

# PURE STRATEGY EQUILIBRIA IN SYMMETRIC TWO-PLAYER ZERO-SUM GAMES\*

Peter Duersch<sup>†</sup>    Jörg Oechssler<sup>‡</sup>    Burkhard C. Schipper<sup>§</sup>

May 11, 2011

## Abstract

We observe that a symmetric two-player zero-sum game has a pure strategy equilibrium if and only if it is not a generalized rock-paper-scissors matrix. Moreover, we show that every finite symmetric quasiconcave two-player zero-sum game has a pure equilibrium. Further sufficient conditions for existence are provided. Our findings extend to general two-player zero-sum games using the symmetrization of zero-sum games due to von Neumann. We point out that the class of symmetric two-player zero-sum games coincides with the class of relative payoff games associated with symmetric two-player games. This allows us to derive results on the existence of finite population evolutionary stable strategies.

**Keywords:** Symmetric two-player games, zero-sum games, Rock-Paper-Scissors, single-peakedness, quasiconcavity, finite population evolutionary stable strategy, saddle point, exact potential games.

**JEL-Classifications:** C72, C73.

---

\*An earlier working paper version has been circulated under the title “Pure Saddle Points and Symmetric Relative Payoff Games”. Some of the material has previously been part of the paper “Unbeatable imitation” presented at the International Conference on Game Theory in Stony Brook, 2009. We thank Bernhard von Stengel, Reinhard John, an associated editor and two anonymous referees for extremely helpful suggestions.

<sup>†</sup>Department of Economics, University of Heidelberg, Email: peter.duersch@awi.uni-heidelberg.de

<sup>‡</sup>Department of Economics, University of Heidelberg, Email: oechssler@uni-hd.de

<sup>§</sup>Department of Economics, University of California, Davis, Email: bcschipper@ucdavis.edu

# 1 Introduction

Many zero-sum games do not have a solution without allowing for mixed strategies. What is the class of zero-sum games possessing pure equilibria? Some answers to these questions have been given by Shapley (1964) and Radzik (1991). For instance, Shapley (1964) showed that a finite two-player zero-sum game has a pure equilibrium if every 2x2 submatrix of the game has a pure equilibrium. Radzik (1991) showed that a two-player zero-sum game whose columns are quasiconcave (i.e. single-peaked) and whose rows are quasiconvex has a pure equilibrium if and only if every submatrix “along the diagonal” has a pure equilibrium. Although both results apply to symmetric two-player zero-sum games, none of these results exploits the symmetry property.

In this paper we are interested in pure equilibria of *symmetric* two-player zero-sum games. It is well known that for instance the rock-paper-scissors game has no pure equilibrium. More generally, we say that a symmetric two-player zero-sum game is a *generalized rock-paper-scissors matrix (gRPS)* if for each column there exists a row with a strictly positive payoff. We observe that a symmetric two-player zero-sum game possesses a pure equilibrium if and only if it is *not* a gRPS. Moreover, we show that every finite symmetric quasiconcave two-player zero-sum game has a pure equilibrium. We also provide sufficient conditions for existence in terms of increasing and decreasing differences, potentials, and additive separability of payoffs. It turns out that in the class of symmetric two-player zero-sum games, increasing and decreasing differences, the existence of an exact potential, and additively separable payoffs are all equivalent.

Brown and von Neumann (1950) and Gale, Kuhn, and Tucker (1950) showed that any general (not necessarily symmetric) two-player zero-sum game can be transformed into a symmetric zero-sum game whose equilibrium induces equilibrium in the underlying game.<sup>1</sup> Using the von Neumann symmetrization, our findings for symmetric two-player zero-sum games can be applied to general two-player zero-sum games. However, our findings are not applicable to the symmetrization by Gale, Kuhn, and Tucker (1950) since it has a special structure that allows only for mixed equilibrium in the symmetrization although such mixed equilibrium may correspond to a pure equilibrium in the underlying game.

Symmetric two-person zero-sum games are often thought to be less relevant to economics. However, in Section 4 we shall argue that they arise naturally when *relative*

---

<sup>1</sup>We thank the editor, Bernhard von Stengel, for drawing our attention to those symmetrizations of two-player zero sum games.

payoffs of arbitrary symmetric two-player games are considered.<sup>2</sup> The reason is simply that relative payoff functions give rise to zero-sum games by construction. Schaffer (1988, 1989) introduced the notion of finite population evolutionary stable strategies (fESS) and observed that a fESS of the original (arbitrary) symmetric game coincides with the Nash equilibrium of the (zero-sum) relative payoff game.<sup>3</sup> Thus, when we apply our equilibrium existence results to the relative payoff game, we also obtain existence results for fESS of the underlying original game. This way, our results can be applied for example to Cournot duopoly, Bertrand duopoly, public goods games, common pool resource games, minimum effort coordination games, synergistic relationships, arms race, Diamond’s search, Nash demand game, or rent seeking. We also show that a symmetric two-player game is an exact potential game if and only if its relative payoff game is an exact potential game. This is useful because the existence of an exact potential of a symmetric zero-sum game is easy to verify.

The fESS of a game is of relevance when evolution operates in a finite (playing-the-field) population. It is also important because frequently it coincides with the stochastically stable states of imitate-the-best dynamics.<sup>4</sup> Thus, when players imitate in such games, we should expect the outcome to be a fESS. The results developed here are also used in our companion paper, Duersch, Oechssler, and Schipper (2011). There we characterize the class of games in which “imitate-if-better” can not be exploited by any other decision rule.

In the next section, we study the existence of pure equilibria in symmetric two-player zero-sum games. Section 3 discusses two-player zero-sum games that are not necessarily symmetric. In Section 4 we apply our results to relative payoff games and the existence of finite population evolutionary stable strategies.

---

<sup>2</sup>There is some experimental evidence that players consider not only their absolute payoffs but also relative payoffs. Early experiments include Nydegger and Owen (1974) and Roth and Malouf (1979). More recently, relative payoff concerns have been studied in behavioral economics and experimental economics under the label of “inequity aversion”.

<sup>3</sup>This relationship between Nash equilibrium and fESS has been analyzed for competitive games by Ania (2008) and for “weakly competitive” games by Hehenkamp et al. (2010).

<sup>4</sup>See e.g. Alós-Ferrer and Ania (2005), Hehenkamp, Leininger, and Possajennikov (2004), Leininger (2006), Matros, Temzelides, and Duffy (2009), Possajennikov (2003), Schipper (2003), Tanaka (2000), and Vega-Redondo (1997)

## 2 Symmetric Zero-Sum Games

We consider a two-player game  $(X, Y, \pi_1, \pi_2)$  consisting of the set of pure actions  $X$  for player 1, the set of pure actions  $Y$  for player 2, and for each player  $i \in \{1, 2\}$  a payoff function  $\pi_i : X \times Y \rightarrow \mathbb{R}$ . The two-player game  $(X, Y, \pi_1, \pi_2)$  is zero-sum if  $\pi_2(x, y) = -\pi_1(x, y)$  for all  $x \in X$  and  $y \in Y$ . The two-player game  $(X, Y, \pi_1, \pi_2)$  is symmetric if  $X = Y$  and  $\pi_1(x, y) = \pi_2(y, x)$  for all  $x, y \in X$ . Thus, if we denote  $\pi = \pi_1$ , then a two-player zero-sum game is symmetric if and only if  $X = Y$  and  $\pi(x, y) = -\pi(y, x)$  for all  $x, y \in X$ .<sup>5</sup> Note that in a symmetric zero-sum game, the payoffs on the main “diagonal” must be zero. We denote by  $(X, \pi)$  a two-player symmetric zero-sum game.

**Definition 1** *In a symmetric two-player zero-sum game  $(X, \pi)$ , a pair of strategies  $(x, y)$  is a pure equilibrium if  $\pi(x, y) = \max_{x' \in X} \pi(x', y) = \min_{y' \in X} \pi(x, y')$ . A pure equilibrium  $(x, y)$  is symmetric if  $x = y$ .*

For obvious reasons, an equilibrium in two-player zero-sum games is often called a “saddle point”. In a symmetric game, if  $(x, y)$  is a pure equilibrium, so is  $(y, x)$ . By the rectangularity of equilibria in two-player zero-sum games, we also have that  $(x, x)$  and  $(y, y)$  are pure equilibria. This yields the following known fact (see e.g. Nash, 1951).

**Remark 2** *A symmetric two-player zero-sum game  $(X, \pi)$  has a pure equilibrium if and only if it has a symmetric pure equilibrium.*

A well known example for a symmetric zero-sum game without a pure equilibrium is the game “Rock-Paper-Scissors”.<sup>6</sup>

$$\begin{array}{c} R \\ P \\ S \end{array} \quad \begin{pmatrix} R & P & S \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

This example can be generalized to the following class of games.

**Definition 3 (Generalized Rock-Paper-Scissors Matrix (gRPS))** *A symmetric zero-sum game  $(X, \pi)$  is a generalized rock-paper-scissors matrix if in each column there exists*

---

<sup>5</sup>The payoff matrix of symmetric zero-sum game is skew-symmetric.

<sup>6</sup>The game was already described by von Neumann (1928, p. 303).

a row with a strictly positive payoff to the row player i.e. if for all  $y \in X$  there exists a  $x \in X$  such that  $\pi(x, y) > 0$ .<sup>7</sup>

**Observation 4** *A symmetric two-player zero-sum game  $(X, \pi)$  possesses a pure equilibrium if and only if it is not a generalized rock-paper-scissors matrix.*

PROOF. If  $(X, \pi)$  has no pure equilibrium (and in particular no pure symmetric equilibrium), then for all  $y \in X$  there is  $x \in X$  such that  $\pi(x, y) > \pi(y, y) = 0$ . Thus,  $(X, \pi)$  is gRPS. Conversely, if  $(X, \pi)$  has a pure equilibrium, by Remark 2 it has a symmetric equilibrium. Thus, there is  $y \in X$  such that  $\pi(x, y) \leq \pi(y, y)$  for all  $x \in X$ , which implies that  $(X, \pi)$  is not gRPS.<sup>8</sup>  $\square$

A symmetric  $2 \times 2$  zero-sum game cannot be a gRPS. If one of the row player's off-diagonal relative payoffs is  $a > 0$ , then the other must be  $-a$  violating the definition of gRPS. "Matching pennies" is not a counter-example because it is not symmetric. Thus we have the following corollary.

**Corollary 5** *Every symmetric  $2 \times 2$  zero-sum game possesses a pure equilibrium.*<sup>9</sup>

In the following we shall study how several properties that are sometimes assumed to hold for games relate to the gRPS property.

**Definition 6 (Quasiconcave)** *A symmetric two-player game  $(X, \pi)$  is quasiconcave (or single-peaked) if there exists a total order  $<$  on  $X$  such that for each  $x, x', x'', y \in X$  and  $x' < x < x''$ , we have that  $\pi(x, y) \geq \min \{\pi(x', y), \pi(x'', y)\}$ .*

That is, a symmetric game is quasiconcave if each column has a single peak. Note that the existence of the order on  $X$  for which the defining inequalities hold does not imply that this is true for any order.

---

<sup>7</sup>In the finite strategy case, an alternative way of characterizing a gRPS game would be to say that the game has a strictly positive minimax with respect to pure strategies, i.e.  $\min_y \max_x \pi(x, y) > 0$ .

<sup>8</sup>We thank the Associate Editor for suggesting this simple proof.

<sup>9</sup>This corollary is also implied by Kaplansky (1945, p. 479) who showed more generally that any symmetric two-player zero-sum game with an even number of actions has always an equilibrium that is not fully mixed. We thank Bernhard von Stengel for drawing our attention to the work of Kaplansky (1945).

**Theorem 7** *Every finite quasiconcave symmetric two-player zero-sum game has a pure equilibrium.*

The proof follows as a corollary from Observation 4 and the following lemma.

**Lemma 8** *A finite quasiconcave symmetric two-player zero-sum game is not a gRPS.*

PROOF. Note first that if  $\pi(\cdot, y)$  is quasiconcave in the first argument then by symmetry  $\pi(x, \cdot)$  is *quasiconvex* in the second argument, i.e.  $x' < x < x''$  implies  $\pi(y, x) \leq \max\{\pi(y, x'), \pi(y, x'')\}$ .

Order actions in  $X$  such that  $(X, \pi)$  is quasiconcave. Let  $\ell$  be the smallest column such that there is a strictly positive payoff above the main diagonale. I.e., there exists  $k < \ell$  with  $\pi(x_k, x_\ell) > 0$ . If there is no such column, then the last column is non-positive and  $(X, \pi)$  is not a gRPS. If there is such a column, then clearly  $\ell > 1$ .

We claim that column  $\ell - 1$  is nonpositive and thus  $(X, \pi)$  is not a gRPS. By the choice of  $\ell$ , we have  $\pi(x_j, x_{\ell-1}) \leq 0$  for all  $j \leq \ell - 1$ . Since  $(X, \pi)$  is a symmetric zero-sum game,  $\pi(x_\ell, x_k) < 0$ . Quasiconcavity in column  $k$  and  $\pi(x_k, x_k) = 0$  implies  $\pi(x, x_k) \leq \pi(x_\ell, x_k)$  for all  $x \geq x_\ell$ . If  $k = \ell - 1$ , then we are done. Otherwise, for each  $x \geq x_\ell$ , quasiconvexity in row  $x$ ,  $\pi(x, x) = 0$  and the previous arguments imply that  $\pi(x, x_{\ell-1}) \leq 0$  as claimed.  $\square$

Note that if the finite zero-sum game is not symmetric but quasiconcave, then it does not need to have a pure equilibrium. A counter example is presented in Radzik (1991, p. 26). Hence, symmetry is crucial for the result.

The converse to Theorem 7 is not true as the following example shows.

**Example 9** *Consider the following “Rock-Paper-Scissors” game augmented by an additional strategy “B”.*

$$\begin{array}{c} R \\ P \\ S \\ B \end{array} \begin{pmatrix} R & P & S & B \\ 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

*Clearly, it is not a gRPS since for column “B” there fails to exist a row yielding a strictly positive payoff. Thus, the game possesses a pure equilibrium,  $(B, B)$ . Yet, no matter how strategies are ordered, the game fails to be quasiconcave. Hence, there are symmetric*

two-player zero-sum games that are neither generalized rock-paper-scissors games nor quasiconcave.

Other “second-order” conditions are commonly explored in the literature when analyzing the existence of pure equilibria. We will consider increasing and decreasing differences, additive separability, and potentials. Surprising to us, it turns out that for symmetric two-player zero-sum games these conditions are all equivalent.

**Definition 10 (Increasing and decreasing differences)** *Let  $X$  be a totally ordered set. A payoff function  $\pi$  has decreasing (resp. increasing) differences on  $X \times X$  if for all  $x'', x', y'', y' \in X$  with  $x'' > x'$  and  $y'' > y'$ ,*

$$\pi(x'', y'') - \pi(x', y'') \leq (\geq) \pi(x'', y') - \pi(x', y').$$

$\pi$  is a valuation if it has both decreasing and increasing differences.<sup>10</sup>

**Definition 11 (Additively Separable)** *We say that a payoff function  $\pi$  is additively separable if  $\pi(x, y) = f(x) + g(y)$  for some functions  $f, g : X \rightarrow \mathbb{R}$ .*

Potential functions are often useful for obtaining results on convergence of learning algorithms to equilibrium, existence of pure equilibrium, and equilibrium selection. The following notion of potential games was introduced by Monderer and Shapley (1996).

**Definition 12 (Exact potential games)** *The symmetric two-player game  $(X, \pi)$  is an exact potential game if there exists a function  $P : X \times X \rightarrow \mathbb{R}$ , called an exact potential, such that for all  $y \in X$  and all  $x, x' \in X$ ,<sup>11</sup>*

$$\begin{aligned} \pi(x, y) - \pi(x', y) &= P(x, y) - P(x', y), \\ \pi(x, y) - \pi(x, y') &= P(y, x) - P(y, x'). \end{aligned}$$

**Proposition 13** *Let  $(X, \pi)$  be an arbitrary symmetric two-player zero-sum game and  $X$  be a totally ordered set. Then the following statements are equivalent:*

(i)  $\pi$  has decreasing differences on  $X \times X$ ,

---

<sup>10</sup>For the two-dimensional case, increasing differences are equivalent to supermodularity, so the two terms can be used interchangeably (see Topkis, 1998, Chapter 2.6).

<sup>11</sup>The second equation is stated using the symmetry of the game.

(ii)  $\pi$  has increasing differences on  $X \times X$ ,

(iii)  $\pi$  is a valuation,

(iv)  $\pi$  is additively separable,

(v)  $(X, \pi)$  has an exact potential.

PROOF. Let  $X$  be a totally ordered set such that  $\pi$  has decreasing differences on  $X \times X$ , that is, for all  $x''', x'', x', x \in X$  with  $x''' > x'$  and  $x'' > x$ ,

$$\pi(x''', x'') - \pi(x', x'') \leq \pi(x''', x) - \pi(x', x).$$

Since  $(X, \pi)$  is a symmetric two-player zero-sum game,  $\pi(x', x) = -\pi(x, x')$  for all  $x, x' \in X$ . Hence, we can rewrite this inequality as

$$-\pi(x'', x''') + \pi(x'', x') \leq -\pi(x, x''') + \pi(x, x'). \quad (1)$$

Rearranging inequality (1) yields the definition of increasing differences,

$$\pi(x'', x') - \pi(x, x') \leq \pi(x'', x''') - \pi(x, x''').$$

Hence (i) if and only if (ii). (iii) follows from the equivalence of (i) and (ii).

By Topkis (1998, Theorem 2.6.4.), a function  $\pi(x, y)$  is additively separable on  $X \times X$  if and only if  $\pi(x, y)$  is a valuation. Thus, (iii) if and only if (iv).

Brânzei, Mallozzi and Tijds (2003, Theorem 1) show that a zero-sum game is an exact potential game if and only if it is additively separable. Hence, (iv) if and only if (v).  $\square$

**Corollary 14** *Let  $(X, \pi)$  be a symmetric two-player zero-sum game for which  $X$  is a nonempty compact subset of a topological space and  $\pi$  is upper semicontinuous. If  $(X, \pi)$  satisfies any of the properties (i) to (v) of Proposition 13, then a pure equilibrium exists.*

PROOF. Since  $X$  is compact and  $\pi$  is upper semicontinuous, any player's best response correspondence of  $(X, \pi)$  is nonempty by Weierstrass' Theorem. Since  $\pi$  is additively separable under any property (i) to (v) by Proposition 13, the best response correspondence is constant. Thus, a pure equilibrium of  $(X, \pi)$  exists.  $\square$

Corollary 14 is implied by Theorem 7 if finite games are considered.

**Remark 15** Let  $X$  be a totally ordered set. If the symmetric zero-sum game  $(X, \pi)$  satisfies any of the properties (i) to (v) of Proposition 13, then  $(X, \pi)$  is quasiconcave.

PROOF. If property (iv) holds then there are some functions  $f, g : X \rightarrow \mathbb{R}$  such that  $\pi(x, y) = f(x) + g(y)$  for all  $x, y \in X$ . Then  $(X, \pi)$  is quasiconcave if and only if  $f(x) \geq \min[f(x'), f(x'')]$ . Since  $X$  is a totally ordered set, we can order it such that  $x' \geq x$  if and only if  $f(x') \geq f(x)$ . Thus,  $(X, \pi)$  is quasiconcave.  $\square$

The converse is not true as the following example shows.

**Example 16** Consider a symmetric two-player  $3 \times 3$  zero-sum game.<sup>12</sup>

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{ccc} A & B & C \\ \left( \begin{array}{ccc} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{array} \right) \end{array}$$

This game is quasiconcave but its payoff function is not a valuation, i.e., payoff differences in own strategies are not constant in the opponent's strategies.

Theorem 7 and Corollary 14 overlap in the important case of  $2 \times 2$  games. It is straight-forward to check that every symmetric  $2 \times 2$  zero-sum game is quasiconcave and satisfies all of the properties (i) to (v) of Proposition 13.

### 3 Application to General Zero-Sum Games

Consider now an arbitrary (not necessarily symmetric) two-player zero-sum game  $(X, Y, \pi_1, \pi_2)$  and define a function  $u : (X \times Y) \times (X \times Y) \rightarrow \mathbb{R}$  by  $u((x, y), (x', y')) = \pi_1(x, y') + \pi_2(x', y)$ . The two-player game  $((X \times Y), u)$  is a two-player symmetric zero-sum game. We call  $((X \times Y), u)$  the *von Neumann symmetrization* of the two-player zero-sum game.

The von Neumann symmetrization of  $(X, Y, \pi_1, \pi_2)$  has a very intuitive interpretation. Suppose that instead of playing the game  $(X, Y, \pi_1, \pi_2)$ , both players play the game twice. First, player 1 plays in the player position 1 and player 2 in the player position 2. Then, in a second stage, player 1 plays in the player position 2 and player 2 plays in the player

---

<sup>12</sup>We thank an anonymous referee for suggesting this example.

position 1. In each stage, players can not observe each other's actions. The payoff of a player is the sum of his payoffs in both stages.<sup>13</sup>

Brown and von Neumann (1950, §4) showed that for any finite two-player zero-sum game an optimal mixed *strategy* in its von Neumann symmetrization induces an optimal mixed strategy *profile* in the two-player zero-sum game (see also Gale, Kuhn and Tucker, 1950, §1). In the context of pure strategies, it is easy to see that the converse is true as well. In particular, when  $(x, y)$  is a pure Nash equilibrium strategy of  $(X \times Y, u)$ ,

$$\begin{aligned} u((x, y), (x', y')) &\geq 0 \text{ for all } (x', y') \in X \times Y, \text{ if and only if} \\ \pi_1(x, y) &\geq \pi_1(x', y) \text{ for all } (x', y') \in X \times Y. \end{aligned}$$

The last inequality is equivalent to both  $\pi_1(x, y) \geq \pi_1(x', y)$  for all  $x' \in X$ , and  $\pi_1(x, y) \geq \pi_1(x, y')$  for all  $y' \in Y$ .

Observation 4 implies that a two-player zero-sum game has a pure equilibrium if and only if its von Neumann symmetrization is not a generalized rock-paper-scissors matrix.

Gale, Kuhn, and Tucker (1950, §2) introduced another symmetrization of two-player zero-sum games that generally leads to smaller payoff matrices than the von Neumann symmetrization. Yet, Observation 4 does not apply to their symmetrization because it has a special structure that by Observation 4 must have equilibria that are mixed. As an example, consider the trivial one-action two-player zero-sum game with payoff matrix  $(1, -1)$ . The Gale-Kuhn-Tucker symmetrization would yield the rock-paper-scissors game, which does not have a pure equilibrium. Yet, Gale, Kuhn, and Tucker (1950, §2) show that generally the mixed equilibrium of their symmetrization induces an (pure or mixed) equilibrium of the underlying game.

## 4 Application to Relative Payoff Games

Consider now more generally a symmetric two-player (not necessarily zero-sum) game  $(X, \pi)$ . When instead of the payoff function  $\pi$  the relative payoffs are considered, then symmetric two-player games give naturally rise to the class of symmetric zero-sum games.

---

<sup>13</sup>Note that in the von Neumann symmetrization, each player chooses strategies in the set  $(X \times Y)$ . However, while for player 1 the strategy  $(x, y)$  means that he chooses  $x$  in the first stage and  $y$  in the second stage, for player 2 the strategy  $(x, y)$  means that he chooses  $y$  in the first stage and  $x$  in the second stage.

**Definition 17 (Relative payoff game)** *Given a symmetric two-player game  $(X, \pi)$ , the associated relative payoff game is  $(X, \Delta)$ , where the relative payoff function  $\Delta : X \times X \rightarrow \mathbb{R}$  is defined by*

$$\Delta(x, y) = \pi(x, y) - \pi(y, x).$$

The relative payoff of a player is the difference between his payoff and the payoff of his opponent.

**Remark 18** *Every relative payoff game is a symmetric zero-sum game. Conversely, for every symmetric zero-sum game, there is a symmetric two-player game for which the relative payoff game is the symmetric zero-sum game.*

PROOF. Note that by definition,  $\Delta(x, y) = \pi(x, y) - \pi(y, x) = -[\pi(y, x) - \pi(x, y)] = -\Delta(y, x)$  and hence  $(X, \Delta)$  is a symmetric zero-sum game. For the converse, if  $(X, \Delta)$  is a symmetric zero-sum game, then  $(X, \pi)$  with  $\pi(x, y) = \frac{1}{2}\Delta(x, y)$  is a symmetric two-player game for which  $(X, \Delta)$  is the relative payoff game. To see this, note that since  $(X, \Delta)$  is a symmetric zero-sum game, we must have that  $(X, \frac{1}{2}\Delta)$  is a symmetric zero-sum game. Note further that  $\Delta(x, y) = \pi(x, y) - \pi(y, x) = \frac{1}{2}\Delta(x, y) - \frac{1}{2}\Delta(y, x) = \frac{1}{2}\Delta(x, y) + \frac{1}{2}\Delta(x, y)$ , where the last equality follows from the fact that  $(X, \frac{1}{2}\Delta)$  is a symmetric zero-sum game.  $\square$

The remark shows that every relative payoff game is a symmetric zero-sum game, and that relative payoff games do not impose any restriction on the class of symmetric zero-sum games. Every symmetric zero-sum game is a relative payoff game of some symmetric two-player game. Note also that different symmetric two-player games may have the same relative payoff game.

Given Remark 18, we can now use our observations for symmetric two-player zero-sum games to study which outcomes in a symmetric two-player game correspond to pure equilibria in its associated relative payoff game. To this extent we introduce the notion of finite population evolutionary stable strategy (Schaffer, 1988, 1989). This concept is appropriate when “playing the field”, i.e. when players are matched against all other players except themselves.

**Definition 19 (fESS)** *A strategy  $x^* \in X$  is a finite population evolutionary stable strategy (fESS) of the game  $(X, \pi)$  if*

$$\pi(x^*, x) \geq \pi(x, x^*) \text{ for all } x \in X. \tag{2}$$

In terms of the associated relative payoff game, inequality (2) is equivalent to

$$\Delta(x^*, x) \geq 0 \text{ for all } x \in X.$$

Schaffer (1988, 1989) observed that  $x^*$  is a fESS of the symmetric game  $(X, \pi)$  if and only if  $(x^*, x^*)$  is a pure Nash equilibrium of the relative payoff game  $(X, \Delta)$ .<sup>14</sup>

Our results in Section 2 provide existence results for fESS of  $(X, \pi)$  when conditions are imposed on the associated relative payoff game  $(X, \Delta)$ . That is, a symmetric game  $(X, \pi)$  has a fESS if and only if its associated relative payoff game  $(X, \Delta)$  is not a gRPS. In particular, every symmetric  $2 \times 2$  game has a fESS. Furthermore, if the relative payoff function  $\Delta$  associated to a finite game  $(X, \pi)$  is quasiconcave, then a fESS exists. Finally, if the relative payoff game  $(X, \Delta)$  associated to  $(X, \pi)$  satisfies the properties of Corollary 14, then a fESS exists.

There is an interesting connection between symmetric two-player games and their relative payoff games with regard to the existence of an exact potential function.

**Theorem 20** *Let  $(X, \pi)$  be a symmetric two-player game with the associated relative payoff game  $(X, \Delta)$ .  $(X, \pi)$  is an exact potential game if and only if  $(X, \Delta)$  is an exact potential game.*

PROOF. If  $P$  is an exact potential function of a symmetric two-player game  $(X, \pi)$ , then  $P$  is symmetric, i.e.  $P(x, y) = P(y, x)$  for all  $x, y \in X$ . To see this note that  $P$  being an exact potential  $(X, \pi)$  implies for all  $x, y \in X$

$$\begin{aligned} \pi(x, y) - \pi(y, y) &= P(x, y) - P(y, y) \text{ and} \\ \pi(x, y) - \pi(y, y) &= P(y, x) - P(y, y) \end{aligned}$$

Hence  $P(y, x) = P(x, y)$ .

By symmetry of  $P$  we obtain for all  $x, x', y, y' \in X$

$$(P(x', y) - P(x, y)) - (P(y, x') - P(y', x')) = (P(x', y') - P(x, y')) - (P(y, x) - P(y', x)).$$

Since  $P$  is an exact potential function of  $(X, \pi)$  we can rewrite this equation

$$(\pi(x', y) - \pi(x, y)) - (\pi(y, x') - \pi(y', x')) = (\pi(x', y') - \pi(x, y')) - (\pi(y, x) - \pi(y', x)).$$

---

<sup>14</sup>See Ania (2008) and Hehenkamp, et al. (2010) for further discussion.

Rearranging terms yields

$$(\pi(x', y) - \pi(y, x')) - (\pi(x, y) - \pi(y, x)) = (\pi(x', y') - \pi(y', x')) - (\pi(x, y') - \pi(y', x)).$$

Using the relative payoff function  $\Delta$ , we obtain

$$\Delta(x', y) - \Delta(x, y) = \Delta(x', y') - \Delta(x, y').$$

Thus  $\Delta$  is a valuation which by Proposition 13 is equivalent to  $(X, \Delta)$  being an exact potential game.  $\square$

Often it is rather difficult to verify the existence of an exact potential function. Theorem 20 and Proposition 13 show that it is straightforward for symmetric two-player games. It is easy to verify whether the relative payoff function associated with the symmetric two-player game is a valuation.

Finally, the following two corollaries provide sufficient conditions imposed on the payoff function  $\pi$  of the underlying game  $(X, \pi)$  for the existence of a fESS.

**Corollary 21** *Consider a symmetric two-player game  $(X, \pi)$  with a compact strategy set  $X$  and a continuous payoff function. If  $(X, \pi)$  is an exact potential game, then a fESS exists.*

The corollary follows since by Theorem 20 the relative payoff game is also an exact potential game. Hence, Corollary 14 implies the existence of a pure equilibrium of  $(X, \Delta)$ , which is a fESS of  $(X, \pi)$ .

**Corollary 22** *Consider a symmetric two-player game  $(X, \pi)$  with a compact strategy set  $X$  and a payoff function that can be written as  $\pi(x, y) = f(x) + g(y) + a(x, y)$  for some continuous functions  $f, g : X \rightarrow \mathbb{R}$  and a symmetric function  $a : X \times X \rightarrow \mathbb{R}$  (i.e.,  $a(x, y) = a(y, x)$  for all  $x, y \in X$ ). Then  $(X, \pi)$  has a fESS.*

The corollary follows since  $\pi(x, y) = f(x) + g(y) + a(x, y)$  implies that in the relative payoff game the term  $a(x, y)$  drops out,  $\Delta(x, y) = f(x) - g(y) - f(y) + g(x)$ . Again, Corollary 14 implies the existence of a fESS of  $(X, \pi)$ .

While at first glance the condition on payoffs in the last corollary looks restrictive, it is satisfied in many well-known textbook examples of two-player games including linear Cournot duopoly, versions of Bertrand competition, public goods games, common pool

resource games, minimum effort coordination games, synergistic relationships, Diamond's search, Nash demand game, and Tullock rent seeking games (for details see Duersch, Oechssler, and Schipper, 2011). In applications, the function  $a$  is often symmetric because it represents just the sum or the product of players' strategies.

## References

- [1] Alós-Ferrer, C. and A.B. Ania (2005). The evolutionary stability of perfectly competitive behavior, *Economic Theory* **26**, 497-516.
- [2] Ania, A. (2008). Evolutionary stability and Nash equilibrium in finite populations, with an application to price competition, *Journal of Economic Behavior and Organization* **65**, 472-488.
- [3] Brânzei, R., Mallozzi, L., and S. Tijs (2003). Supermodular games and potential games, *Journal of Mathematical Economics* **39**, 39-49.
- [4] Brown, G.W. and J. von Neumann (1950). Solutions of games by differential equations, in: Kuhn, H.W. and A.W. Tucker (eds.), *Contributions to the theory of games*, Annals of Mathematics Studies No. 24, Princeton University Press, 81-87.
- [5] Duersch, P., Oechssler, J., and B.C. Schipper (2011). Unbeatable imitation, mimeo., University of Heidelberg and the University of California, Davis.
- [6] Gale, D., Kuhn, H.W. and A.W. Tucker (1950). On symmetric games, in: Kuhn, H.W. and A.W. Tucker (eds.), *Contributions to the theory of games*, Annals of Mathematics Studies, Princeton University Press, 81-87.
- [7] Hehenkamp, B., Possajennikov, A., and T. Guse (2010). On the equivalence of Nash and evolutionary equilibrium in finite populations, *Journal of Economic Behavior and Organization* **73**, 254-258.
- [8] Kaplansky, I. (1945). A contribution to von Neumann's theory of games, *Annals of Mathematics* **46**, 474-479.
- [9] Leininger, W. (2006). Fending off one means fending off all: evolutionary stability in quasi-submodular games, *Economic Theory* **29**, 713-719.
- [10] Matros, A., Temzelides, T., and J. Duffy (2009). Competitive behavior in market games: Evidence and theory, mimeo.

- [11] Monderer, D. and L.S. Shapley (1996). Potential games, *Games and Economic Behavior* **14**, 124-143.
- [12] Nash, J. (1951). Non-cooperative games, *Annals of Mathematics* **54**, 286-295.
- [13] von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele, *Mathematische Annalen* **100**, 295-320.
- [14] Nydegger, R.V. and G. Owen (1974). Two-person bargaining: An experimental test of the Nash axioms, *International Journal of Game Theory* **3**, 239-249.
- [15] Possajennikov, A. (2003). Evolutionary foundation of aggregative-taking behavior, *Economic Theory* **21**, 921-928.
- [16] Radzik, T. (1991). Saddle point theorems, *International Journal of Game Theory* **20**, 23-32.
- [17] Roth, A.E. and M.W.K. Malouf (1979). Game-theoretic models and the role of information in bargaining, *Psychological Review* **86**, 574-594.
- [18] Schaffer, M.E. (1989). Are profit-maximizers the best survivors?, *Journal of Economic Behavior and Organization* **12**, 29-45.
- [19] Schaffer, M.E. (1988). Evolutionary stable strategies for a finite population and a variable contest size, *Journal of Theoretical Biology* **132**, 469-478.
- [20] Shapley, L.S. (1964). Some topics in two-person games, in: Dresher, M., Shapley, L.S. and A.W. Tucker (eds.), *Advances in Game Theory, Annals of Mathematical Studies* **52**, 1-28.
- [21] Tanaka, Y. (2000). A finite population ESS and a long run equilibrium in an n-players coordination game, *Mathematical Social Sciences* **39**, 195-206.
- [22] Topkis, D. M. (1998). *Supermodularity and complementarity*, Princeton, New Jersey: Princeton University Press.
- [23] Vega-Redondo, F. (1997). The evolution of Walrasian behavior, *Econometrica* **65**, 375-384.