega : t_i(\omega)(E) = p\}$ is nonempty, and otherwise set $[t_i(E) = p] := \emptyset^{S(E)}$.

Lemma 1 *$[t_i(E) = p]$ is a $S(E)$-based event.*

Proof. $[t_i(E) = p] = B_i^p(E) \cap B_i^{1-p}(\neg E)$. Hence the claim follows from Proposition 2.□

The following proposition is a generalization of the standard *"No-Agreeing-to-Disagree" theorem* (Aumann, 1976):

Proposition 3 *Let $\underline{S}$ be an unawareness belief structure for which there exists a positive common prior, G be an event, and $p_i \in [0,1]$, for $i \in I$. If $CB^1(\bigcap_{i \in I}[t_i(G) = p_i])$ is nonempty, then $p_i = p_j$, for all $i, j \in I$.*

The proposition asserts that, even under unawareness, if individuals have a positive common prior and common certainty of posteriors for an event (and thus common awareness of that event), then the posteriors must agree among all individuals. So individuals with a positive common prior cannot agree-to-disagree on the posteriors of events which they are all aware of.

As mentioned previously, the assumption of a *positive* common prior is a technical assumption akin to the assumption of a prior that puts strict positive probability on each partition cell in Aumann (1976). It can be weakened further considerably by requiring only a common prior on a space $S \succeq S(G)$ satisfying $P^S(CB^1(\bigcap_{i \in I}[t_i(G) = p_i])) > 0$. In the appendix, we actually prove this more general version of the "No-agreement-to-disagree" theorem and show that this condition is implied by a positive common prior.

3.4 Discussion of the Common Prior Assumption

How could a *prior* under unawareness be interpreted? Following the discussion of the notion of a prior in standard Bayesian analysis by Savage (1954), Morris (1995), and Samet (1999), we like to distinguish three interpretations: First, a prior is interpreted verbally as a player's subjective belief at a prior stage. Second, the prior is a coherence condition on the player's types. Third, the prior is the long run relative frequency of repeated events observed by the player in the past.

Consider the first interpretation. A prior is a subjective belief at a prior stage before the player received further information which led her to the interim belief $t_i(\omega)$. With unawareness, this interpretation is nonsensical. One would have to imagine that the player had been aware of all relevant aspects of reality at the prior stage, but then became unaware of some of them (while nevertheless having received more information regarding other aspects).

In standard Bayesian analysis, Samet (1999) put forward a second interpretation of a prior as a coherence condition on types: For every event $E \in \Sigma$ and every $p \in [0,1]$, every type of the player answers affirmatively to the question "Given that tomorrow you will assign to the event E probability at least p, do you assign to E probability at least p now?" This interpretation is conceptually valid also for unawareness belief structures with an important qualification: Every type of the player is asked these questions only for events of which she is aware because otherwise a question by itself may make the type aware of an event of which she was previously unaware. While this qualification is vacuous in standard Bayesian analysis - because of the implicit assumption of full awareness - it implies for unawareness belief structures that each type is "aware" only of the prior restricted to the events that she is aware of. Moreover, every type can only perceive the beliefs of her alternative types of which she is aware. This emphasizes that the prior is derived from types rather than being a primitive of the model.

The third interpretation views the prior as the relative frequency of events observed previously by the individual as history goes to infinity and before receiving information which led to her interim belief $t_i(\omega)$. Again, with unawareness such a interpretation is nonsensical. One would have to imagine that the player had been measuring all events in history, but then became unaware of some of them (while nevertheless having received more information regarding other events). To recapture the validity of the frequentist interpretation, we must assume that every player can observe only events that she is aware of interim. This assumption is quite reasonable since a player can only measure what she is aware of. For instance, meteorologists were unable to measure ozone before they became aware of it. Yet, the applicability of the frequentist interpretation may be limited since we allow also for conditioning on unobservable events (such as types of other players), a caveat that applies not only to unawareness belief structures but to belief structures in general.

What are the implications of the absence of speculation on the priors? For standard type-spaces, the converse to the "No-speculative-trade" theorem characterizes the common prior assumption through the absence of speculative trade (Morris, 1994, Bonanno and Nehring, 1999, Feinberg, 2000, Halpern, 2002, Heifetz, 2006). Example 1 shows that we cannot characterize positive common priors or even just common priors on unawareness belief structures by the absence of common certainty of strict preference to trade. Note that our notion of "No-speculative-trade" is slightly different from the literature: For instance, Feinberg (2000) characterizes the common prior by the absence of common certainty of speculation for *some* states. We show that a positive common prior implies the absence of common certainty of speculation for *all* states. Hence, our notion of "No-speculative-trade" implies Feinberg's notion of "No-speculative-trade".[9] Note,

[9]We opted for our notion of "local" speculation because intuitively one is interested to know whether

however, that the impossibility of the converse to a "No-speculative-trade" theorem for unawareness belief structures is not due to the different notion of "No-speculative-trade" employed. To see this, consider once again Example 1. At state ω_6 it is not common certainty that players want to speculate. Yet, we noticed already that there is no common prior in this model. Hence, also "No-speculative-trade" in the sense of Feinberg does not imply a common prior in unawareness belief structures. To sum up, we show that it is still possible to define the common prior assumption under unawareness. Moreover, our "No-speculative-trade" theorem demonstrates that the common prior assumption enhanced by positivity imposes discipline. Yet, contrary to standard type-spaces the common prior assumption is not "provable" by the absence of speculation under unawareness, it just remains (in principle) "falsifiable". The possibility of characterizing a common prior by absence of speculation in the standard type-space versus the impossibility of such characterization in unawareness belief structures illustrates an important difference between unawareness belief structures and standard type-spaces.

4 Universal Unawareness Belief Type Space

In this section, we investigate the foundations of unawareness belief structures. Our aim is to construct unawareness belief structures explicitly from hierarchies of beliefs, and show the existence of a universal unawareness type space that contains all hierarchies of beliefs analogous to Mertens and Zamir (1985) for type spaces without unawareness.

4.1 Hierarchical Construction

The lattice of spaces of states of nature is

$$\left\{ Z^L = \prod_{d \in L} Z_d \right\}_{L \subseteq D}$$

where each Z_d is a Hausdorff topological space, and represents one dimension of reality. The set of dimensions D is finite or countable.[10] (For $L = \emptyset$ we maintain the convention that $Z^\emptyset = \{\emptyset\}$.)

there are *some* states (as opposed to *all* states) where players speculate. Our notion of "No-speculative-trade" coincides with Feinberg's notion on belief closed subsets.

[10]We could start with a different complete lattice of Hausdorff topological (state) spaces $\langle \{Z_\alpha\}_{\alpha \in \mathcal{A}}, \succeq \rangle$ and surjective continuous projections between them in accordance with the lattice order, as long as the properties of Sections 2.1 and 2.2 are satisfied.

For each subset of dimensions $L \subseteq D$ we will construct the space of player i's hierarchies of beliefs when i is aware only of the dimensions in L (but may believe that other players are aware of less dimensions, and so forth recursively).

The basic domain of belief for such a player is

$$Y_0^{i,L} = \prod_{d \in L} Z_d = Z^L.$$

The first-order beliefs of the player,

$$M_1^{i,L} = \Delta\left(Y_0^{i,L}\right)$$

is the space of regular Borel probability measures on $Y_0^{i,L}$, endowed with the topology of weak convergence.

The domain of second-order beliefs of the player is

$$Y_1^{i,L} = \prod_{d \in L} Z_d \times \prod_{j \neq i} \left(\bigcup_{L' \subseteq L} M_1^{j,L'} \right).$$

For $y_1^{i,L} = \left((z_d)_{d \in L}, (m_1^{j,L_j})_{j \neq i}\right)$ define $p_0^{i,L,1}\left(y_1^{i,L}\right) := (z_d)_{d \in L}$.

If $p_{n-1}^{i,L,n} : Y_n^{i,L} \longrightarrow Y_{n-1}^{i,L}$ has already been defined, define for $\mu_{n+1}^{i,L} \in \Delta(Y_n^{i,L})$:

$$\left(\operatorname{marg}_{Y_{n-1}^{i,L}}\left(\mu_{n+1}^{i,L}\right)\right)(\cdot) := \mu_{n+1}^{i,L}\left(\left(p_{n-1}^{i,L,n}\right)^{-1}(\cdot)\right).$$

Inductively, suppose that for $k = 1, \ldots, n$, for every subset of dimensions $L \subseteq D$, and for every player $i \in I$, when she is aware only of the subset of dimensions in L, we have already defined the spaces $M_k^{i,L}$ of k-level hierarchies, as well as her domain of $(k+1)$-order beliefs $Y_k^{i,L}$. Define

$$M_{n+1}^{i,L} = \left\{ \left(\left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}\right), \mu_{n+1}^{i,L}\right) \in M_n^{i,L} \times \Delta\left(Y_n^{i,L}\right) : \operatorname{marg}_{Y_{n-1}^{i,L}} \mu_{n+1}^{i,L} = \mu_n^{i,L} \right\}.$$

$$q_n^{i,L,n+1} : M_{n+1}^{i,L} \longrightarrow M_n^{i,L}$$

is now naturally defined as

$$q_n^{i,L,n+1}\left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}, \mu_{n+1}^{i,L}\right) := \left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}\right).$$

Now, define

$$Y_{n+1}^{i,L} := \prod_{d \in L} Z_d \times \prod_{j \neq i} \left(\bigcup_{L' \subseteq L} M_{n+1}^{j,L'} \right)$$

and

$$p_n^{i,L,n+1} : Y_{n+1}^{i,L} \longrightarrow Y_n^{i,L}$$

is defined as follows:

$$p_n^{i,L,n+1}\left((z_d)_{d\in L}, \left(m_{n+1}^{j,L_j}\right)_{j\neq i}\right) := \left((z_d)_{d\in L}, \left(q_n^{j,L_j,n+1}(m_{n+1}^{j,L_j})\right)_{j\neq i}\right).$$

In the limit, define

$$M_\infty^{i,L} = \left\{\left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}, \ldots\right) : \left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}\right) \in M_n^{i,L} \text{ for all } n \in N\right\}$$

and

$$Y_\infty^{i,L} = \prod_{d\in L} Z_d \times \prod_{j\neq i}\left(\bigcup_{L'\subseteq L} M_\infty^{j,L'}\right).$$

Each $m_\infty^{i,L} = \left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}, \ldots\right) \in M_\infty^{i,L}$ is a projective system of probability measures, and by the Kolmogorov extension theorem it has an inverse limit, that is, a unique probability measure

$$t_\infty^{i,L}\left(m_\infty^{i,L}\right) \in \Delta\left(Y_\infty^{i,L}\right)$$

whose marginal on $Y_n^{i,L}$ is $\mu_{n+1}^{i,L}$ for every $n \geq 0$. Conversely, any probability measure μ on $Y_\infty^{i,L}$ induces a unique projective system in $M_\infty^{i,L}$ whose projective limit is μ. The measure $t_\infty^{i,L}\left(m_\infty^{i,L}\right)$ induces $m_\infty^{i,L}$.

Hence, the map

$$t_\infty^{i,L} : M_\infty^{i,L} \longrightarrow \Delta\left(Y_\infty^{i,L}\right)$$

is a bijection. But also, by standard arguments (see, for example, the proof of Theorem 9 in Heifetz, 1993), this map is a homeomorphism (i.e., continuous with a continuous inverse).

Next, we define projections from higher to lower levels of awareness. For any subsets of dimensions $\hat{L} \subseteq L \subseteq D$ define the (Borel measurable) projection

$$\rho_0^{i,L,\hat{L}} : Y_0^{i,L} \longrightarrow Y_0^{i,\hat{L}}$$

between i's domain of belief $Y_0^{i,L}$ and the poorer domain $Y_0^{i,\hat{L}}$ by

$$\rho_0^{i,L,\hat{L}}\left((z_d)_{d\in L}\right) = (z_d)_{d\in\hat{L}}.$$

Define also the marginal

$$\eta_1^{i,L,\hat{L}} : M_1^{i,L} \longrightarrow M_1^{i,\hat{L}}$$

by

$$\left(\eta_1^{i,L,\hat{L}}\left(\mu_1^{i,L}\right)\right)(\cdot) := \mu_1^{i,L}\left(\left(\rho_0^{i,L,\hat{L}}\right)^{-1}(\cdot)\right) = \left(\mathrm{marg}_{\left(\prod_{d\in\hat{L}} Z_d\right)}\mu_1^{i,L}\right)(\cdot).$$

Inductively, define the (Borel measurable) projection between i's domains of belief across her different awareness levels

$$\rho_{n+1}^{i,L,\hat{L}} : Y_{n+1}^{i,L} \longrightarrow Y_{n+1}^{i,\hat{L}}$$

by

$$\rho_{n+1}^{i,L,\hat{L}}\left((z_d)_{d\in L}, \left(m_{n+1}^{j,L'_j}\right)_{j\neq i}\right) = \left((z_d)_{d\in\hat{L}}, \left(\eta_{n+1}^{j,L'_j,(L'_j\cap\hat{L})}\left(m_{n+1}^{j,L'_j}\right)\right)_{j\neq i}\right)$$

and the marginal

$$\eta_{n+1}^{i,L,\hat{L}} : M_{n+1}^{i,L} \longrightarrow M_{n+1}^{i,\hat{L}}$$

by

$$\eta_{n+1}^{i,L,\hat{L}}\left(\left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}\right), \mu_{n+1}^{i,L}\right) = \left(\mu_1^{i,L}\left(\rho_0^{i,L,\hat{L}}\right)^{-1}, \ldots, \mu_n^{i,L}\left(\rho_{n-1}^{i,L,\hat{L}}\right)^{-1}, \mu_{n+1}^{i,L}\left(\rho_n^{i,L,\hat{L}}\right)^{-1}\right).$$

One can check that

$$\left(\mathrm{marg}_{Y_{n-1}^{i,\hat{L}}}\left(\mu_{n+1}^{i,L}\right)\right)\left(\left(\rho_n^{i,L,\hat{L}}\right)^{-1}(\cdot)\right) = \mu_n^{i,L}\left(\left(\rho_{n-1}^{i,L,\hat{L}}\right)^{-1}(\cdot)\right),$$

that is, the above definition makes sense.

In the limit, define the marginal

$$\eta_\infty^{i,L,\hat{L}} : M_\infty^{i,L} \longrightarrow M_\infty^{i,\hat{L}}$$

by

$$\eta_\infty^{i,L,\hat{L}}\left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}, \ldots\right) = \left(\mu_1^{i,L}\left(\rho_0^{i,L,\hat{L}}\right)^{-1}, \ldots, \mu_n^{i,L}\left(\rho_{n-1}^{i,L,\hat{L}}\right)^{-1}, \ldots\right)$$

and the projections between i's domains of belief across her different awareness levels

$$\rho_\infty^{i,L,\hat{L}} : Y_\infty^{i,L} \longrightarrow Y_\infty^{i,\hat{L}}$$

by

$$\rho_\infty^{i,L,\hat{L}}\left((z_d)_{d\in L}, \left(m_\infty^{j,L'_j}\right)_{j\neq i}\right) := \left((z_d)_{d\in\hat{L}}, \left(\eta_\infty^{j,L'_j,(L'_j\cap\hat{L})}\left(m_\infty^{j,L'_j}\right)\right)_{j\neq i}\right).$$

One can verify that the beliefs of i's types commute with these projections and marginals:

$$\left(t_\infty^{i,\hat{L}} \circ \eta_\infty^{i,L,\hat{L}}\left(m_\infty^{i,L}\right)\right)(\cdot) = \left(t_\infty^{i,L}\left(m_\infty^{i,L}\right)\right)\left(\left(\rho_\infty^{i,L,\hat{L}}\right)^{-1}(\cdot)\right). \quad (3)$$

Finally, define the lattice of spaces

$$Y^L = \prod_{d \in L} Z_d \times \prod_{i \in I} \left(\bigcup_{L' \subseteq L} M_\infty^{i,L'} \right)$$

partially ordered by the partial inclusion order of subsets of dimensions $L \subseteq D$. Redefine the beliefs of each type $m_\infty^{i,L'}$ to be on $Y^{L'}$ (rather than on the above "personalized" domain $Y_\infty^{i,L'}$) so as to express the idea that each type is introspective, as follows: For every

$$y^L = \left((z_d)_{d \in L}, \left(m_\infty^{j,L'_j} \right)_{j \neq i}, m_\infty^{i,L'} \right) \in Y^L, \quad L \supseteq L'$$

define

$$t_i \left(y^L \right) \in \Delta \left(Y^{L'} \right)$$

by

$$t_i \left(y^L \right)(E) := t_\infty^{i,L'} \left(m_\infty^{i,L'} \right) \left(\left\{ \left((z'_d)_{d \in L'}, \left(m_\infty^{j,L'_j} \right)_{j \neq i} \right) \in Y_\infty^{i,L'} : \left((z'_d)_{d \in L'}, \left(m_\infty^{j,L'_j} \right)_{j \neq i}, m_\infty^{i,L'} \right) \in E \right\} \right)$$

for every Borel subset $E \subseteq Y^{L'}$.

For $L \supseteq \hat{L}$, let

$$r_{\hat{L}}^L : Y^L \longrightarrow Y^{\hat{L}}$$

be the natural projection defined by

$$r_{\hat{L}}^L \left((z_d)_{d \in L}, \left(m_\infty^{i,L'_i} \right)_{i \in I} \right) := \left((z_d)_{d \in \hat{L}}, \left(\eta_\infty^{i,L'_i,(L'_i \cap \hat{L})} \left(m_\infty^{i,L'_i} \right) \right)_{i \in I} \right)$$

For $y^L \in Y^L$ we will denote

$$y_{\hat{L}}^L = r_{\hat{L}}^L \left(y^L \right).$$

Notice that these projections and the beliefs t_i satisfy the following properties:

(0) Confinement: $t_i \left(y^L \right) \in \Delta \left(Y^{L'} \right)$ for some $Y^{L'}$, with $L' \subseteq L$.

(1) If $\ddot{L} \supseteq \dot{L} \supseteq L$, $y^{\ddot{L}} \in Y^{\ddot{L}}$ and $t_i \left(y^{\ddot{L}} \right) \in \Delta \left(Y^L \right)$ then

$$t_i \left(y_{\dot{L}}^{\ddot{L}} \right) = t_i \left(y^{\ddot{L}} \right).$$

(2) If $\ddot{L} \supseteq \dot{L} \supseteq L$, $y^{\ddot{L}} \in Y^{\ddot{L}}$ and $t_i \left(y^{\ddot{L}} \right) \in \Delta \left(Y^{\dot{L}} \right)$ then

$$t_i \left(y_L^{\ddot{L}} \right) = t_i \left(y^{\ddot{L}} \right) \left(r_L^{\dot{L}} \right)^{-1}.$$

(3) If $\ddot{L} \supseteq \dot{L} \supseteq L$, $y^{\ddot{L}} \in Y^{\ddot{L}}$ and $t_i\left(y^{\ddot{L}}_{\dot{L}}\right) \in \Delta\left(Y^L\right)$ then $t_i\left(y^{\ddot{L}}\right) \in \Delta\left(Y^{\hat{L}}\right)$ for some $\hat{L} \supseteq L$.

Property (2) follows from Equation (3). The Properties (0) – (3) are the properties of the type mappings in the definition of unawareness belief structures (Definition 1 of this paper).

For each $L \subseteq D$ and $\dot{L} \supseteq L$, $Y^{\dot{L}}$ has obviously-defined projections onto Y^L. They are such, that the properties in Sections 2.1 and 2.2 are satisfied. Also, the introspection property is satisfied in the resulting structure.

We say that

$$\underline{\mathcal{Y}} := \left\langle \left\{Y^L\right\}_{L\subseteq D}, (r^L_{\hat{L}})_{\hat{L}\subseteq L}, (t_i)_{i\in I} \right\rangle$$

is the universal space with unawareness (for the lattice of spaces of states of nature $\left\{Z^L\right\}_{L\subseteq D}$ and the set of players I), in a sense that we will make precise in the next section.

From what we have remarked above, it follows that this structure is an unawareness belief space.

4.2 Category of Unawareness Type Spaces and Universality

In this section we sketch the existence of a universal unawareness type space.

Having fixed Hausdorff spaces $\{Z_d\}_{d\in D}$, let $\mathcal{L} \subseteq 2^D$ be a collection of subsets of D, that forms a complete lattice with set theoretic union as join and set theoretic intersection as meet. We call such a collection $\mathcal{L}$ eligible.

An *unawareness type space* $\underline{S} = \left\langle \{S^L\}_{L\in\mathcal{L}}, (r^L_{\hat{L}})_{\hat{L}\subseteq L}, (t_i)_{i\in I}, (\theta^L)_{L\in\mathcal{L}} \right\rangle$ is a lattice of Hausdorff topological spaces $\left\{S^L\right\}_{L\in\mathcal{L}}$, with Borel measurable maps

$$\theta^L : S^L \longrightarrow \prod_{d\in L} Z_d,$$

for $L \in \mathcal{L}$, specifying the state of nature (in the corresponding set of dimensions L) and Borel measurable introspective belief maps t_i, $i \in I$, such that for every $\omega^L \in S^L$, $t_i\left(\omega^L\right) \in \Delta\left(S^{L'}\right)$ for some $L' \subseteq L$; and with commuting projections

$$r^L_{\hat{L}} : S^L \longrightarrow S^{\hat{L}}, \quad \hat{L} \subseteq L, \quad \hat{L}, L \in \mathcal{L}$$

which commute also with θ^L, and satisfy properties (0)–(3) above (with ω-s instead of y-s).

In particular, the hierarchical construction $\underline{\mathcal{Y}}$ from the previous section is an unawareness type space.

Let $\mathcal{L} \subseteq \dot{\mathcal{L}}$ be eligible collections, and let $\underline{\mathcal{S}} = \left\langle \{S^L\}_{L\in\mathcal{L}}, (r^L_{\hat{L}})_{\hat{L}\subseteq L}, (t_i)_{i\in I}, (\theta^L)_{L\in\mathcal{L}} \right\rangle$ and $\underline{\dot{\mathcal{S}}} = \left\langle \{\dot{S}^L\}_{L\in\dot{\mathcal{L}}}, (\dot{r}^L_{\hat{L}})_{\hat{L}\subseteq L}, (\dot{t}_i)_{i\in I}, (\dot{\theta}^L)_{L\in\dot{\mathcal{L}}} \right\rangle$ be two unawareness type spaces (with corresponding mappings and projections, denoted with and without a dot, respectively). We say that the collection of mappings

$$\varphi^L : S^L \to \dot{S}^L, \quad L \in \mathcal{L}$$

is a *type morphism* if these mappings preserve the state of nature

$$\dot{\theta}^L \varphi^L = \theta^L$$

and the beliefs of the players: If $t_i\left(\omega^L\right) \in \Delta\left(S^{L'}\right)$ then $\dot{t}_i\left(\varphi^L\left(\omega^L\right)\right) \in \Delta\left(\dot{S}^{L'}\right)$ and

$$\left(\dot{t}_i\left(\varphi^L\left(\omega^L\right)\right)\right)(\cdot) = t_i\left(\omega^L\right)\left(\left(\varphi^{L'}\right)^{-1}(\cdot)\right)$$

Any unawareness type space $\underline{\mathcal{S}}$ admits the following inductively defined mappings into the spaces $M_n^{i,L}$, which unfold the players' beliefs in states $\omega^{\dot{L}}$ in which $t_i\left(\omega^{\dot{L}}\right) \in \Delta\left(S^L\right)$, as follows:

$$\left(h_1^{i,L}\left(\omega^{\dot{L}}\right)\right)(\cdot) := t_i\left(\omega^{\dot{L}}\right)\left(\left(\theta^L\right)^{-1}(\cdot)\right) \in \Delta\left(Y_0^{i,L}\right)$$

and inductively

$$h_{n+1}^{i,L}\left(\omega^{\dot{L}}\right) := \left(h_n^{i,L}\left(\omega^{\dot{L}}\right), t_i\left(\omega^{\dot{L}}\right)\left(\theta^L, \left(\cup_{L'\subseteq L} h_n^{j,L'}\right)_{j\neq i}\right)^{-1}\right).$$

In the limit, define the entire unfolding of player i's belief at $\omega^{\dot{L}}$ to be

$$h_\infty^{i,L}\left(\omega^{\dot{L}}\right) := \left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}, \ldots\right)$$

such that

$$\left(\mu_1^{i,L}, \ldots, \mu_n^{i,L}\right) = h_n^{i,L}\left(\omega^{\dot{L}}\right),$$

for all $n \geq 1$.

Combining this map for all the players and for the state of nature at $\omega^{\dot{L}}$, for all the states $\omega^{\dot{L}}$ in the unawareness type space $\underline{\mathcal{S}}$, constitutes the unique type morphism into $\underline{\mathcal{Y}}$ (since type morphisms preserve this explicit-description unfolding). As there can be at most one universal space, this establishes that $\underline{\mathcal{Y}}$ is universal.

5 Related Literature

There is a growing literature on unawareness both in economics and computer science. The independent parallel work of Sadzik (2006) is closest to ours. Building to a certain extent on our earlier work, Heifetz, Meier, and Schipper (2006), he presents a framework of unawareness with probabilistic beliefs in which the common prior on the upmost space is a primitive. In contrast, we take types as primitives and define a prior on the entire unawareness belief structure as a convex combination of the type's beliefs.

In a companion paper, Meier and Schipper (2012a) apply unawareness belief structures to develop Bayesian games with unawareness, define Bayesian Nash equilibrium, and prove existence. Moreover, they investigate the robustness of equilibria in strategic games to uncertainty about opponents' unawareness of actions.

Feinberg (2009) discusses games with unawareness by modeling games and many views thereof, each (mutual) view being a finite sequence of player names $i_1, ..., i_n$ with the interpretation that this is how i_1 views how .... how i_n views the game. This differs from our unawareness belief structures in which each state "encapsulates" the views of the players, their views about other players' views etc. in a standard and parsimonious way.

Halpern and Rêgo (2006), Rêgo and Halpern (2012), Li (2006), Heifetz, Meier, and Schipper (2011a,b), Meier and Schipper (2012b), and Feinberg (2009) present models of extensive-form games with unawareness and analyze solution concepts for them. Li (2006) is based on Li (2009), in which she presents a set theoretic model with knowledge and non-trivial unawareness. A state-space is a product set where each dimension corresponds to an issue. A decision maker may be unaware of some issues by "living in" a space with less dimensions. Modica (2008) studies the updating of probabilities and argues that new information may change posteriors more if it implies also a higher level of awareness. A dynamic framework for a single decision maker with unawareness is introduced by Grant and Quiggin (2011) who also discuss heuristic approaches in the face of awareness of unawareness. Ewerhart (2001) studies the possibility of agreement under a notion of unawareness different from the aforementioned literature.

More recently we learned that Board and Chung (2011) presented a different model of unawareness in which they also study speculative trade under what they term living in "denial" and "paranoia". The precise connection to our results is yet to be explored.

The literature on unawareness is related to the recent work in behavioral economics, finance, and macroeconomics that discusses the economic relevance of peoples inattention for various economic outcomes such as retirement savings, portfolio choice, choice of health care plans, etc. This literature focuses on questions such as how to design optimally economic policies to "nudge" people's attention (Thaler and Sunstein, 2008) or how to optimally allocate (voluntary) inattention (e.g. Sims, 2010, Van Nieuwerburgh and Veldkamp, 2010). While the notions of inattention discussed in behavioral economics, finance, and macroeconomics may not correspond exactly to the well-defined epistemic notion of unawareness and may additionally involve biases and features of bounded rationality, we believe that unawareness may be one component of those notions of inattention. Our "No-speculative-trade" result can be viewed as showing the absence of speculative trade with rational but involuntarily inattentive agents.

Appendices

A Properties of Belief and Awareness

Proposition 4 *Let E and F be events, $\{E_l\}_{l=1,2,...}$ be an at most countable collection of events, and $p, q \in [0,1]$. The following properties of belief obtain:*

(o) $B_i^p(E) \subseteq B_i^q(E)$, *for* $q \leq p$,

(i) Necessitation: $B_i^1(\Omega) = \Omega$,

(ii) Additivity: $B_i^p(E) \subseteq \neg B_i^q(\neg E)$, *for* $p + q > 1$,

(iiia) $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) \subseteq \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiib) for any decreasing sequence of events $\{E_l\}_{l=1}^{\infty}$, $B_i^p\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^p(E_l)$,

(iiic) $B_i^1\left(\bigcap_{l=1}^{\infty} E_l\right) = \bigcap_{l=1}^{\infty} B_i^1(E_l)$,

(iv) Monotonicity: $E \subseteq F$ *implies* $B_i^p(E) \subseteq B_i^p(F)$,

(va) Introspection: $B_i^p(E) \subseteq B_i^1 B_i^p(E)$,

(vb) Introspection II: $B_i^p B_i^q(E) \subseteq B_i^q(E)$, *for* $p > 0$.

In our unawareness belief structure, Necessitation means that an individual always is certain of the universal event Ω, i.e., she is certain of "tautologies with the lowest expressive power." (ii) means that if an individual believes an event E with at least probability p, then she can not believe the negation of E with any probability strictly greater than $1-p$. Property (iii a - c) are variations of conjunction, i.e., if an individual believes a conjunction of events with probability at least p, then she p-believes each of the events. The interpretation of monotonicity is: If an event E implies an event F, then p-believing the event E implies that the individual also p-believes the event F. Property (v) concerns the introspection of belief: If an individual believes the event E with at least probability p then she is certain that she believes the event E with at least probability p. Also, if she believes with positive probability that she p-believes an event, the she actually p-believes this event.

The following properties of awareness and belief obtain.

Proposition 5 *Let E be an event and $p, q \in [0,1]$. The following properties of awareness and belief obtain: 1. Plausibility: $U_i(E) \subseteq \neg B_i^p(E) \cap \neg B_i^p \neg B_i^p(E)$, 2. Strong Plausibility: $U_i(E) \subseteq \bigcap_{n=1}^{\infty} \left(\neg B_i^p\right)^n (E)$, 3. B^pU Introspection: $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0,1]$ and $B_i^0 U_i(E) = A_i(E)$, 4. AU Introspection: $U_i(E) = U_i U_i(E)$, 5. Weak Necessitation: $A_i(E) = B_i^1 \left(S(E)^{\uparrow}\right)$, 6. $B_i^p(E) \subseteq A_i(E)$ and $B_i^0(E) = A_i(E)$, 7. $B_i^p(E) \subseteq A_i B_i^q(E)$, 8. Symmetry: $A_i(E) = A_i(\neg E)$, 9. A Conjunction: $\bigcap_{\lambda \in L} A_i(E_\lambda) = A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$, 10. AB^p Self Reflection: $A_i B_i^p(E) = A_i(E)$, 11. AA Self Reflection: $A_i A_i(E) = A_i(E)$, and 12. $B_i^p A_i(E) = A_i(E)$.*

These properties are analogous to the properties in unawareness knowledge structures (Heifetz, Meier, and Schipper, 2006, 2008). Properties 1 to 5 have been suggested by Dekel, Lipman, and Rustichini (1998), and 8 to 11 by Fagin and Halpern (1988), Modica and Rustichini (1999) and Halpern (2001).

Note that properties 3, 4, 5, 8, 9, 11, and 12 hold also for non-measurable events, because even if E is not measurable, by 5. $A_i(E)$ is measurable.

Definition 11 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 6 *For every event $F \in \Sigma$:*

(i) $CB^1(F)$ is evident, that is $CB^1(F) \subseteq B_i^1(CB^1(F))$ for all $i \in I$.

(ii) There exists an evident event E such that $\omega \in E$ and $E \subseteq B_i^1(F)$ for all $i \in I$, if and only if $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

Proposition 7 *Let E be an event and $p, q \in [0,1]$. The following multi-person properties obtain:*

1. $A_i(E) = A_i A_j(E)$,
2. $A_i(E) = A_i B_j^p(E)$,
3. $B_i^p(E) \subseteq A_i B_j^q(E)$,
4. $B_i^p(E) \subseteq A_i A_j(E)$,
5. $CA(E) = A(E)$,
6. $CB^1(E) \subseteq CA(E)$,
7. $B^p(E) \subseteq CA(E)$, $B^0(E) = CA(E)$,
8. $B^p(E) \subseteq A(E)$, $B^0(E) = A(E)$,
9. $A(E) = B^1(S(E)^{\uparrow})$,
10. $CA(E) = B^1(S(E)^{\uparrow})$,
11. $CB^1(S(E)^{\uparrow}) \subseteq A(E)$,
12. $CB^1(S(E)^{\uparrow}) \subseteq CA(E)$,

Note that properties 1, 5, 9, 10, 11, and 12 also hold for non-measurable events.

B The Connection to Standard Type Spaces

In this section, we show how to derive a standard type-space from our unawareness structure by "flattening" our lattice of spaces. "Flattening" the belief structure is a purely mathematical procedure that essentially "erases" the "language" required to identify events that agents could be unaware of. Since a flattened structure is a standard type-space, the Dekel-Lipman-Modica-Rustichini critique applies. Hence unawareness is trivial in the flattened structure. We also mention a simple example that demonstrates that not every standard type-space can be derived from a non-trivial unawareness belief structure.

Definition 12 *$G \subseteq \Omega$ is a* measurable set *if and only if for all $S \in \mathcal{S}$, $G \cap S \in \mathcal{F}_S$.*

Notice that a measurable set is not necessarily an event in our special event structure.

Remark 3 *The collection of measurable sets forms a sigma-algebra on Ω.*

Remark 4 *Let $\mathcal{S}$ be at most countable and G be a measurable set, $p \in [0,1]$ and $i \in I$. Then $\{\omega \in \Omega : t_i(\omega)(G) \geq p\}$ is a measurable set.*

Let $\underline{S}$ be an unawareness belief structure. We define the *flattened type-space* associated with the unawareness belief structure $\underline{S}$ by

$$F(\underline{S}) := \langle \Omega, \mathcal{F}, (t_i^F)_{i \in I} \rangle,$$

where $\Omega = \bigcup_{S \in \mathcal{S}} S$ is the union of all state-spaces in the unawareness belief structure $\underline{\mathcal{S}}$, $\mathcal{F}$ is the collection of all measurable sets in $\underline{\mathcal{S}}$, and $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is defined by

$$t_i^F(\omega)(E) \quad := \quad \begin{cases} t_i(\omega)(E \cap S_{t_i(\omega)}) & \text{if } E \cap S_{t_i(\omega)} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

A *standard type-space* on X for the player set I is a tuple

$$\underline{X} := \left\langle X, \mathcal{F}_X, (t_i)_{i \in I} \right\rangle,$$

where X is a nonempty set, $\mathcal{F}_X$ is a sigma-field on X, and for $i \in I$, t_i is a $\mathcal{F}_X - \mathcal{F}_{\Delta(X)}$ measurable function from X to $\Delta(X, \mathcal{F}_X)$, the space of countable additive probability measures on $(X, \mathcal{F}_X)$, such that for all $\omega \in X$ and $E \in \mathcal{F}_X$: $[t_i(\omega)] \subseteq E$ implies $t_i(\omega)(E) = 1$, where $[t_i(\omega)] := \{\omega' \in X : t_i(\omega') = t_i(\omega)\}$.

Proposition 8 *If $\underline{\mathcal{S}}$ is an unawareness belief structure, then $F(\underline{\mathcal{S}})$ is a standard type-space. Moreover, it has the following property: For every $p > 0$, measurable set $E \in \mathcal{F}$, and $i \in I$: $\{\omega \in \Omega : t_i(\omega)(E) \geq p\} = \{\omega \in \Omega : t_i^F(\omega)(E) \geq p\}$.*

A flattened unawareness structure is just a standard type-space. To derive such a type-space, one extends a player's type mapping by assigning probability zero to measurable sets for which the player's belief was previously undefined. Of course, once an unawareness structure is flattened, there is no way to analyze reasoning about unawareness anymore since by Dekel, Lipman, and Rustichini (1998) unawareness is trivial.

Note that the converse to Proposition 8 is not true. I.e., given a standard type-space, it is not always possible to find some unawareness structure with non-trivial unawareness. For instance, let $X = \{\omega_1, \omega_2, \omega_3\}$ with $t_i(\omega_1) = t_i(\omega_2) = t_i(\omega_3) = \tau_i$ and $\tau_i(\{\omega_1\}) = \tau_i(\{\omega_2\}) = \frac{1}{2}$ and $\tau_i(\{\omega_3\}) = 0$. If $\Omega = S = X$, then by Dekel, Lipman, and Rustichini (1998) the unawareness structure has trivial unawareness only. Any non-trivial partition of X into separate spaces yields either no projections or violates properties (0) to (3). We conclude that not every standard types-space with zero probability can be used to model unawareness. We understand the contribution of our work as making restrictions required for modeling unawareness precise in unawareness belief structures.

If an unawareness belief structure has a common prior, then the associated flattened model has a common prior. To see this, note that the common prior always induces a common prior on the smallest space, which implies that there is a common prior in the flattened model. If an unawareness belief structure has a positive common prior, then it does not follow that there is a positive common prior in the flattened model. To see this consider once again Figure 3. A common prior in the associated flattened model must ascribe probability zero to all states in S'. Such common prior clearly violates the positivity assumption of Definition 9. Again, this example demonstrates a difference between unawareness belief structures and standard type-spaces.

We showed in Example 1 that our notion of "No-speculative-trade" does not imply the existence of common prior in unawareness belief structures. Does our notion of "No-speculative-trade" imply at least the existence of a common prior in the flattened model? Recall from the discussion section that our notion of "No-speculative-trade" is slightly different from Feinberg (2000) who characterizes the common prior by the absence of common certainty of speculation for *some* states. We show that a positive common prior implies the absence of common certainty of speculation for *all* states. Hence, our notion of "No-speculative-trade" implies Feinberg's notion of "No-speculative-trade". Since Feinberg showed that his notion of "No-speculative-trade" implies a common prior for standard type-spaces, the existence of a common prior for the flattened model of an unawareness belief structure follows then from his result. Again, this demonstrates a difference between unawareness belief structures and standard type-spaces.

C Proofs

C.1 Proof of Remark 1

Let $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$. We have to show that $t_i(\omega_{S'}) = t_i(\omega)$:

Because of (0) and (3), we have that $S_{t_i(\omega_{S'})} \preceq S = S_{t_i(\omega)}$. Because of (2), we have $t_i(\omega_S) = t_i(\omega)_{|S} = t_i(\omega)$, and therefore $t_i(\omega_S) \in \Delta(S)$. But $(\omega_{S'})_S = \omega_S$. Thus (3) implies that $S \preceq S_{t_i(\omega_{S'})}$. So we must have $S_{t_i(\omega_{S'})} = S$. Now, (2) implies that $t_i(\omega) = t_i(\omega_S) = t_i((\omega_{S'})_S) = t_i(\omega_{S'})_{|S} = t_i(\omega_{S'})$. $\square$

C.2 Proof of Remark 2

Define $D := \{\omega' \in S_{t_i(\omega)} : t_i(\omega') = t_i(\omega)\}$. I.e., $D = Ben_i(\omega) \cap S_{t_i(\omega)}$. We need to show that $D^{\uparrow} = Ben_i(\omega)$.

Consider first "$\subseteq$": If $\omega' \in D^{\uparrow}$ then $\omega'_{S_{t_i(\omega)}} \in Ben_i(\omega)$. This is equivalent to $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega) \in \triangle(S_{t_i(\omega)})$. By (3) we have $S_{t_i(\omega')} \succeq S_{t_i(\omega)}$. By (2), $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega')_{|S_{t_i(\omega)}}$. It follows that $t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Thus $\omega' \in Ben_i(\omega)$.

"$\supseteq$": $\omega' \in Ben_i(\omega)$ if and only if $t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Hence for $\omega' \in Ben_i(\omega)$, we have $S_{t_i(\omega')} \succeq S_{t_i(\omega)}$. By (2) $t_i(\omega'_{S_{t_i(\omega)}}) = t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega)$. Hence $\omega'_{S_{t_i(\omega)}} \in D$. Thus $\omega' \in D^{\uparrow}$. $\square$

C.3 Proof of Proposition 1

$A_i(E)$ is an $S(E)$-based event if there exists a subset $D \subseteq S(E)$ s.t. $D^{\uparrow} = A_i(E)$.

Assume that $A_i(E)$ is non-empty. Define $D := \{\omega \in S(E) : t_i(\omega) \in \Delta(S(E))\}$. By definition of the awareness operator, $D = A_i(E) \cap S(E)$. We show that $D^{\uparrow} = A_i(E)$.

Let $\omega \in D^{\uparrow}$, that is $\omega \in S'$ for some $S' \succeq S(E)$ and $\omega_{S(E)} \in D$. This is equivalent to $t_i(\omega_{S(E)}) \in \Delta(S(E))$. By (0) follows $S' \succeq S_{t_i(\omega)}$. By (3) we have $S_{t_i(\omega)} \succeq S(E)$. Thus $\omega \in A_i(E)$. (Note that $A_i(E) = \{\omega \in \Omega : S_{t_i(\omega)} \succeq S(E)\}$.)

In the reverse direction, let $\omega \in A_i(E)$, i.e., $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$. By (0), $\omega \in S'$ with $S' \succeq S$. Consider $\omega_{S(E)}$. By (2), $t_i(\omega_{S(E)}) = t_i(\omega)_{|S(E)}$. Hence $\omega_{S(E)} \in D$. Thus $\omega \in D^{\uparrow}$.

Finally, if $A_i(E)$ is empty, then by definition of the awareness operator, we have $A_i(E) = \emptyset^{S(E)}$. $\square$

C.4 Proof of Proposition 2

$B_i^p(E)$ is an $S(E)$-based event if there exists a subset $D \subseteq S(E)$ s.t. $D^{\uparrow} = B_i^p(E)$. Assume that $B_i^p(E)$ is non-empty. Define $D := \{\omega \in S(E) : t_i(\omega)(E) \geq p\}$. By definition of the p-belief operator, $D = B_i^p(E) \cap S(E)$. We show that $D^{\uparrow} = B_i^p(E)$.

Let $\omega \in D^{\uparrow}$, that is $\omega \in S'$ for some $S' \succeq S(E)$ and $\omega_{S(E)} \in D$. This is equivalent to $t_i(\omega_{S(E)})(E) \geq p$. By (0), $S_{t_i(\omega_{S(E)})} = S(E)$. By (3), we have $S_{t_i(\omega)} \succeq S(E)$. By (2), it follows that $p \leq t_i(\omega_{S(E)})(E) = t_i(\omega)_{|S(E)}(E)$. Hence $t_i(\omega)(E) \geq p$. Thus $\omega \in B_i^p(E)$.

In the reverse direction, let $\omega \in B_i^p(E)$, i.e., $t_i(\omega)(E) \geq p$. Since $t_i(\omega)(E) \geq p$ it follows that $S_{t_i(\omega)} \succeq S(E)$. Let $\omega \in S'$. By (0), $S' \succeq S_{t_i(\omega)}$. Consider $\omega_{S(E)}$. By (2), $t_i(\omega_{S(E)})(E) = t_i(\omega)(E)_{|S(E)} \geq p$. Hence $\omega_{S(E)} \in D$. Thus $\omega \in D^{\uparrow}$.

Finally, if $B_i^p(E)$ is empty, then by definition of the p-belief operator, we have $B_i^p(E) = \emptyset^{S(E)}$. $\square$

C.5 Proof of Theorem 1

Before we prove the theorem, we state the following definition and observations. Some of it will be also used for the proof of Proposition 3.

Definition 13 *We define:*

(i) A probability measure $P_i^S \in \Delta(S)$ a prior for player i on S *if for every event $E \in \Sigma$ with $S(E) \preceq S$ equation (1u) is satisfied, i.e.,*

$$P_i^S(E \cap S \cap A_i(E)) = \int_{S \cap A_i(E)} t_i(\cdot)(E) dP_i^S(\cdot). \tag{4}$$

(ii) A common prior P^S on S is a prior for player i on S, for all $i \in I$.

(iii) A positive common prior P^S on S is a common prior on S such that for all $i \in I$ and $\omega \in \Omega$: if $t_i(\omega) \in \Delta(S')$ for some $S' \preceq S$, then $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$ and $P^S\left(([t_i(\omega)] \cap S')^{\uparrow} \cap S\right) > 0$.

Note that a projective system of priors for player i on $S \in \mathcal{S}$, common priors on $S \in \mathcal{S}$, and positive common priors on $S \in \mathcal{S}$ is a prior for player i, common prior, and positive common prior, respectively.

Remark 5 *If $P = \left(P^S\right)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ is a positive (common) prior, then also $P^S \in \Delta(S)$ is positive (common) prior on S for every $S \in \mathcal{S}$.*

Remark 6 *If $\mu_i \in \Delta(S)$ is a* positive *prior for player i on S and $S' \preceq S$, then the marginal of μ_i on S', $\left(\mu_i^S\right)_{|S'}$ is a* positive *prior for player i on S'.*

Lemma 2 *Let P^S be a positive common prior on the state space S and let $i \in I$ and $\omega \in \Omega$ be such that $t_i(\omega) \in \Delta(S)$. Moreover, let E be a measurable event such that $S(E) \preceq S$. Then $[t_i(\omega)] \cap S \cap E$ and $[t_i(\omega)] \cap S$ are measurable, $P^S([t_i(\omega)] \cap S) > 0$, and we have $t_i(\omega)(S \cap E) = \frac{P^S([t_i(\omega)] \cap S \cap E)}{P^S([t_i(\omega)] \cap S)}$.*

Proof. That $[t_i(\omega)] \cap S \cap E$ and $[t_i(\omega)] \cap S$ are measurable, and $P^S([t_i(\omega)] \cap S) > 0$, follows from the definition of a positive common prior on S. Recall that $A_i(E) = S(E)^{\uparrow}$. This implies, since $S(E) \preceq S$ and $S\left(([t_i(\omega)] \cap S)^{\uparrow}\right) = S$, that $S \cap A_i\left(([t_i(\omega)] \cap S)^{\uparrow} \cap E\right) = S$. We also have $[t_i(\omega)] \cap S = ([t_i(\omega)] \cap S)^{\uparrow} \cap S$.

Since $[t_i(\omega)] \cap S \cap E$ is measurable, introspection implies that $t_i(\omega')([t_i(\omega)] \cap S \cap E) = 0$, for $\omega' \notin [t_i(\omega)] \cap S$: Recall that $Ben_i(\omega) = ([t_i(\omega)] \cap S)^{\uparrow}$ and that $[t_i(\omega)] \cap S \cap E$ is measurable and disjoint from $Ben_i(\omega')$, for $\omega' \in S$ with $\omega' \notin [t_i(\omega)] \cap S$.

Also, for $\omega' \in [t_i(\omega)] \cap S$, we have $t_i(\omega')([t_i(\omega)] \cap S \cap E) = t_i(\omega)([t_i(\omega)] \cap S \cap E)$.

By definition of a prior on S, and all the above mentioned facts, it follows that:

$$\begin{aligned}
P^S([t_i(\omega)] \cap S \cap E) &= P^S\left(([t_i(\omega)] \cap S)^{\uparrow} \cap E \cap S \cap A_i\left(([t_i(\omega)] \cap S)^{\uparrow} \cap E\right)\right) \\
&= \int_{S \cap A_i\left(([t_i(\omega)] \cap S)^{\uparrow} \cap E\right)} t_i(\omega')\left(([t_i(\omega)] \cap S)^{\uparrow} \cap E\right) dP^S(\omega') \\
&= \int_S t_i(\omega')([t_i(\omega)] \cap S \cap E)\, dP^S(\omega') \\
&= \int_{[t_i(\omega)] \cap S} t_i(\omega)([t_i(\omega)] \cap S \cap E)\, dP^S(\omega') \\
&= t_i(\omega)(S \cap E) \int_{[t_i(\omega)] \cap S} dP^S(\omega') \\
&= t_i(\omega)(S \cap E)\, P^S([t_i(\omega)] \cap S).
\end{aligned}$$

The fact that $P^S([t_i(\omega)] \cap S) > 0$, implies now the desired equation. □

The next lemma follows directly from Lemma 2 above.

Lemma 3 *Let P^S be a positive common prior on some finite state-space S and let $i \in I$ and $\omega \in \Omega$ such that $t_i(\omega) \in \triangle(S)$. Then we have for all $\omega' \in [t_i(\omega)] \cap S$ that $t_i(\omega)(\{\omega'\}) = \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)}$.*

Proof of the Theorem. The idea of the proof is follows: First, if the set of states in which there is common certainty that the first player's expectation is strictly above α and the second player's expectations is weakly below α is nonempty, there is a minimal state-space such that the common certainty event restricted to this space is nonempty. Second, this restricted common certainty event is a belief closed subset in which beliefs are stationary. Third, this set, together with the restriction of types to this set constitutes a standard state-space to which a standard no-speculative-trade argument can be applied.

Note that $E_1^{>\alpha}$ and $E_2^{\leq\alpha}$ may not be events in our unawareness belief structure. The definition of the belief operator as well as Proposition 4 and 6 can be extended to measurable subsets of Ω. The proofs are analogous and thus omitted.

Suppose that $CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right)$ is non-empty. Then fix a $\preceq$-minimal state-space S such that $W := CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S \neq \emptyset$. Such a space S exists by the finiteness of Σ.

By Remark 5, since P is a positive common prior, P^S is a positive common prior on S.

Since $W = CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S \subseteq S \cap B_i^1\left(CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right)\right)$, the minimality of S implies that for each $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$ we do have $S_{t_i(\omega)} = S$ and $t_i(\omega)(W) = 1$.

By the definition, $t_i(\omega)([t_i(\omega)] \cap S) = 1$, for each $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$. Since $t_i(\omega)(W) = 1$, we have $t_i(\omega)(([t_i(\omega)] \cap S) \setminus W) = 0$.

By Lemma 3, this implies that $P^S(\{\omega'\}) = 0$, for $\omega' \in ([t_i(\omega)] \cap S) \setminus W$ such that $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$. It follows that $P^S(([t_i(\omega)] \cap S) \setminus W) = 0$ and hence, $P^S(([t_i(\omega)] \cap S) \cap W) = P^S([t_i(\omega)] \cap S) - P^S(([t_i(\omega)] \cap S) \setminus W) = P^S([t_i(\omega)] \cap S) > 0$. So, we do have $P^S(W) > 0$.

The fact that $P^S(\{\omega'\}) = 0$, for $\omega' \in ([t_i(\omega)] \cap S) \setminus W$ such that $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S = W$ implies the following: For any random variable x, we have $\sum_{\omega' \in [t_i(\overline{\omega})] \cap S} x(\omega') P^S(\{\omega'\}) = \sum_{\omega' \in W \cap [t_i(\overline{\omega})] \cap S} x(\omega') P^S(\{\omega'\})$, if $[t_i(\overline{\omega})] \cap W \neq \emptyset$. And also $\sum_{\omega \in W} x(\omega) P^S(\{\omega\}) = \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} x(\omega) P^S(\{\omega\})$. This is so, because there is a $\omega \in [t_i(\overline{\omega})] \cap W$ and for this ω, we do have $\omega \in CB^1\left(E_1^{>\alpha} \cap E_2^{\leq\alpha}\right) \cap S$ and $[t_i(\omega)] = [t_i(\overline{\omega})]$ and this implies $P^S(([t_i(\overline{\omega})] \cap S) \setminus W) = 0$.

For $i = 1, 2$ we have

$$
\begin{aligned}
& \sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \, t_i(\omega)(\{\omega'\}) \\
= & \sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\omega)] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega \in [t_i(\overline{\omega})] \cap S} P^S(\{\omega\}) \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\omega)] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} P^S([t_i(\overline{\omega})] \cap S) \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \frac{P^S(\{\omega'\})}{P^S([t_i(\overline{\omega})] \cap S)} \\
= & \sum_{[t_i(\overline{\omega})] \cap W \neq \emptyset} \sum_{\omega' \in [t_i(\overline{\omega})] \cap S} v(\omega') \, P^S(\{\omega'\}) \\
= & \sum_{\omega' \in W} v(\omega') \, P^S(\{\omega'\}).
\end{aligned}
$$

But by the assumptions, we have $\sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_1(\omega)] \cap S} v(\omega') \, t_1(\omega)(\{\omega'\}) > \alpha P^S(W)$ and $\sum_{\omega \in W} P^S(\{\omega\}) \sum_{\omega' \in [t_2(\omega)] \cap S} v(\omega') \, t_2(\omega)(\{\omega'\}) \leq \alpha P^S(W)$, a contradiction, since $P^S(W) > 0$. □

C.6 Proof of Proposition 3

Proposition 9 *Let $\underline{S}$ be an unawareness belief structure, G be an event, and $p_i \in [0,1]$, for $i \in I$. Suppose that there exists a common prior P^S on a space $S \succeq S(G)$ such that $P^S(CB^1(\bigcap_{i\in I}[t_i(G) = p_i])) > 0$. Then $p_i = p_j$, for all $i, j \in I$.*

Before we prove the result, we show:

Remark 7 *The conditions of Proposition 3 imply the conditions of Proposition 9. Hence, Proposition 9 implies Proposition 3 (since they have the same conclusions).*

Proof of the Remark. Let $\left(P^{S'}\right)_{S'\in\mathcal{S}}$ be a positive common prior. Let $E := CB^1\left(\bigcap_{i\in I}[t_i(G) = p_i]\right)$ be nonempty. Choose a state $\omega \in E$ and a player $i \in I$. By Proposition 6 (i), $\omega \in B_i^1(E)$, that is, $t_i(\omega)(E) = 1$. In particular, $t_i(\omega) \in \Delta(S)$ for some $S \succeq S(E) = S(G)$. (That $S(E) = S(G)$ follows from Lemma 1, the definition of intersection of events, and what was remarked after the definition of common belief in Section 2.11.) Since P^S is a positive common prior on S, we have by Lemma 2 that $P^S([t_i(\omega)] \cap S) > 0$ and that $1 = t_i(\omega)(S \cap E) = \frac{P^S([t_i(\omega)]\cap S\cap E)}{P^S([t_i(\omega)]\cap S)}$. Hence $P^S(E) > 0$. □

Remark 8 *For any $\omega \in \Omega$, $t_i(\omega)(E \cap A_i(E)) = t_i(\omega)(E)$ for any event E s.t. $S(E) \preceq S_{t_i(\omega)}$.*

Proof of the Remark: Let E be an event and $t_i(\omega)$ be such that $S(E) \preceq S_{t_i(\omega)}$. Since $E = (E \cap A_i(E)) \cup (E \cap U_i(E))$ and $A_i(E) \cap U_i(E) = \emptyset^{S(E)}$, we have $(E \cap A_i(E)) \cap (E \cap U_i(E)) = \emptyset^{S(E)}$. Since $t_i(\omega)$ is an additive probability measure, $t_i(\omega)(E) = t_i(\omega)(E \cap A_i(E)) + t_i(\omega)(E \cap U_i(E))$. Since $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0,1]$ (B^pU-Introspection in Proposition 5), we must have $t_i(\omega)(E \cap U_i(E)) = 0$. □

The following lemma says that if there is a prior on a state-space then the marginal on a lower space is a prior as well.

Lemma 4 *If $\mu \in \Delta(S')$ is a prior for player i on S' and $S \preceq S'$, then $(\mu)_{|S}$ (the marginal of μ on S) is a prior for player i on S.*

Proof of the Lemma. Let E be an event with $S(E) \preceq S$ and let μ be individual i's prior probability measure on S' with $S' \succeq S$. We have to show that $\mu\left((r^{S'}_{S})^{-1}(E \cap S \cap A_i(E))\right) = \int_{S \cap A_i(E)} t_i(\cdot)(E) d\mu(\cdot)$. Since $S(E) \preceq S$, and by Proposition 1, $S(A_i(E)) = S(E)$, it follows that $(r^{S'}_{S})^{-1}(E \cap S \cap A_i(E)) = E \cap S' \cap A_i(E)$, and therefore $\mu_{|S}(E \cap S \cap A_i(E)) = \mu(E \cap S' \cap A_i(E))$. So it remains to show that $\int_{S \cap A_i(E)} t_i(\cdot)(E \cap A_i(E)) d(\mu_{|S})(\cdot) = \int_{S' \cap A_i(E)} t_i(\cdot)\left(E \cap A_i(E)\right) d\mu(\cdot)$.

We first show the following **Claim:** *Let $\omega \in S(E) \preceq S \preceq S'$ such that $\omega \in A_i(E)$. Then $t_i(\omega)(E \cap A_i(E)) = t_i(\omega_S)(E \cap A_i(E))$.*

Proposition 1, $\omega \in A_i(E)$ and $S(E) \preceq S$ imply that $\omega_S \in A_i(E)$. We have that $\omega \in A_i(E)$ implies $t_i(\omega)(E \cap A_i(E)) = t_i(\omega)(E \cap A_i(E) \cap S_{t_i(\omega)})$. By (3) of Definition 1, we have $S_{t_i(\omega_S)} \preceq S_{t_i(\omega)}$. And by (1) of Definition 1 $t_i(\omega_S)(E \cap A_i(E)) = t_i(\omega_S)(E \cap A_i(E) \cap S_{t_i(\omega_S)}) = t_i(\omega_{S_{t_i(\omega_S)}})(E \cap A_i(E) \cap S_{t_i(\omega_S)})$. By (2) of Definition 1, we have $t_i(\omega_{S_{t_i(\omega_S)}})(E \cap A_i(E) \cap S_{t_i(\omega_S)}) = t_i(\omega)((r^{S_{t_i(\omega)}}_{S_{t_i(\omega_S)}})^{-1}(E \cap A_i(E) \cap S_{t_i(\omega_S)})) = t_i(\omega)(E \cap A_i(E) \cap S_{t_i(\omega)}) = t_i(\omega)(E \cap A_i(E))$. Hence the claim is proved.

We have

$$\begin{aligned} \int_{A_i(E) \cap S} t_i(\cdot)(A_i(E) \cap E) d(\mu_{|S})(\cdot) &= \int_{A_i(E) \cap S'} t_i(r^{S'}_{S}(\cdot))(A_i(E) \cap E) d\mu(\cdot) \\ &= \int_{A_i(E) \cap S'} t_i(\cdot)(A_i(E) \cap E) d\mu(\cdot), \end{aligned}$$

where the first equation follows from the definition of marginal and the second from the above claim. □

Remark 9 *Let $\hat{S}$ be the upmost state-space in the lattice $\mathcal{S}$, and let $(P^S_i)_{S \in \mathcal{S}} \in \prod_{S \in \mathcal{S}} \Delta(S)$ be a tuple of probability measures. Then $(P^S_i)_{S \in \mathcal{S}}$ is a prior for player i if and only if $P^{\hat{S}}_i$ is a prior for player i on $\hat{S}$ and P^S_i is the marginal of $P^{\hat{S}}_i$ for every $S \in \mathcal{S}$.*

This remark together with Lemma 4 implies the following:

Remark 10 *A common prior (Definition 8) induces a common prior on S, for any $S \in \mathcal{S}$. The converse is not necessarily true unless S is the upmost state-space of the lattice. Note that it is possible that players have different priors, but at some space S (below the upmost space) the priors on S coincide. Hence, in such a case they have different priors, but a common prior on S (and by Lemma 4 also a common prior on spaces less expressive than S).*

Proof of Proposition 9. By Proposition 6, $\omega \in CB^1(F)$ if and only if there exists an event E that is evident such that $\omega \in E \subseteq B^1(F)$.

Since for an evident E we have $E \subseteq B_i^1(E) \subseteq A_i(E)$ for all $i \in I$. It follows that $P^S(E \cap A_i(E)) = P^S(E)$ for $S \succeq S(E)$. Set $F = \bigcap_{i \in I}[t_i(G) = p_i]$ and let $E = CB^1(F)$. By Proposition 1, $S(E) = S(G)$. By Lemma 4 and the properties imposed on t_i, we consider w.l.o.g. a common prior $P^{S(G)}$ on $S(G)$.

$$\begin{aligned} P^{S(G)}(E) &= \int_{S(G)\cap A_i(E)} t_i(\cdot)(E)dP^{S(G)}(\cdot) \\ &= \int_{E\cap S(G)\cap A_i(E)} t_i(\cdot)(E)dP^{S(G)}(\cdot) + \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E)dP^{S(G)}(\cdot). \end{aligned}$$

We have

$$\int_{E\cap S(G)\cap A_i(E)} t_i(\cdot)(E)dP^{S(G)}(\cdot) = \int_{E\cap S(G)\cap A_i(E)} 1dP^{S(G)}(\cdot) = P^{S(G)}(E).$$

The second last equation above follows from the fact that E is evident. So, we have $E \subseteq B_i^1(E)$, that is $t_i(\cdot)(E) = 1$, for $\omega \in E$. It follows that

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E)dP^{S(G)}(\cdot) = 0. \tag{5}$$

$$\int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G)dP^{S(G)}(\cdot) = \int_{E\cap A_i(E)\cap S(G)} p_i dP^{S(G)}(\cdot) = p_i P^{S(G)}(E)$$

If $\omega \in E = CB^1(F)$, then $\omega \in E \subseteq B_i^1(F) \subseteq B_i^1([t_i(G) = p_i])$. Note that $[t_i(G) = p_i] = B_i^{p_i}(G) \cap B_i^{1-p_i}(\neg G)$. Therefore, by monotonicity $B_i^1([t_i(G) = p_i]) \subseteq B_i^1(B_i^{p_i}(G)) \cap B_i^1(B_i^{1-p_i}(\neg G))$. Introspection II implies now that $\omega \in B_i^{p_i}(G) \cap B_i^{1-p_i}(\neg G) = [t_i(G) = p_i]$. So we have $t_i(\omega)(G) = p_i$, for $\omega \in E$.

$$\begin{aligned} \int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G)dP^{S(G)}(\cdot) &= \int_{E\cap A_i(E)\cap S(G)} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) \\ &= \int_{S(G)\cap A_i(E)} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) \\ &\quad - \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot). \end{aligned}$$

Since by the monotonicity of probability measures

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) \le \int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(E)dP^{S(G)}(\cdot),$$

we must have by equation (5) and non-negativity of probability measures

$$\int_{(S(G)\cap A_i(E))\setminus E} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot) = 0.$$

Note that $P^{S(G)}(G\cap E) = \int_{S(G)\cap A_i(E)} t_i(\cdot)(G\cap E)dP^{S(G)}(\cdot)$.

Note further that $P^{S(G)}(E) = P^{S(G)}(E\cap A_i(E))$ for all $i\in N$ since $E = CB^1(F)\subseteq A_i(E)$ for all $i\in N$. Similarly, $P^{S(G)}(G\cap E) = P^{S(G)}(G\cap E\cap A_i(E))$ for all $i\in N$.

Thus

$$p_i P^{S(G)}(E) = P^{S(G)}(G\cap E). \tag{6}$$

Note that by assumption $P^{S(G)}(E) > 0$.

Since equation (6) holds for all $i\in I$, we must have $p_i = p_j$, for all $i,j\in I$. $\square$

C.7 Proof of Proposition 4

(0) $B_i^p(E)\subseteq B_i^q(E)$ for $p,q\in[0,1]$ with $q\le p$ is trivial.

(i) $B_i^1(\Omega)\subseteq\Omega$ holds trivially. In the reverse direction, note that $t_i(\omega)(\Omega) = t_i(\omega)(\Omega\cap S_{t_i(\omega)}) = t_i(\omega)(S_{t_i(\omega)}) = 1$ for all $\omega\in\Omega$. Thus $\Omega\subseteq B_i^1(\Omega)$.

(ii) $\omega\in B_i^p(E)$ if and only if $t_i(\omega)(E)\ge p$. Since $t_i(\omega)$ is an additive probability measure, $t_i(\omega)(\neg E)\le 1-p$. Hence $\omega\in\neg B_i^q(\neg E)$ for $q>1-p$.

(iiia) $\omega\in B_i^p\left(\bigcap_{l=1}^\infty E_l\right)$ if and only if $t_i(\omega)\left(\bigcap_{l=1}^\infty E_l\right)\ge p$. Monotonicity of the probability measure $t_i(\omega)$ implies $t_i(\omega)(E_l)\ge p$ for all $l=1,2,...$, which is equivalent to $\omega\in\bigcap_{l=1}^\infty B_i^p(E_l)$.

(iiib) It is enough to show that any sequence of events $\{E_l\}_{l=1}^\infty$ with $E_l\supseteq E_{l+1}$ for $l=1,2,...$ we have $B_i^p\left(\bigcap_{l=1}^\infty E_l\right)\supseteq\bigcap_{l=1}^\infty B_i^p(E_l)$. $\omega\in\bigcap_{l=1}^\infty B_i^p(E_l)$ if and only if $t_i(\omega)(E_l)\ge p$ for $l=1,2,....$ Since $t_i(\omega)$ is a countable additive probability measure, it is continuous from above. That is, if $E_l\supseteq E_{l+1}$ for $l=1,2,...$, we have $\lim_{l\to\infty} t_i(\omega)(E_l) = t_i(\omega)\left(\bigcap_{l=1}^\infty E_l\right)$. Since for every $l=1,2,...$, $t_i(\omega)(E_l)\ge p$, we have $p\le\lim_{l\to\infty} t_i(\omega)(E_l) = t_i(\omega)\left(\bigcap_{l=1}^\infty E_l\right)$. Hence $\omega\in B_i^p\left(\bigcap_{l=1}^\infty E_l\right)$.

(iiic) It is enough to show that $B_i^1\left(\bigcap_{l=1}^\infty E_l\right)\supseteq\bigcap_{l=1}^\infty B_i^1(E_l)$. $\omega\in\bigcap_{l=1}^\infty B_i^1(E_l)$ if and only if $t_i(\omega)(E_l)=1$ for $l=1,2,....$ Since $t_i(\omega)$ is a countable additive probability measure, it satisfies Bonferroni's Inequality. I.e., $t_i(\omega)\left(\bigcap_{l=1}^\infty E_l\right)\ge 1-\sum_{l=1}^\infty 1-t_i(\omega)(E_l)$. Since $t_i(\omega)(E_l)=1$ for all $l=1,2,...$, we have $1-t_i(\omega)(E_l)=0$ for all $l=1,2,...$, and hence $\sum_{l=1}^\infty 1-t_i(\omega)(E_l)=0$. It follows that $t_i(\omega)\left(\bigcap_{l=1}^\infty E_l\right)=1$. We conclude that $\omega\in B_i^1\left(\bigcap_{l=1}^\infty E_l\right)$.

(iv) Since $t_i(\omega)$ is a probability measure (satisfying monotonicity) for any $\omega \in \Omega$, $E \subseteq F$ implies that if $t_i(\omega)(E) \geq p$ then $t_i(\omega)(F) \geq p$.

(va) Let $\omega \in B_i^p(E)$. Then $t_i(\omega)(E) \geq p$. It follows that for all $\omega' \in Ben_i(\omega)$ we have $t_i(\omega')(E) \geq p$. Hence $Ben_i(\omega) \subseteq B_i^p(E)$. Thus $t_i(\omega)(B_i^p(E)) = 1$, which implies that $\omega \in B_i^1 B_i^p(E)$.

(vb) Let $\omega \in B_i^p(B_i^q(E))$, for some $p \in (0,1]$ and assume by contradiction that $\omega \notin B_i^q(E)$. Then, since by Propositions 1 and 2 $\omega \in A_i(E)$, we must have $q > 0$ and $\omega \in B_i^{1-r}(\neg E)$ for some r with $q > r \geq 0$. By (va), we have $\omega \in B_i^1\left(B_i^{1-r}(\neg E)\right)$. Note that $B_i^{1-r}(\neg E)$ and $B_i^q(E)$ are disjoint because of (ii), and hence $B_i^{1-r}(\neg E) \subseteq \neg B_i^q(E)$. Monotonicity implies now that $\omega \in B_i^1\left(\neg B_i^q(E)\right)$, hence, by (ii) $\omega \in \neg B_i^p(B_i^q(E))$ a contradiction to $\omega \in B_i^p(B_i^q(E))$. □

C.8 Proof of Proposition 5

1. This property is equivalent to $B_i^p(E) \cup B_i^p \neg B_i^p(E) \subseteq A_i(E)$. By Property 5. we have $B_i^p(E) \subseteq A_i(E)$. To see that $B_i^p \neg B_i^p(E) \subseteq A_i(E)$, note that $\omega \in B_i^p \neg B_i^p(E)$ if and only if $t_i(\omega)(\neg B_i^p(E)) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(\neg B_i^p(E)) = S(E)$. The last equality follows by Property 8 and Proposition 2. Hence $\omega \in A_i(E)$.

2. The proof is analogous to 1. The is property is equivalent to $\bigcap_{n=1}^{\infty} B_i^p\left(\neg B_i^p\right)^{n-1}(E) \subseteq A_i(E)$. $\omega \in B_i^p\left(\neg B_i^p\right)^{n-1}(E)$ for any $n = 1, 2, ...$ if and only it $t_i(\omega)\left(\left(\neg B_i^p\right)^{n-1}(E)\right) \geq p$ for any $n = 1, 2, ...$ It follows that $S_{t_i(\omega)} \succeq S\left(\left(\neg B_i^p\right)^{n-1}(E)\right)$ for any $n = 1, 2, ...$ By Proposition 2, $S\left(\left(\neg B_i^p\right)^{n-1}(E)\right) = S(E)$ for any $n = 1, 2, ...$ Hence $\omega \in A_i(E)$.

3. First, we show $B_i^p U_i(E) \subseteq A_i(E)$. $\omega \in B_i^p U_i(E)$ if and only if $t_i(\omega)(U_i(E)) \geq p$. It implies $S_{t_i(\omega)} \succeq S(U_i(E))$. By Proposition 1, $S(U_i(E)) = S(E)$. Hence $S_{t_i(\omega)} \succeq S(E)$ which is equivalent to $\omega \in A_i(E)$.

Second, we show that $B_i^p U_i(E) = \emptyset^{S(E)}$ for $p \in (0,1]$. Since $B_i^p U_i(E) \subseteq A_i(E)$ we have by monotonicity $B_i^1 B_i^p U_i(E) \subseteq B_i^1 A_i(E)$. By introspection, $B_i^p U_i(E) \subseteq B_i^1 B_i^p U_i(E) \subseteq B_i^1 A_i(E)$. By additivity, we have $B_i^p U_i(E) \subseteq \neg B_i^1 A_i(E)$. Hence $B_i^p U_i(E) = \emptyset^{S(E)} = \neg B_i^1 A_i(E) \cap B_i^1 A_i(E)$.

Third, we show that $B_i^0 U_i(E) = A_i(E)$. $\omega \in A_i(E)$ if and only if $\omega \in A_i U_i(E)$ since by AA-self-reflection $A_i(E) = A_i A_i(E)$ and by symmetry $A_i A_i(E) = A_i U_i(E)$. Hence, if $\omega \in A_i(E)$ then $t_i(\omega)(U_i(E))$ is defined. Therefore $\omega \in B_i^0 U_i(E)$, and hence $A_i(E) \subseteq B_i^0 U_i(E)$. Together with the first part of the proof, we conclude $B_i^0 U_i(E) = A_i(E)$.

4. This property is equivalent to $A_i U_i(E) = A_i(E)$. $\omega \in A_i U_i(E)$ if and only if

$S_{t_i(\omega)} \succeq S(U_i(E)) = S(A_i(E)) = S(E)$ by Proposition 1. Hence $\omega \in A_iU_i(E)$ if and only if $\omega \in A_i(E)$.

5. $\omega \in A_i(E)$ if and only if $S_{t_i(\omega)} \succeq S(E)$. For any $t_i(\omega)$, we have $S_{t_i(\omega)} \succeq S(E)$ if and only if $1 = t_i(\omega)(S(E)^{\uparrow})$. This is equivalent to $\omega \in B_i^1(S(E)^{\uparrow})$.

6. First, we show $B_i^p(E) \subseteq A_i(E)$. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$, which is equivalent to $\omega \in A_i(E)$.

Second, we show for $p = 0$, $A_i(E) \subseteq B_i^0(E)$. $\omega \in A_i(E)$ if and only if $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$. Hence $t_i(\omega)(E) \geq 0$, which implies that $\omega \in B_i^0(E)$.

7. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$. By Proposition 2, it is equivalent to $S_{t_i(\omega)} \succeq S(B_i^q(E))$, which is equivalent to $\omega \in A_iB_i^q(E)$.

8. By the definition of negation, $S(E) = S(\neg E)$. Hence for $t_i(\omega) \in \triangle(S)$, $S \succeq S(E)$ if and only if $S \succeq S(\neg E)$.

9. $\omega \in \bigcap_{\lambda \in L} A_i(E_\lambda)$ if and only if $S_{t_i(\omega)} \succeq S(E_\lambda)$ for all $\lambda \in L$. This is equivalent to $S_{t_i(\omega)} \succeq \sup_{\lambda \in L} S(E_\lambda) = S\left(\bigcap_{\lambda \in L} E_\lambda\right)$, which is equivalent to $\omega \in A_i\left(\bigcap_{\lambda \in L} E_\lambda\right)$.

10. By Proposition 2, $S(E) = S(B_i^p(E))$. Hence, $\omega \in A_i(E)$ if and only if $\omega \in A_iB_i^p(E)$.

11. By Proposition 1, $S(E) = S(A_i(E))$. Hence $\omega \in A_i(E)$ if and only if $\omega \in A_iA_i(E)$.

12. $\omega \in B_i^pA_i(E)$ if and only if $t_i(\omega)(A_i(E)) \geq p$. This implies $S_{t_i(\omega)} \succeq S(A_i(E))$. By Proposition 1, $S(A_i(E)) = S(E)$. Thus $\omega \in A_i(E)$. To see the converse, by weak necessitation and introspection, $A_i(E) = B_i^1(S(E)^{\uparrow}) \subseteq B_i^1B_i^1(S(E)^{\uparrow}) = B_i^1A_i(E)$. By Proposition 4 (o), $B_i^1A_i(E) \subseteq B_i^pA_i(E)$. □

C.9 Proof of Proposition 7

1. By Proposition 1, $S(E) = S(A_j(E))$. Hence $\omega \in A_i(E)$ if and only if $\omega \in A_iA_j(E)$.

2. By Proposition 2, $S(E) = S(B_j^p(E))$. Hence, $\omega \in A_i(E)$ if and only if $\omega \in A_iB_j^p(E)$.

3. $\omega \in B_i^p(E)$ if and only if $t_i(\omega)(E) \geq p$. This implies that $S_{t_i(\omega)} \succeq S(E)$. By Proposition 2, this is equivalent to $S_{t_i(\omega)} \succeq S(B_j^q(E))$, which is equivalent to $\omega \in A_iB_j^q(E)$.

4. The proof is analogous to 3.

5. We show by induction that $A^n(E) = A(E)$, for all $n \geq 1$. We have $\omega \in A(A^n(E))$ if and only if $S_{t_i(\omega)} \succeq S(A^n(E))$, for all $i \in I$, which, by the induction hypothesis, is the case if and only if $S_{t_i(\omega)} \succeq S(A(E))$, for all $i \in I$. By the definition of "$\cap$", it is the case that $S(A(E)) = \sup_{i \in I} S(A_i(E))$. By Proposition 1, we have $S(A_i(E)) = S(E)$ and

hence $S(A(E)) = S(E)$. It follows that $S_{t_i(\omega)} \succeq S(A(E))$ if and only if $S_{t_i(\omega)} \succeq S(E)$. But $S_{t_i(\omega)} \succeq S(E)$ if and only if $\omega \in A_i(E)$. Hence we have $A^n(E) = A(E)$, for all $n \geq 1$, and therefore $CA(E) = A(E)$.

6. $\omega \in CB^1(E)$ implies $\omega \in B_i^1(E)$ for all $i \in I$. This is equivalent to $t_i(\omega)(E) = 1$ for all $i \in I$, which implies $S_{t_i(\omega)} \succeq S(E)$ for all $i \in I$. Hence, by 5. we have $\omega \in A(E) = CA(E)$.

7. First, we show that $B^p(E) \subseteq A(E)$. $\omega \in B^p(E)$ if and only if $t_i(\omega)(E) \geq p$ for all $i \in I$. Hence $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, for all $i \in I$. This implies that $\omega \in A_i(E)$, for all $i \in I$. It follows that $\omega \in A(E)$.

Second, we show that $A(E) = B^0(E)$. $\omega \in A(E)$ if and only if $\omega \in A_i(E)$ for all $i \in I$ if and only if (by 6. of Proposition 5) $\omega \in B_i^0(E)$ for all $i \in I$ if and only if $\omega \in B^0(E)$.

8. The proof follows from 7. and 5.

9. By weak necessitation, $A(E) := \bigcap_{i \in I} A_i(E) = \bigcap_{i \in I} B_i^1(S(E)^{\uparrow}) := B^1(S(E)^{\uparrow})$.

10. The proof follows from 9. and 5.

11. By definition of common certainty, $CB^1(S(E)^{\uparrow}) \subseteq B^1(S(E)^{\uparrow})$. By 9., $B^1(S(E)^{\uparrow}) = A(E)$.

12. The proof follows from 11. and 5. □

C.10 Proof of Proposition 8

We only have to show:

1. $t_i^F : \Omega \longrightarrow \Delta(\Omega, \mathcal{F})$ is measurable, where $\Delta(\Omega, \mathcal{F})$ is endowed with the sigma-algebra generated by sets $\{\mu \in \Delta(\Omega, \mathcal{F}) : \mu(E) \geq p\}$ for $p \in [0, 1]$ and $E \in \mathcal{F}$.
2. For all $\omega \in \Omega$, $i \in I$, and $E \in \mathcal{F}$: If $[t_i^F(\omega)] = \{\omega' \in \Omega : t_i^F(\omega') = t_i^F(\omega)\} \subseteq E$, then $t_i^F(\omega)(E) = 1$.

But both properties follow directly from the respective properties in the unawareness belief structure $\underline{\mathcal{S}}$. □

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