Speculative Trade under Unawareness: The Infinite Case*

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Abstract

We prove a "no-speculative-trade" theorem under unawareness for the infinite case. This generalizes the result for the finite case by Heifetz, Meier, and Schipper (2013).

Keywords: Awareness, unawareness, speculation, trade, agreement, common prior, common certainty.

JEL-Classifications: C70, C72, D53, D80, D82, D84.

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1 Introduction

Unawareness refers to the lack of conception rather than to the lack of information. It is natural to presume that asymmetric unawareness may lead to speculative trade. Indeed, Heifetz, Meier, and Schipper (2013) present a simple example of speculation under unawareness in which there is common certainty of willingness to trade but agents have a strict preference to trade despite the existence of a common prior.[1] This is impossible in standard state-space structures with a common prior. In standard "no-speculative-trade" theorems, if there is common certainty of willingness to trade, then agents are necessarily indifferent to trade (Milgrom and Stokey, 1982). Somewhat surprisingly, Heifetz, Meier, and Schipper (2013) also prove a "no-speculative-trade" result according to which under a common prior there cannot be common certainty of strict preference to trade. This means that arbitrarily small transaction costs rule out speculation under asymmetric unawareness. The "no-speculative-trade" result in Heifetz, Meier, and Schipper (2013) was stated for finite unawareness belief structures. In this note we generalize the result to infinite unawareness belief structures. Such a generalization is relevant since the space of underlying uncertainties may be large. Especially if it is large, agents may be unaware of some of them. Moreover, the generalization serves as a robustness check for our "no-speculative-trade" result for finite unawareness belief structures. It shows that the result in Heifetz, Meier, and Schipper (2013) is not an artefact of the finiteness assumption but holds more generally. The topological unawareness belief structure introduced in this paper to prove our result may be of independent interest for other applications.

Board and Chung (2011) present a different model of unawareness, in which unawareness is about "objects" rather than events. Intuitively, we model an agent's unawareness of events like "penicillium rubens has antibiotic properties" while in their model the agent's corresponding unawareness would be about "penicillium rubens". Board and Chung (2011) also prove a "no-speculative-trade" result for finite spaces. This suggests that the "no-speculative-trade" result in Heifetz, Meier, and Schipper (2013) remains robust if unawareness of events is replaced by unawareness of objects. Further, Board and Chung (2011) study awareness of unawareness of objects. This allows them to show that the "no-speculative-trade" result obtains both under "living in denial" and "living under paranoia", where former refers to the situation in which every agent is certain that there is nothing they are unaware of while latter refer to the situation in which every agent is certain that there is something they are unaware of. Grant and Quiggin

[1] This example is a probabilistic version of the speculation example in Heifetz, Meier, and Schipper (2006).

(2013) discuss the speculative trade example of Heifetz, Meier, and Schipper (2006) in the context of awareness of unawareness. The authors argue that agents should induce from previous experiences of becoming aware and from differences in awareness across agents that they themselves could be unaware of something. This awareness of unawareness may be coupled with a version of a precautionary principle which may make them reluctant to engage in speculative trade.

The paper is organized as follows. The next section introduces topological unawareness belief structures. The general "no-speculative-trade" theorem is stated in Section 3. Finally, Section 4 contains the proof of the theorem.

2 Topological Unawareness Belief Structures

We consider an unawareness belief structure as defined in Heifetz, Meier, and Schipper (2013) but with additional topological properties.

2.1 Compact Hausdorff State-Spaces

Let $\mathcal{S} = \{S\}$ be a complete lattice of disjoint *state-spaces*, with respect to the partial order $\preceq$ on $\mathcal{S}$. If S' and S'' are such that $S'' \succeq S'$ we say that S'' is more expressive than S'. $(\mathcal{S}, \preceq)$ is well-founded, that is, every non-empty subset $\mathcal{X} \subseteq \mathcal{S}$ contains a $\preceq$-minimal element. That is, there is a $S' \in \mathcal{X}$ such that for all $S \in \mathcal{X}$: if $S \preceq S'$, then $S = S'$. Each state-space $S \in \mathcal{S}$ is a non-empty compact Hausdorff space with a Borel σ-field $\mathcal{F}_S$. Denote by $\Omega = \bigcup_{S' \in \mathcal{S}} S'$ the union of these spaces. Ω is endowed with the disjoint-union topology: $O \subseteq \Omega$ is open if and only if $O \cap S$ is open in S for all $S \in \mathcal{S}$.

As we will see later, the lattice of spaces is useful to model unawareness with multiple agents. If an agent forms beliefs about events of a space with low expressive power and not about some events of a space with higher expressive power, then she is unaware of latter events. In a multi-agent context, agents should also form beliefs about other agent's awareness. In particular, at some events that agent i forms beliefs over, agent j may form beliefs about events in a space with lower expressive power. It means that i thinks that j is unaware of some aspects of the reality. Interactive unawareness is captured by forming beliefs about beliefs etc. down the lattice. We will make this precise once we introduce agents' type mappings in Subsection 2.8.

2.2 Continuous Projections

For every S and S' such that $S' \succeq S$, there is a continuous surjective projection $r_S^{S'}: S' \to S$, where r_S^S is the identity. Note that the cardinality of S is smaller than or equal to the cardinality of S'. We require the projections to commute: if $S'' \succeq S' \succeq S$ then $r_S^{S''} = r_S^{S'} \circ r_{S'}^{S''}$. If $\omega \in S'$, denote $\omega_S = r_S^{S'}(\omega)$. If $D \subseteq S'$, denote $D_S = \{\omega_S : \omega \in D\}$.

2.3 Events

For $D \subseteq S$, denote $D^{\uparrow} = \bigcup_{S' \in \mathcal{S}: S' \succeq S} \left(r_S^{S'}\right)^{-1}(D)$. An *event* is a pair (E, S), where $E = D^{\uparrow}$ with $D \subseteq S$, where $S \in \mathcal{S}$. D is called the *base* and S the *base-space* of (E, S), denoted by $S(E)$. If $E \neq \emptyset$, then S is uniquely determined by E and, abusing notation, we write E for (E, S). Otherwise, we write $\emptyset^S$ for $(\emptyset, S)$. Note that not every subset of Ω is an event.

Let Σ be the set of *measurable events* of Ω, i.e., $D^{\uparrow}$ such that $D \in \mathcal{F}_S$, for some state-space $S \in \mathcal{S}$. Note that unless $\mathcal{S}$ is a singleton, Σ is not an algebra because it contains distinct $\emptyset^S$ for all $S \in \mathcal{S}$.

2.4 Negation

If $(D^{\uparrow}, S)$ is an event where $D \subseteq S$, the negation $\neg(D^{\uparrow}, S)$ of $(D^{\uparrow}, S)$ is defined by $\neg(D^{\uparrow}, S) := ((S \setminus D)^{\uparrow}, S)$. Note, that by this definition, the negation of a (measurable) event is a (measurable) event. Abusing notation, we write $\neg D^{\uparrow} := (S \setminus D)^{\uparrow}$. Note that by our notational convention, we have $\neg S^{\uparrow} = \emptyset^S$ and $\neg \emptyset^S = S^{\uparrow}$, for each space $S \in \mathcal{S}$. $\neg D^{\uparrow}$ is typically a proper subset of the complement $\Omega \setminus D^{\uparrow}$. That is, $(S \setminus D)^{\uparrow} \subsetneqq \Omega \setminus D^{\uparrow}$.

Intuitively, there may be states in which the description of an event $D^{\uparrow}$ is both expressible and valid – these are the states in $D^{\uparrow}$; there may be states in which its description is expressible but invalid – these are the states in $\neg D^{\uparrow}$; and there may be states in which neither its description nor its negation are expressible. The definition of negation is crucial for modeling unawareness in our structures. Intuitively, if an agent considers the negation of an event E, then she does not necessarily considers everything but E. Rather, she considers everything but E *given her awareness level* as defined by the space on which her beliefs are defined on.

2.5 Conjunction and Disjunction

If $\left\{\left(D_\lambda^\uparrow, S_\lambda\right)\right\}_{\lambda\in L}$ is a finite or countable collection of events (with $D_\lambda \subseteq S_\lambda$, for $\lambda \in L$), their conjunction $\bigwedge_{\lambda\in L}\left(D_\lambda^\uparrow, S_\lambda\right)$ is defined by $\bigwedge_{\lambda\in L}\left(D_\lambda^\uparrow, S_\lambda\right) := \left(\left(\bigcap_{\lambda\in L} D_\lambda^\uparrow\right), \sup_{\lambda\in L} S_\lambda\right)$. Note, that since $\mathcal{S}$ is a *complete* lattice, $\sup_{\lambda\in L} S_\lambda$ exists. If $S = \sup_{\lambda\in L} S_\lambda$, then we have $\left(\bigcap_{\lambda\in L} D_\lambda^\uparrow\right) = \left(\bigcap_{\lambda\in L}\left(\left(r_{S_\lambda}^S\right)^{-1}(D_\lambda)\right)\right)^\uparrow$. Again, abusing notation, we write $\bigwedge_{\lambda\in L} D_\lambda^\uparrow := \bigcap_{\lambda\in L} D_\lambda^\uparrow$ (we will therefore use the conjunction symbol $\wedge$ and the intersection symbol $\cap$ interchangeably).

We define the relation $\subseteq$ between events (E,S) and (F,S'), by $(E,S)\subseteq(F,S')$ if and only if $E\subseteq F$ as sets *and* $S'\preceq S$. If $E\neq\emptyset$, we have that $(E,S)\subseteq(F,S')$ if and only if $E\subseteq F$ as sets. Note however that for $E=\emptyset^S$ we have $(E,S)\subseteq(F,S')$ if and only if $S'\preceq S$. Hence we can write $E\subseteq F$ instead of $(E,S)\subseteq(F,S')$ as long as we keep in mind that in the case of $E=\emptyset^S$ we have $\emptyset^S\subseteq F$ if and only if $S\succeq S(F)$. It follows from these definitions that for events E and F, $E\subseteq F$ is equivalent to $\neg F\subseteq\neg E$ only when E and F have the same base, i.e., $S(E)=S(F)$.

The disjunction of $\left\{D_\lambda^\uparrow\right\}_{\lambda\in L}$ is defined by the de Morgan law $\bigvee_{\lambda\in L} D_\lambda^\uparrow := \neg\left(\bigwedge_{\lambda\in L}\neg\left(D_\lambda^\uparrow\right)\right)$. Typically $\bigvee_{\lambda\in L} D_\lambda^\uparrow \subsetneqq \bigcup_{\lambda\in L} D_\lambda^\uparrow$, and if all D_λ are nonempty we have that $\bigvee_{\lambda\in L} D_\lambda^\uparrow = \bigcup_{\lambda\in L} D_\lambda^\uparrow$ holds if and only if all the $D_\lambda^\uparrow$ have the same base-space. Note, that by these definitions, the conjunction and disjunction of (at most countably many measurable) events is a (measurable) event.

Apart from the topological conditions and the well-foundedness assumption, the event-structure outlined so far is analogous to Heifetz, Meier, and Schipper (2006, 2008, 2013).

2.6 Regular Borel Probability Measures

Here and in what follows, the term 'events' always means measurable events in Σ unless otherwise stated.

For each $S\in\mathcal{S}$, $\Delta(S)$ is the set of regular Borel probability measures on $(S,\mathcal{F}_S)$. We consider this set itself as a measurable space which is endowed with the topology of weak convergence.[2]

[2]The topology of weak convergence is generated by the sub-basis of sets $\{\mu\in\Delta(S):\mu(O)>r\}$ where $O\subseteq S$ is open and $r\in\mathbb{R}$ (see e.g. Billingsley (1968), appendix III). This topology coincides with the weak* topology when S is normal (and in particular compact and/or metric); the weakest topology

$\bigcup_{S\in\mathcal{S}}\Delta(S)$ is endowed with the disjoint-union topology: $O_\Delta \subseteq \bigcup_{S\in\mathcal{S}}\Delta(S)$ is open if and only if $O_\Delta \cap \Delta(S)$ is open in $\Delta(S)$ for all $S \in \mathcal{S}$.

Note that although each S and each $\Delta(S)$ are compact, if $\mathcal{S}$ is infinite, Ω and $\bigcup_{S\in\mathcal{S}}\Delta(S)$ are *not* compact.

2.7 Marginals

For a probability measure $\mu \in \Delta(S')$, the marginal $\mu_{|S}$ of μ on $S \preceq S'$ is defined by

$$\mu_{|S}(D) := \mu\left(\left(r_S^{S'}\right)^{-1}(D)\right), \quad D \in \mathcal{F}_S.$$

Let S_μ be the space on which μ is a probability measure. Whenever $S_\mu \succeq S(E)$ then we abuse notation slightly and write

$$\mu(E) = \mu(E \cap S_\mu).$$

If $S(E) \not\preceq S_\mu$, then we say that $\mu(E)$ is undefined.

2.8 Continuous Type Mappings

Let I be a nonempty finite or countable set of individuals. For every individual, each state gives rise to a probabilistic belief over states in some space.

Definition 1 *For each individual $i \in I$ there is a continuous type mapping $t_i : \Omega \longrightarrow \bigcup_{S\in\mathcal{S}}\Delta(S)$. We require the type mapping t_i to satisfy the following properties:*[3]

(0) Confinement: If $\omega \in S'$ then $t_i(\omega) \in \triangle(S)$ for some $S \preceq S'$.

(1) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S)$ then $t_i(\omega_{S'}) = t_i(\omega)$.

(2) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega) \in \triangle(S')$ then $t_i(\omega_S) = t_i(\omega)_{|S}$.

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for which the mapping

$$\mu \longrightarrow \int_S f d\mu$$

is continuous for every continuous real-valued function f on S (see Heifetz, 2006, Fn. 3).

[3]As is shown in Heifetz, Meier, and Schipper (2013), Property (1) of the type mappings is implied by the Properties (0),(2), and (3).

(3) If $S'' \succeq S' \succeq S$, $\omega \in S''$, and $t_i(\omega_{S'}) \in \triangle(S)$ then $S_{t_i(\omega)} \succeq S$.

$t_i(\omega)$ represents individual i's belief at state ω. Properties (0) to (3) guarantee the consistent fit of beliefs and awareness at different state-spaces. *Confinement* means that at any given state $\omega \in \Omega$ an individual's belief is concentrated on states that are all described with the same "vocabulary" - the "vocabulary" available to the individual at ω. This "vocabulary" may be less expressive than the "vocabulary" used to describe statements in the state ω." Properties (1) to (3) compare the types of an individual in a state $\omega \in S'$ and its projection to ω_S, for some $S \preceq S'$. Property (1) and (2) mean that at the projected state ω_S the individual believes everything she believes at ω given that she is aware of it at ω_S. Property (3) means that at ω an individual cannot be unaware of an event that she is aware of at the projected state $\omega_{S'}$.

Define[4]

$$Ben_i(\omega) := \left\{ \omega' \in \Omega : t_i(\omega')_{|S_{t_i(\omega)}} = t_i(\omega) \right\}.$$

This is the set of states at which individual i's type or the marginal thereof coincides with her type at ω. Such sets are events in our structure:

Remark 1 *For any $\omega \in \Omega$, $Ben_i(\omega)$ is a $S_{t_i(\omega)}$-based event, which is not necessarily measurable.*[5]

Assumption 1 *If $Ben_i(\omega) \subseteq E$, for a measurable event E, then $t_i(\omega)(E) = 1$.*

This assumption implies introspection (Property (va) in Proposition 4 in Heifetz, Meier, and Schipper, 2013). Note, that if $Ben_i(\omega)$ is measurable, then Assumption 1 implies $t_i(\omega)(Ben_i(\omega)) = 1$.

Definition 2 *We denote by $\underline{\Omega} := \left\langle \mathcal{S}, \left(r_{S'}^{S''}\right)_{S'' \succeq S'}, (t_i)_{i \in I} \right\rangle$ a topological unawareness belief structure.*

Topological unawareness belief structures are analogous to unawareness belief structures in Heifetz, Meier, and Schipper (2013) except for the additional topological properties and the well-foundedness asumption. See Heifetz, Meier, and Schipper (2013) for further details and interpretations.

[4]The name "Ben" is chosen analogously to the "ken" in knowledge structures.

[5]Even in a standard type-space, if the σ-algebra is not countably generated, then the set of states where a player is of a certain type might not be measurable.

2.9 Belief, Common Certainty, and Awareness

For $i \in I$, $p \in [0,1]$ and an event E, the p *-belief operator* is defined by

$$B_i^p(E) := \{\omega \in \Omega : t_i(\omega)(E) \geq p\},$$

if there is a state ω such that $t_i(\omega)(E) \geq p$, and by

$$B_i^p(E) := \emptyset^{S(E)}$$

otherwise. The mutual p-belief operator on events is defined by

$$B^p(E) = \bigcap_{i \in I} B_i^p(E).$$

The common certainty operator on events is defined by

$$CB^1(E) = \bigcap_{n=1}^{\infty} \left(B^1\right)^n (E).$$

These are standard definitions (e.g. see Monderer and Samet, 1989) adapted to our unawareness structures.

As in Heifetz, Meier, and Schipper (2013) we define for every $i \in I$ the awareness operator

$$A_i(E) := \{\omega \in \Omega : t_i(\omega) \in \Delta(S) \text{ for some } S \succeq S(E)\},$$

for every event E, if there is a state ω such that $t_i(\omega) \in \Delta(S)$ with $S \succeq S(E)$, and by

$$A_i(E) := \emptyset^{S(E)}$$

otherwise.

In Heifetz, Meier, and Schipper (2013, Proposition 1 and 2) we show that $A_i(E)$, $B_i^p(E)$, $B^p(E)$, and $CB^1(E)$ are all $S(E)$-based measurable events. We also show in Heifetz, Meier, and Schipper (2013, Proposition 4) that standard properties of belief obtain. Moreover, in Heifetz, Meier, and Schipper (2013, Proposition 5) we show "standard" properties of awareness. One of those properties is *weak necessitation*, i.e., for any event $E \in \Sigma$, $A_i(E) = B_i^1(S(E)^{\uparrow})$. This property will be used later in the proof.

Definition 3 *An event E is evident if for each $i \in I$, $E \subseteq B_i^1(E)$.*

Proposition 1 *For every event $F \in \Sigma$:*

(i) $CB^1(F)$ *is evident, that is* $CB^1(F) \subseteq B_i^1(CB^1(F))$ *for all* $i \in I$.

(ii) There exists an evident event E *such that* $\omega \in E$ *and* $E \subseteq B_i^1(F)$ *for all* $i \in I$, *if and only if* $\omega \in CB^1(F)$.

The proof is analogous to Proposition 3 in Monderer and Samet (1989) for a standard state-space and thus omitted.

3 A Generalized "No-speculative-trade" Theorem

Definition 4 (Prior) *A prior for player* i *is a system of probability measures* $P_i = \left(P_i^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ *such that*

1. *The system is projective: If* $S' \preceq S$ *then the marginal of* P_i^S *on* S' *is* $P_i^{S'}$. *(That is, if* $E \in \Sigma$ *is an event whose base-space* $S(E)$ *is lower or equal to* S', *then* $P_i^S(E) = P_i^{S'}(E)$.*)*
2. *Each probability measure* P_i^S *is a mixture of* i*'s beliefs in* S*: for every event* $E \in \Sigma$ *such that* $S(E) \preceq S$,

$$P_i^S\left(E \cap S \cap A_i\left(E\right)\right) = \int_{S\cap A_i(E)} t_i\left(\cdot\right)\left(E\right) dP_i^S\left(\cdot\right). \tag{1}$$

We call any probability measure $\mu_i \in \Delta(S)$ *satisfying equation (1) in place of* P_i^S *a prior of player* i *on* S.

Roughly the probability assigned by agent i's prior to the event $E \cap A_i(E)$ is the mixture of agent i's beliefs $t_i(\omega)(E)$ over states ω in which i is aware of E weighted by the prior. This is analogous to the definition of prior in standard belief structures (see Samet, 1999). See Heifetz, Meier, and Schipper (2013) for further discussions and examples.

Definition 5 (Common Prior) $P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ *(resp.* $P^S \in \Delta(S)$*) is a common prior (resp. a common prior on* S*) if* P *(resp.* P^S*) is a prior (resp. a prior on* S*) for every player* $i \in I$.

Denote by $[t_i(\omega)] := \{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\}$.

Definition 6 *A common prior* $P = \left(P^S\right)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ *(resp. a common prior* P^S *on* S*) is* positive *if and only if for all* $i \in I$ *and* $\omega \in \Omega$*: if* $t_i(\omega) \in \triangle(S')$*, then* $P^S\left(([t_i(\omega)] \cap S')^{\uparrow} \cap S\right) > 0$ *for all* $S \succeq S'$ *(resp. for* S*).*

Note that with this positivity assumption, $Ben_i(\omega)$ is measurable for every $\omega \in \Omega$ and i. Note further that by Lemma 3 below, $[t_i(\omega)] \cap S' \in \mathcal{F}_{S'}$.

Recall Remark 7 in Heifetz, Meier, and Schipper (2013) according to which if $\hat{S}$ is the upmost state-space in the lattice $\mathcal{S}$, and $(P_i^S)_{S\in\mathcal{S}} \in \prod_{S\in\mathcal{S}} \Delta(S)$ is a tuple of probability measures, then $(P_i^S)_{S\in\mathcal{S}}$ is a prior for player i if and only if $P_i^{\hat{S}}$ is a prior for player i on $\hat{S}$ and P_i^S is the marginal of $P_i^{\hat{S}}$ for every $S \in \mathcal{S}$.

Speculative trade between agents could occur at a state if it is common certainty that agents form different expectations of a random variable (e.g. stock returns).

Definition 7 *Let* x_1 *and* x_2 *be real numbers and* v *a continuous random variable*[6] *on* Ω*. Define the sets* $E_1^{\leq x_1} := \left\{\omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot)\, d(t_1(\omega))(\cdot) \leq x_1\right\}$ *and* $E_2^{\geq x_2} := \left\{\omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot)\, d(t_2(\omega))(\cdot) \geq x_2\right\}$*. We say that at* ω*, conditional on his information, player 1 (resp. player 2) believes that the expectation of* v *is weakly below* x_1 *(resp. weakly above* x_2*) if and only if* $\omega \in E_1^{\leq x_1}$ *(resp.* $\omega \in E_1^{\geq x_2}$*).*

Since we endowed Ω with the disjoint union topology, we have that a function $v : \Omega \longrightarrow \mathbb{R}$ is continuous if and only if for each $S \in \mathcal{S}$ the restriction of v to S is a continuous function from S to $\mathbb{R}$.

Note that the sets $E_1^{\leq x_1}$ or $E_1^{\geq x_2}$ may not be events in our unawareness belief structure, because $v(\omega) \neq v(\omega_S)$ is allowed, for $\omega \in S' \succ S$. Yet, we can define p-belief, mutual p-belief, and common certainty for measurable subsets of Ω, and show that the properties

[6] One may wonder to what extent the assumption of a continuous random variable would preclude situations in which for instance the "returns from investing in eurozone sovereign debt seems to dependent discontinuously on the question of whether the eurozone stays intact". We view continuity more as a technical assumption. The intuitive notion of continuity referred to in the eurozone example does not necessarily correspond to the mathematical notion of continuity. To see this, note that since we do not assume that state-spaces are connected, we allow for a states-space to be a disjoint union of two open subsets, call them A and B, with A standing for the event that the eurozone stays intact while B corresponding to the event that the eurozone falls apart. Then a random variable, which restricted to A is continuous and larger than x while restricted to B being also continuous but smaller than y with $x > y$, is a continuous random variable although it depends "discontinuously" (i.e., more precisely it "jumps") on the event of whether or not the eurozone stays intact or not.

stated in Heifetz, Meier, and Schipper (2013, Propositions 4 and 6) obtain as well. The proofs are analogous and thus omitted.

Theorem 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure and P a positive common prior. Then there is no state $\tilde{\omega} \in \Omega$, continuous random variable $v : \Omega \longrightarrow \mathbb{R}$, and $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ such that: at $\tilde{\omega}$ it is common certainty that conditional on her information, player 1 believes that the expectation of v is weakly below x_1 and, conditional on his information, player 2 believes that the expectation of v is weakly above x_2.*

This general "no-speculative-trade" theorem implies our "no-speculative-trade" theorem for finite unawareness belief structures (Heifetz, Meier, and Schipper, 2013).

We should clarify how the additional assumptions on the lattice and the state-spaces allow us to generalize the "no-speculative-trade" theorem for finite unawareness belief structures in Heifetz, Meier, and Schipper (2013) to the infinite case. One assumption imposed on topological unawareness structures is that the complete lattice of spaces is well-founded. That is, we impose that each nonempty subset $\mathcal{X} \subseteq \mathcal{S}$ of the lattice contains at least one $\preceq$-minimal space. If the join is an element of the subset $\mathcal{X}$, then it is the unique $\preceq$-minimal space of $\mathcal{X}$. Otherwise well-foundedness implies that there must be several incomparable $\preceq$-minimal spaces. The assumption that the lattice of spaces is well-founded is used in an important step of the proof. If the set of states in which there is common certainty that the first player's expectation is strictly above x and the second player's expectations is weakly below x is nonempty, we need to find a minimal state-space such that the common certainty event restricted to this space is nonempty. Assuming that the lattice of spaces is well-founded allows us to find such a minimal space, which is key to extending the "no-speculative-trade" theorem to unawareness. Otherwise, it could be the case that agent 1 may believe that agent 2 is less aware and believes that agent 1 is even less aware and believes that agent 2 is even less aware and believes ...

The assumption that state-spaces are compact Hausdorff might not be strictly necessary. This is because our assumption of a positive common prior (Definition 6) implies that for each agent there can be at most a countable number of types. This fact should facilitate the development of results in a purely measure theoretic setting. Yet, our assumptions make the proofs of Lemmata 1, 2, and 3, and Theorem 1 simpler. Moreover, the topological unawareness belief structure could be of independent interest. For instance, it might prepare for other versions of the "no-speculative-trade" results, where the positivity assumption is dropped (and hence any player could have uncountably many types). Feinberg (2000) and Heifetz (2006) prove results on the absence of common certainty of speculative-trade in *some* states without the positivity assumption. We focus on the absence of common certainty of speculative-trade in *all* states. Hence, our notion of "no-speculative-trade" implies Feinberg's notion of "no-speculative-trade". Although this comes at the cost of the positivity assumption, we opted for our notion because intuitively one is interested to know whether there are *some* states (as opposed to *all* states) where agents speculate.

On the feasibility of a converse to Theorem 1, we note that Heifetz, Meier, and Schipper (2013) show by example that the converse of the "no-speculative-trade" theorem does not hold even in the finite case.

4 Proof of Theorem 1

4.1 Preliminary Definitions and Results

We define $G \subseteq \Omega$ to be a *measurable set* if and only if for all $S \in \mathcal{S}$, $G \cap S \in \mathcal{F}_S$. The collection of measurable sets forms a sigma-algebra on Ω.

Let $\underline{\Omega}$ be an unawareness belief structure. As in Heifetz, Meier, and Schipper (2012, Appendix B), we define the *flattened type-space* associated with the unawareness belief structure $\underline{\Omega}$ by

$$F(\underline{\Omega}) := \langle \Omega, \mathcal{F}, (t_i^F)_{i \in I} \rangle,$$

where Ω is the union of all state-spaces in the unawareness belief structure $\underline{\Omega}$, $\mathcal{F}$ is the collection of all measurable sets in $\underline{\Omega}$, and $t_i^F : \Omega \longrightarrow \Delta(\Omega)$ is defined by

$$t_i^F(\omega)(E) \quad := \quad \begin{cases} t_i(\omega)(E \cap S_{t_i(\omega)}) & \text{if } E \cap S_{t_i(\omega)} \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

The definition of the belief operator as well as standard properties of belief and Proposition 1 can be extended to measurable subsets of Ω. The proofs are analogous and thus omitted.

In this article, a *standard topological type space* (see for instance, Heifetz, 2006) is a compact Hausdorff space Ω such that for every individual $i \in I$ there is a continuous type mapping $t_i : \Omega \longrightarrow \Delta(\Omega)$ from Ω to the the space of regular Borel probability measures $\Delta(\Omega)$ endowed with the topology of weak convergence.

Let $\underline{\Omega}$ be a topological unawareness belief structure and P a positive common prior. For the proof of the theorem, we have to show that there is no evident measurable set

$E \in \mathcal{F}$ such that $\tilde{\omega} \in E$ and

$$\int_\Omega v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_\Omega v(\cdot) d(t_2(\omega))(\cdot)$$

for all $\omega \in E$.

We need the following lemmata:

Lemma 1 *Let $\underline{\Omega}$ be a topological unawareness belief structure, $v : \Omega \longrightarrow \mathbb{R}$ be a continuous random variable, and $x \in \mathbb{R}$. Then $\left\{\omega \in \Omega : \int_\Omega v(\cdot) d(t_i(\omega))(\cdot) \geq x\right\}$ and $\left\{\omega \in \Omega : \int_\Omega v(\cdot) d(t_i(\omega))(\cdot) \leq x\right\}$ are closed subsets of Ω.*[7]

Proof of the Lemma. Since for every $S \in \mathcal{S}$, the topology on $\Delta(S)$ coincides with the weak* topology and since in particular, $v : S \longrightarrow \mathbb{R}$ is continuous, $\left\{\mu \in \Delta(S) : \int_S v(\cdot) d\mu(\cdot) < x\right\}$ is open in $\Delta(S)$. Hence $\left\{\nu \in \bigcup_{S \in \mathcal{S}} \Delta(S) : \int_S v(\cdot) d\nu(\cdot) < x\right\}$ is open in $\bigcup_{S \in \mathcal{S}} \Delta(S)$.

By the continuity of $t_i : \Omega \longrightarrow \bigcup_{S \in \mathcal{S}} \Delta(S)$, it follows that $\left\{\omega \in \Omega : \int_\Omega v(\cdot) d(t_i(\omega))(\cdot) < x\right\}$ is open in Ω and hence its relative complement with respect to Ω, $\left\{\omega \in \Omega : \int_\Omega v(\cdot) d(t_i(\omega))(\cdot) \geq x\right\}$ is closed in Ω. □

Lemma 2 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let E be a closed subset of Ω. Then $CB^1(E)$ is a closed subset of Ω.*

Proof of the Lemma. The relative complement of E with respect to Ω, $\Omega \setminus E$, is open, and hence for every $S \in \mathcal{S}$, $(\Omega \setminus E) \cap S = S \setminus (E \cap S)$ is open in S. Therefore $\{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. It follows that $\bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}$ is open. Hence for every $i \in I$, $\left\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(S \setminus (E \cap S)) > 0\}\right\}$ is open. It follows that its relative complement with respect to Ω, $B_i^1(E) = \left\{\omega \in \Omega : t_i(\omega) \in \bigcup_{S \in \mathcal{S}} \{\mu \in \Delta(S) : \mu(E \cap S) = 1\}\right\}$ is closed. Since an arbitrary intersection of closed sets is closed, the Lemma follows by induction. □

Lemma 3 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Then for every $\omega \in \Omega$, every state-space $S \in \mathcal{S}$ and every player $i \in I$, the set $\{\omega' \in \Omega : t_i(\omega') = t_i(\omega)\} \cap S$ is closed in S.*

[7] Note that we abuse notation and write $\int_\Omega v(\cdot) d(t_i(\omega))(\cdot)$ instead of $\int_{S_{t_i(\omega)}} v(\cdot) d(t_i(\omega))(\cdot)$.

Proof of the Lemma. Since $\Delta(S_{t_i(\omega)})$ is the set of regular Borel probability measures on $S_{t_i(\omega)}$ endowed with the topology of weak convergence, $\{t_i(\omega)\}$ is closed in $\Delta(S_{t_i(\omega)})$, and hence $\{t_i(\omega)\}$ is closed in $\bigcup_{S\in\mathcal{S}}\Delta(S)$. Therefore, by continuity of t_i, $t_i^{-1}(\{t_i(\omega)\}) = [t_i(\omega)]$ is closed in Ω. Hence, $[t_i(\omega)] \cap S$ is closed in S. $\square$

Lemma 4 *Let $\underline{\Omega}$ be a topological unawareness belief structure. Let P^S be a positive (common) prior on the state-space S, and let $\omega \in S$ be such that $t_i(\omega) \in \Delta(S)$. Then, for every $E \in \mathcal{F}_S$, we do have $t_i(\omega)(E) = t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E\cap[t_i(\omega)])}{P^S(S\cap[t_i(\omega)])}$.*

Proof. We have $t_i(\omega)(S \cap [t_i(\omega)]) = 1$ and hence $t_i(\omega)(E) = t_i(\omega)(E \cap S \cap [t_i(\omega)]) = t_i(\omega)(E \cap [t_i(\omega)])$. Since P^S is positive, we do have $P^S(S \cap [t_i(\omega)]) > 0$.

Since $S((E \cap [t_i(\omega)])^{\uparrow}) = S$ and since $\omega' \in [t_i(\omega)]$ implies $t_i(\omega') \in \Delta(S)$, we do have $(E\cap[t_i(\omega)])^{\uparrow} \cap A_i((E\cap[t_i(\omega)])^{\uparrow}) = (E\cap[t_i(\omega)])^{\uparrow}$. We also have $(S\cap[t_i(\omega)])^{\uparrow} \subseteq A_i(S^{\uparrow}) = A_i((E \cap [t_i(\omega)])^{\uparrow})$. The last equality follows from weak necessitation. We have - by the definition of a common prior - the following (with our abuse of notation):

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S\cap A_i((E\cap[t_i(\omega)])^{\uparrow})} t_i(\cdot)(E \cap [t_i(\omega)])dP^S(\cdot) \\ &= \int_{S\cap[t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)])dP^S(\cdot) \\ &\quad + \int_{(S\cap A_i(S^{\uparrow}))\setminus(S\cap[t_i(\omega)])} t_i(\cdot)(E \cap [t_i(\omega)])dP^S(\cdot) \end{aligned}$$

But if $\omega' \in (S\cap A_i((E\cap[t_i(\omega)])^{\uparrow})) \setminus (S\cap[t_i(\omega)])$, then $t_i(\omega')(E\cap[t_i(\omega)]) = 0$, and hence, we have

$$\begin{aligned} P^S(E \cap [t_i(\omega)]) &= \int_{S\cap[t_i(\omega)]} t_i(\cdot)(E \cap [t_i(\omega)])dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)]) \int_{S\cap[t_i(\omega)]} 1dP^S(\cdot) \\ &= t_i(\omega)(E \cap [t_i(\omega)])P^S(S \cap [t_i(\omega)]). \end{aligned}$$

Since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $t_i(\omega)(E \cap [t_i(\omega)]) = \frac{P^S(E\cap[t_i(\omega)])}{P^S(S\cap[t_i(\omega)])}$. $\square$

4.2 Proof of the Theorem

Suppose by contradiction that there are $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ and a continuous random variable $v : \Omega \longrightarrow \mathbb{R}$ such that $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$, where

$$E_1^{\leq x_1} := \left\{ \omega \in \Omega : \int_{S_{t_1(\omega)}} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 \right\}, \text{ and}$$
$$E_2^{\geq x_2} := \left\{ \omega \in \Omega : \int_{S_{t_2(\omega)}} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2 \right\}.$$

Let S be a $\preceq$-minimal state-space with the property that $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \neq \emptyset$.

By standard properties of beliefs, we have $CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \subseteq B_i^1(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$ for $i = 1, 2$. This implies that for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ and $i = 1, 2$, we have $t_i(\omega)(CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, which by the minimality of S implies that $t_i(\omega) \in \Delta(S)$ and $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$.

By Lemma 2, $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ is closed in S. Therefore it is easy to verify that if flattened, $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$, that is $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ with the induced structure, is a standard topological type-space (as in Heifetz, 2006), since for each $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$ for $i = 1, 2$.

Since P^S is a positive prior on S, we have that $P^S(S \cap [t_i(\omega)]) > 0$, for each $\omega \in S$.

For $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ we also have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = 1$, and by Lemma 4, we have $t_i(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = \frac{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)])}{P^S(S \cap [t_i(\omega)])}$.

Hence, since $P^S(S \cap [t_i(\omega)]) > 0$, it follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap [t_i(\omega)]) = P^S(S \cap [t_i(\omega)]) > 0$. It follows that $P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) > 0$. Therefore it is easy to check that $\frac{P^S(\cdot)}{P^S(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))}$ is a common prior on $F(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))$.

Claim: Let $\omega \in CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \cap S$. Then $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1$ and $\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

We prove the second inequality, the first is analogous to the second one. We know already that $t_2(\omega) \in \Delta(S)$. By the definitions $\omega \in S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ implies $\omega \in S \cap B_2^1(E_2^{\geq x_2})$, and therefore $t_2(\omega)([t_2(\omega)] \cap E_2^{\geq x_2} \cap S) = 1$. It follows that $[t_2(\omega)] \cap E_2^{\geq x_2} \cap S$ is non-empty. Let $\omega' \in [t_2(\omega)] \cap E_2^{\geq x_2} \cap S$. Then we have $\int_S v(\cdot) d(t_2(\omega'))(\cdot) \geq x_2$. But we have $t_2(\omega) = t_2(\omega')$ and therefore $\int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$.

Since S is compact and $v : S \longrightarrow \mathbb{R}$ is continuous, there is a $\bar{v} \in \mathbb{R}$ such that $|v(\tilde{\omega})| \leq \bar{v}$ for all $\tilde{\omega} \in S$.

Since $t_2(\omega)(S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})) = 1$, we have

$$\begin{aligned}
\left| \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} v(\cdot) d(t_2(\omega))(\cdot) \right| &\leq \bar{v} \int_{S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))} 1 d(t_2(\omega))(\cdot) \\
&= \bar{v} \quad t_2(\omega)(S \setminus (S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}))) \\
&= 0.
\end{aligned}$$

Hence, we have

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot) = \int_S v(\cdot) d(t_2(\omega))(\cdot) \geq x_2$$

and this finishes the proof of the claim.

It follows that we have found a standard topological type-space $S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})$ in the sense of Heifetz (2006) with a common prior and a continuous random variable $v : S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2}) \longrightarrow \mathbb{R}$ such that

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_1(\omega))(\cdot) \leq x_1 < x_2 \leq \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) d(t_2(\omega))(\cdot).$$

Note that if we replace $v(\cdot)$ by $v(\cdot) - \frac{x_1 + x_2}{2}$, we get

$$\int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_1(\omega))(\cdot) < 0 < \int_{S \cap CB^1(E_1^{\leq x_1} \cap E_2^{\geq x_2})} v(\cdot) - \frac{x_1 + x_2}{2} d(t_2(\omega))(\cdot).$$

But this is a contradiction to Feinberg's (2000) Theorem (Proposition 1 in Heifetz, 2006). Hence this completes the proof of the theorem. □

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