The Pure Theory of International Trade and Investment

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Preface

In the present volume I have sought to provide graduate students and advanced undergraduates with a fairly rigorous account of the barter theory of international trade and investment. I hope to supplement it later with a companion volume dealing with the monetary aspects of international economics.

The treatment is, I think, systematic and moderately comprehensive. There are nevertheless three important omissions. At first I had hoped to include a couple of chapters on preferential trading relationships (including, as special cases, customs unions and free trade associations). However I have not been able to compress an adequate discussion into fifty pages; and, in any case, I hope to publish soon an extensive treatment in book form. There is lacking also an adequate analysis of the gains from trade and investment. Within its frame of reference the discussion in Chaps. 12 and 13 is fairly complete; but the point of view adopted in those chapters is uncompromisingly static. Until a late stage in the preparation of the manuscript I had planned to include two companion chapters entitled, respectively, “The Gain from Trade and Investment Dynamically Considered” and “Optimal Trade and Investment Dynamically Considered.” In those chapters I had hoped to discuss the implications for welfare of factor endowments and technologies which systematically change over time, and to examine the problem of formulating an optimal commercial policy over time. In the end, however, the plan was abandoned, partly out of a growing
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In each case, the subscript runs from 1 to n. In addition we shall employ the following vector notation:

\[ a_i = (a_{i1}, \ldots, a_{in}) \]
\[ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \]
\[ w = (w_1, \ldots, w_n) \]
\[ p = (p_1, \ldots, p_n) \]
\[ V = (V_1, \ldots, V_n) \]
\[ x = (x_1, \ldots, x_n) \]
\[ \theta = (\log w_1, \ldots, \log w_n) \]
\[ \pi = (\log p_1, \ldots, \log p_n) \]
\[ c = (c_1, \ldots, c_n) \]

In terms of this notation, the jth production function can be written

\[ l = F_j(a_j) \]

Given \( w \), producers select those \( a_j \)'s which minimize the unit cost of product \( j \). Thus if \( x_j = \{a_j | f_j(a_j) = 1\} \) then

\[ c_j(w) = \min_{a \in x_j} \sum w_i a_{ij} \] (1A.1)

where \( c_j(w) \) is homogeneous of degree one and concave. From Eq. (1A.1),

\[ \sum_i w_i (\partial a_{ij} / \partial w_k) = 0; \text{ hence} \]

\[ \frac{\partial c_j}{\partial w_i} = a_{ij} + \sum_k w_k (\frac{\partial a_{ij}}{\partial w} ) = a_{ij} \]

Under competition, with all goods produced, therefore,

\[ \sum_i w_i a_{ij}(w) = c_j(w) = p_j \]

or, in vector notation,

\[ p = c(w) \] (1A.2)

with the Jacobian

\[ \frac{\partial c}{\partial w} = (a) = A \] (1A.3)

It will be more convenient, however, to work with the logarithms of prices. Eqs. (1A.2) and (1A.3) then become

\[ \pi = \log c(e^\theta) = \varphi(\theta) \] (1A.2')

---

APPENDIX*

Generalizations of the Stolper–Samuelson and Samuelson–Rybczynski Theorems

Not all theorems proved for the case of two products and two factors of production admit of straightforward generalization; and, even where a generalization is highly plausible, the proof may be quite difficult. To illustrate these propositions we consider now the problem of extending the Stolper–Samuelson and Samuelson–Rybczynski Theorems so that they cope with any number of factors and products.


We begin by introducing a slightly revised notation and by reviewing the material of Sec. 2. The new notation is as follows:

\( a_{ij} \) = the amount of factor \( i \) used in the production of a unit of commodity \( j \)
\( w_i \) = the money reward (rental) of the \( i \)th factor of production
\( p_j \) = the money price of the \( j \)th product
\( V_j \) = the community's endowment of the \( j \)th factor of production
\( \theta_j \) = log \( w_i \)
\( \pi_j \) = log \( p_j \)
\( C_j \) = the unit cost of producing the \( j \)th commodity

* This appendix was written jointly with Professor Leon L. Wegge of the University of California at Davis.
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and
\[
\frac{\partial \psi}{\partial \theta} = \psi'(\theta) = \beta^{-1} A \beta
\]

(1A.3')

respectively, where \( \beta \) and \( \theta \) are diagonal matrices so that the \((i,j)\)th element of \( \psi' \) is
\[
\alpha_{ij} = \frac{w_i a_{ij}}{p_j}
\]

(1A.4)

Let \((\alpha_{ij})\), the matrix of distributive shares, be denoted by \( S \). It is obvious that the column sums of \( S \) are one; that is, \( S \) is a stochastic matrix. It follows that the inverse \( S^{-1} \) also has unit column

sums.(14)

Stolper and Samuelson, in their consideration of the two-by-two case, assumed that
\[
\frac{a_{11}}{a_{21}} > \frac{a_{12}}{a_{22}} \quad \text{and} \quad \frac{\alpha_{11}}{\alpha_{21}} > \frac{\alpha_{12}}{\alpha_{22}}
\]

(1A.5)

In words, the first factor is used relatively intensively in the first industry, and the second factor is used relatively intensively in the second industry. It follows from inequalities (1A.5) that \( A^{-1} \) and \( S^{-1} \) possess negative off-diagonal elements and positive diagonal elements:
\[
S^{-1} = \begin{pmatrix} + & - \\ - & + \end{pmatrix}
\]

(1A.6)

And since the column sums of \( S^{-1} \) are equal to one the diagonal elements must be greater than one. In fact (1A.5) is necessary as well as sufficient for the Stolper–Samuelson result:
\[
S^{-1} = \begin{pmatrix} + & - \\ - & + \end{pmatrix} \quad \text{if and only if} \quad \frac{\alpha_{11}}{\alpha_{21}} > \frac{\alpha_{12}}{\alpha_{22}}
\]

(1A.7a)

A2. ARBITRARY BUT EQUAL NUMBERS OF PRODUCTS AND FACTORS

In view of (1A.7a) one is tempted to seek conditions on the \( \alpha_{ij} \), which contain (1A.5) as a special case and which are necessary and sufficient for

\( \frac{\partial \psi}{\partial \theta} = \psi'(\theta) = \beta^{-1} A \beta \)

(1A.3')

the inverse of \( S^{-1} \) to have the sign pattern

\[
S^{-1} = \begin{pmatrix} + & - & \cdots & - \\ - & + & \cdots & - \\ \cdots & \cdots & \cdots & \cdots \\ - & - & \cdots & + \end{pmatrix}
\]

(1A.8a)

For then, clearly, one would be able to say that an increase in the \( j \)th product price is associated with a more than proportionate increase in the \( j \)th factor reward (and therefore with an unambiguous increase in the real reward of the \( j \)th factor) and with a decline in the (money and real) rewards of all other factors. This is in fact the path to generalization we shall follow.

Notice, however, that by simply renumbering factors or products, the Stolper–Samuelson Theorem can be given the equivalent form:
\[
S^{-1} = \begin{pmatrix} - & + \\ + & - \end{pmatrix} \quad \text{if and only if} \quad \frac{\alpha_{11}}{\alpha_{12}} < \frac{\alpha_{21}}{\alpha_{22}}
\]

(1A.7b)

This formulation suggests the alternative possibility of generalizing the theorem by placing restrictions on the \( \alpha_{ij} \) necessary and sufficient for the inverse of \( S^{-1} \) to have the sign pattern

\[
S^{-1} = \begin{pmatrix} + & \cdots & - & \cdots & + \\ - & + & \cdots & - & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ - & - & \cdots & + & - \\ + & \cdots & - & \cdots & + \end{pmatrix}
\]

(1A.8b)

For \( n > 2 \), it is impossible by simply renumbering factors or products to reduce the sign pattern (1A.8b) to the pattern (1A.8a). The two paths to generalization are, therefore, genuine alternatives. Generalizations obtained by the two paths are equivalent only for \( n \leq 2 \).

As already noted, we shall confine our attention to generalizations reached by the first path. The reader interested in the destination of the second path may consult Inada [2] and Kemp and Wegge [8].

We seek first a generalization of the concept of relative factor intensity; that is, we seek to generalize (1A.5) so that it may be interpreted as requiring that the \( i \)th factor of production is used relatively intensively in the \( i \)th industry. Evidently nonsingularity of the matrix \( S \) is not enough; further restrictions are necessary. There are several ways in which the concept might be generalized, but it seems most natural to say that factor \( i \) is used

(14) Let 1 be the row vector consisting of \( n \) ones. Then if \( S \) has column sums equal to one, \( 1S = 1 \), whence \( 1S^{-1} = 1(1S^{-1}) = 1 \). The proof is Chipman's [1].

(15) Cf. Chipman [1], Minabe [9], and Uekawa [16]. Johnson ([3] p. 30) appears to deny the possibility of generalization.
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relatively intensively in the \( i \)th industry, and is associated with that industry, if and only if

\[
\max_s \left( \frac{\alpha_i}{\alpha_{is}} \right) = \frac{\alpha_i}{\alpha_{is}} \quad j = 1, \ldots, n \tag{1A.9}
\]

If (1A.9) holds for all \( i \), so that\(^{(18)}\)

\[
\frac{\alpha_i}{\alpha_{is}} > \frac{\alpha_i}{\alpha_{is}} \quad i \neq j, \quad i \neq s, \quad i = 1, \ldots, n \tag{1A.10}
\]

we have the required generalization. In the special two-by-two case, (1A.10) reduces to (1A.5).

**Necessary Conditions**

We begin by establishing three necessary conditions for the sign pattern (1A.8a). As far as possible, only the most elementary properties of matrices and determinants will be used in the proofs. In one or two places, however, appeal will be made to Jacobi's theorem on determinants\(^{(18)}\) and to well-known properties of Minkowski matrices.\(^{(20)}\)

**Theorem 1A.1:** If \( S^{-1} \) has the sign pattern (1A.8a) then every principal minor of \( S \) is positive, that is, \( S \) is a P-matrix.

**Proof:** Partition \( S \) and \( S^{-1} \) so that

\[
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}
\]

\[^{(18)}\] For \( n > 2 \) the inequalities (1A.10) may be written more compactly as

\[
\frac{\alpha_i}{\alpha_{is}} > \frac{\alpha_i}{\alpha_{is}} \quad i \neq j, \quad i \neq s, \quad i = 1, \ldots, n
\]

The "missing" relation, with \( s = j \), follows from \( \alpha_{is}/\alpha_{it} > \alpha_{is}/\alpha_{is} \) and \( \alpha_{is}/\alpha_{is} > \alpha_{is}/\alpha_{is} \). Thus (1A.10) contains just \( n(n-1) \) \( (n-2) \) independent relations.

It is worth noting also that (1A.10) holds if and only if

\[
\frac{\alpha_i}{\alpha_{is}} > \frac{\alpha_i}{\alpha_{is}} \quad i \neq j, \quad i \neq s, \quad i = 1, \ldots, n
\]

\[^{(18)}\] Cf. Aitken [20], pp. 97ff.

\[^{(20)}\] A matrix is called Minkowski if its diagonal elements are nonnegative and its off-diagonal elements non-positive and if all of its column (row) sums are non-negative. If all column (row) sums are strictly positive the matrix is called proper Minkowski; in all other cases it is said to be improper. The determinant of a proper Minkowski matrix is positive and the inverse of an irreducible Minkowski matrix is positive. For proofs, further properties, and for references, see Ostrowski [23].

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\[
S^{-1} = \begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix}
\]

where \( S_{11} \) and \( (S^{-1})_{11} \) are \( k \times k \). The essence of the proof consists in the demonstration that

\[
[S_{11}] = \frac{[S^{-1}]_{11}}{|S|} \tag{1A.11}
\]

Once (1A.11) is proved, we can appeal to the fact that, since \( S^{-1} \) and \( (S^{-1})_{11} \) are proper Minkowski matrices, both \( |S^{-1}| \) and \( |(S^{-1})_{11}| \), and therefore \( |S_{11}| \), are positive. The proof is then completed by noting that \( k \) can take the values \( 1, \ldots, n - 1 \).

By easy steps,

\[
|S^{-1}| = \begin{vmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{vmatrix} = |S^{-1}|_{21} \cdot \begin{vmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{vmatrix} I = -((S^{-1})_{12})^{-1}((S^{-1})_{11})^{-1}
\]

If \( (S^{-1})_{11} \) is nonsingular

\[
|S^{-1}| = |S^{-1}|_{21} \cdot \begin{vmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{vmatrix} I = 0 \]

\[
|S^{-1}| = |(S^{-1})_{11} - (S^{-1})_{12}(S^{-1})_{22}^{-1}(S^{-1})_{21}| (S^{-1})_{12}((S^{-1})_{22})^{-1} \]

On the other hand,

\[
\begin{pmatrix} (S^{-1})_{11} & (S^{-1})_{12} \\ (S^{-1})_{21} & (S^{-1})_{22} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\]

hence

\[
(S^{-1})_{11}S_{11} + (S^{-1})_{12}S_{12} = I \tag{1A.13a}
\]

and

\[
(S^{-1})_{21}S_{11} + (S^{-1})_{22}S_{22} = 0 \tag{1A.13b}
\]

From Eq. (1A.13b),

\[
S_{21} = -(S^{-1})_{21}^{-1}(S^{-1})_{22}S_{11}
\]
Substituting in Eq. \((1A.13a)\),
\[
S_{11} = ((S^{-1})_{11} - (S^{-1})_{12}(S^{-1})_{22})^{-1}(S^{-1})_{21}\]
whence
\[
|S_{11}| = |(S^{-1})_{11} - (S^{-1})_{12}(S^{-1})_{22}|^{-1}(S^{-1})_{21}\]
(1A.14)

Combining Eqs. \((1A.12)\) and \((1A.14)\), we obtain Eq. \((1A.11)\). Q.E.D.

**Theorem 1A.2**: If \(S^{-1}\) has the sign pattern \((1A.8a)\), the inequalities \((1A.10)\) are satisfied.

**Proof**: Suppose that \(S^{-1}\) has the required sign pattern and is therefore an irreducible Minkowski matrix. Then every principal submatrix of \(S^{-1}\) is an irreducible Minkowski matrix. Hence \(S\) and the inverse of every principal submatrix of \(S^{-1}\), is positive. From Jacobi's theorem on determinants, for every determinant \(\Delta\)
\[
\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21} = \Delta_{11,22}\Delta
\]
Hence
\[
\Delta_{11}(-1)^{1+2}\Delta_{22} - (-1)^{1+2}\Delta_{12}\Delta_{21} = (-1)^{1+2}\Delta_{11,22}\Delta_{11}\Delta
\]
(1A.15)

Applying Eq. \((1A.15)\) to \(S^{-1}\), and noting that \((-1)^{1+2}\Delta_{11,22}\Delta_{11}\) is an element of the inverse of an \((n-1)\)-dimensional principal submatrix of \(S^{-1}\) and therefore positive, we conclude that \(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} > 0\). The same argument may be applied to each principal sub-matrix of \(S^{-1}\). Q.E.D.

**Theorem 1A.3**: If \(S^{-1}\) has the sign pattern \((1A.8a)\), \(\alpha_{ii} > \alpha_{ij} \) \(i \neq j\).

**Proof**: From Theorem 1A.2, the inequalities \((1A.10)\) are satisfied. Hence
\[
\alpha_{ii}\alpha_{jj} - \alpha_{ij}\alpha_{ji} > 0 \quad s \neq j \quad i \neq j
\]
Summing over \(s \neq i\),
\[
\alpha_{ii} \sum_{s \neq i} \alpha_{jj} - \alpha_{ij} \sum_{s \neq i} \alpha_{ji} > 0
\]
That is, since column sums are one,
\[
\alpha_{ii}(1 - \alpha_{jj}) - \alpha_{ij}(1 - \alpha_{ji}) > 0
\]
Hence \(\alpha_{ii} > \alpha_{jj}\). Q.E.D.
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if all denominators are positive, we may write

\[
\frac{\lambda_1 a_{12}}{\lambda_2 a_{12}} < \frac{\lambda_1 a_{14} + \lambda_2 a_{14}}{\lambda_2 a_{14}}
\]

\[
= \frac{\lambda_1 a_{12}}{\lambda_1 a_{12} + \lambda_2 a_{12}}
\]

\[
= \frac{\lambda_1 a_{12}}{\lambda_1 a_{12} + \lambda_2 a_{12}}
\]

[from Eq. (1A.17)]

in contradiction of (1A.10). Suppose, alternatively, that \((\lambda_1 a_{12} + \lambda_2 a_{12})\) is nonpositive or, from Eq. (1A.17), that \(-(\lambda_1 a_{12} + \lambda_2 a_{12})\) is nonpositive. From Eq. (1A.17) we have also

\[
\begin{pmatrix}
\alpha_{13} & \alpha_{14} \\
\alpha_{23} & \alpha_{24}
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix}
= \begin{pmatrix}
-\lambda_1 a_{23} - \lambda_2 a_{23} \\
-\lambda_1 a_{24} - \lambda_2 a_{24}
\end{pmatrix}
\]

so that

\[
\begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix}
= \begin{pmatrix}
\alpha_{13} & \alpha_{14} \\
\alpha_{23} & \alpha_{24}
\end{pmatrix}^{-1}
\begin{pmatrix}
\alpha_{13} & \alpha_{14} \\
\alpha_{23} & \alpha_{24}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix}
= \begin{pmatrix}
\lambda_1 a_{23} & \lambda_2 a_{23} \\
\lambda_1 a_{24} & \lambda_2 a_{24}
\end{pmatrix}
\]

By assumption, \(\lambda_1 > 0\) and \(-\lambda_1 a_{23} - \lambda_2 a_{23} \leq 0\); hence

\[
\lambda_1 a_{23} + \lambda_2 a_{23} < 0
\]

(1A.20)

Now, from (1A.10),

\[
\lambda_1 a_{23} + \lambda_2 a_{23} \leq 0
\]

Hence

\[
\begin{pmatrix}
\lambda_1 a_{23} & \lambda_2 a_{23} \\
\lambda_1 a_{24} & \lambda_2 a_{24}
\end{pmatrix}
\]

Hence

\[
\begin{pmatrix}
\lambda_1 a_{23} & \lambda_2 a_{23} \\
\lambda_1 a_{24} & \lambda_2 a_{24}
\end{pmatrix}
\]

from Eqs (1A.19) and (1A.20)

\[
\begin{pmatrix}
\lambda_1 a_{23} & \lambda_2 a_{23} \\
\lambda_1 a_{24} & \lambda_2 a_{24}
\end{pmatrix}
\]

[from Eq. (1A.17)]

\[
\begin{pmatrix}
\lambda_1 a_{23} & \lambda_2 a_{23} \\
\lambda_1 a_{24} & \lambda_2 a_{24}
\end{pmatrix}
\]

in contradiction of (1A.10). We conclude that \(|S|\) cannot be zero.

To complete the proof that \(|S| > 0\), we define the matrix

\[
S(t) = \begin{pmatrix}
\alpha_{13} & \alpha_{14} & \alpha_{23} & \alpha_{24} \\
\alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\
\alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\epsilon & -\epsilon \\
-d & d - \delta
\end{pmatrix}
\geq
\begin{pmatrix}
\epsilon & -\epsilon \\
-d & d - \delta
\end{pmatrix}
\]

(1A.19)
where

\[ s_x = \left( 1 - t \sum_{i=1}^{n} \alpha_{ij} \right) / \alpha_{ii} \]

Evidently \(|S(t)|\) is a continuous function of \(t\), with \(|S(0)| > 0\). Moreover, for \(0 < t \leq 1\), \(S(t)\) is a stochastic matrix satisfying the inequalities (1A.10), so that \(|S(1)| \neq 0\). It follows that \(|S| = |S(1)| > 0\). Q.E.D.

\textbf{Proof of (b):} It suffices to show that the first diagonal element is greater than one. Let the \((i, j)\)th cofactor of \(S\) be represented by \(S_{ij}\). Then

\[ |S| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_{a1} & \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \\ \alpha_{a1} & \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \\ \alpha_{a1} & \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \end{vmatrix} [\text{Add all other rows to the first and recall that} \ S \text{ is a stochastic matrix.}]
\]

\[ = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \alpha_{a1} & \alpha_{a2} - \alpha_{a1} & \alpha_{a3} - \alpha_{a1} & \alpha_{a4} - \alpha_{a1} \\ \alpha_{a1} & \alpha_{a2} - \alpha_{a1} & \alpha_{a3} - \alpha_{a1} & \alpha_{a4} - \alpha_{a1} \\ \alpha_{a1} & \alpha_{a2} - \alpha_{a1} & \alpha_{a3} - \alpha_{a1} & \alpha_{a4} - \alpha_{a1} \end{vmatrix} [\text{Subtract the first column from each of the remaining columns.}]
\]

\[ = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \\ \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \\ \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \end{vmatrix}
\]

\[ - \alpha_{a1} \begin{vmatrix} 1 & 1 & 1 \\ \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \\ \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \end{vmatrix}
\]

\[ = S_{11} - \alpha_{a2} \alpha_{a3} - \alpha_{a3} \alpha_{a4} - \alpha_{a4} \alpha_{a1}, \text{say.}
\]

Now the first diagonal element of \(S^{-1}\) is \(S_{11} / |S|\). Evidently this element is greater than one if and only if \((\alpha_{a1} \alpha_{a2} + \alpha_{a1} \alpha_{a3} + \alpha_{a1} \alpha_{a4})\) is positive. Adding the first two terms, we obtain

\[ \alpha_{a2} \alpha_{a3} + \alpha_{a3} \alpha_{a4} = (\alpha_{a1} \alpha_{a2} - \alpha_{a1} \alpha_{a3} + \alpha_{a1} \alpha_{a4} - \alpha_{a2} \alpha_{a4})(\alpha_{a4} - \alpha_{a3}) + (\alpha_{a1} \alpha_{a3} - \alpha_{a1} \alpha_{a4} + \alpha_{a1} \alpha_{a2})(\alpha_{a4} - \alpha_{a3}) \]

(1A.21)

The first of the four bracketed expressions may be written

\[ \xi = \alpha_{a1} \alpha_{a2} (1 - \mu_2) - \alpha_{a1} \alpha_{a3} (1 - \mu_3) \]

where \(\mu_2 = \alpha_{a2}/\alpha_{a3}\) and \(\mu_3 = \alpha_{a3}/\alpha_{a2}\). Now the inequalities (1A.10) imply that the diagonal element is the largest element in each row (see the proof of Theorem 1A.3); hence \(\mu_2 < 1\). Moreover, it follows from (1A.10) that \(\mu_3 < \mu_3\). Evidently \(\xi\) is positive if \(\mu_2 \geq 1\); but from (1A.10) and the fact that \((1 - \mu_3) < (1 - \mu_3)\), it is positive also if \(\mu_3 < 1\). The third bracketed component of (1A.21) has a structure similar to that of the first; hence it too is positive. The second and fourth bracketed expressions are already known to be positive. Hence \((\alpha_{a1} \alpha_{a2} + \alpha_{a1} \alpha_{a4})\) is positive. By similar reasoning, \((\alpha_{a1} \alpha_{a2} + \alpha_{a1} \alpha_{a4})\) and \((\alpha_{a1} \alpha_{a2} + \alpha_{a1} \alpha_{a4})\) are positive. It follows by addition that \((\alpha_{a1} \alpha_{a2} + \alpha_{a1} \alpha_{a4} + \alpha_{a1} \alpha_{a4} + \alpha_{a1} \alpha_{a4})\) is positive. Q.E.D.

\textbf{Proof of (e):} It suffices to show that, if the \((i, 2)\)th element of \(S^{-1}\) is positive, the \((1, 3)\)th and the \((1, 4)\)th elements must be negative. In view of (a) above, this amounts to showing that, if \(A_{31}\) is positive, both \(A_{32}\) and \(A_{34}\) must be negative, where \(A_{ij}\) is the cofactor of the \((i, j)\)th element of \(S\). We first prove the following lemma.

\textbf{Lemma:} \(\alpha_{xj} > \alpha_{xj} / \alpha_{x0}\) implies \(A_{ii} < 0\) \(i \neq j \neq r \neq s\).

\textbf{Proof:} Without loss of generality, we may concentrate on the case \(i = 1, j = 2\). Suppose first that \(\alpha_{a1} > \alpha_{a1} / \alpha_{a4}\). We seek to show that \(A_{31}\) is negative. Dividing each column of \(A_{31}\) by its first element, the second row by \(\alpha_{a1}/\alpha_{a4}\) and the third row by \(\alpha_{a1}/\alpha_{a4}\), we obtain

\[ A_{31} = -\beta \begin{vmatrix} 1 & 1 & 1 \\ \alpha_{a2} \alpha_{a1} & \alpha_{a2} \alpha_{a4} & \alpha_{a2} \alpha_{a4} \\ \alpha_{a2} \alpha_{a1} & \alpha_{a2} \alpha_{a4} & \alpha_{a2} \alpha_{a4} \end{vmatrix}
\]

\[ \beta = \alpha_{a2} \alpha_{a1} / \alpha_{a2} > 0\). From the inequalities (1A.10), \(\alpha_{a1} \alpha_{a2} / \alpha_{a2} \alpha_{a4}\) is greater than one and \((\alpha_{a2} \alpha_{a1} - \alpha_{a2} \alpha_{a4})\) positive. Hence

\[ A_{31} = -\beta \begin{vmatrix} 1 & 1 & 1 \\ \alpha_{a2} \alpha_{a1} & \alpha_{a2} \alpha_{a4} & \alpha_{a2} \alpha_{a4} \\ \alpha_{a2} \alpha_{a1} & \alpha_{a2} \alpha_{a4} & \alpha_{a2} \alpha_{a4} \end{vmatrix}
\]

\[ = -\beta A_{31}, \text{ say.}\]
Suppose that $\alpha_{23}\alpha_{34} < \alpha_{12}\alpha_{43}$. Then, since $\alpha_{33}\alpha_{50}/\alpha_{13}\alpha_{34} > 1$ [from (1A.10)],

$$A_{21} = \begin{vmatrix} 1 & 1 & 1 \\ * & * & * \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence

$$A_{21} < 0 \text{ if } \alpha_{23}\alpha_{34} < \alpha_{12}\alpha_{43} \quad (1A.22a)$$

Applying the same argument to the transpose of $A_{21}$ we find that

$$A_{21} < 0 \text{ if } \alpha_{12}\alpha_{43} > \alpha_{14}\alpha_{23} \quad (1A.22b)$$

That establishes the Lemma.

We return to the proof of (c). Suppose $A_{21} > 0$. Then, in view of the Lemma,

$$\alpha_{13} > \alpha_{12}\alpha_{43}/\alpha_{42} \quad \text{and} \quad \alpha_{44} > \alpha_{12}\alpha_{43}/\alpha_{32}$$

Applying the Lemma to these inequalities, we conclude that $A_{21}$ and $A_{41}$ must be negative. Q.E.D.

Proof of (d): In view of the above numerical example, it suffices to show that the sum of a diagonal and an off-diagonal element in the same row of $S^{-1}$ must be positive. In particular, it suffices to show that

$$A_{11} + A_{21} = \begin{vmatrix} \alpha_{23} & \alpha_{23} & \alpha_{34} \\ \alpha_{32} & \alpha_{32} & \alpha_{34} \\ \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix} - \begin{vmatrix} \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_{23} - \alpha_{12} & \alpha_{23} - \alpha_{13} & \alpha_{34} - \alpha_{14} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

$$\begin{vmatrix} \alpha_{23} - \alpha_{12} & \alpha_{23} - \alpha_{13} & \alpha_{34} - \alpha_{14} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

is positive. Dividing the second row by $\alpha_{22}$, the third row by $\alpha_{42}$, the second column by $\alpha_{32}/\alpha_{42}$, and the third column by $\alpha_{34}/\alpha_{42}$, we obtain

$$A_{11} + A_{21} = \alpha_{33}\alpha_{43}$$

$$\begin{vmatrix} \alpha_{23} - \alpha_{12} & \alpha_{23} - \alpha_{13} & \alpha_{34} - \alpha_{14} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

$$(1A.23)$$

Perhaps the most interesting feature of the original (two-by-two) Stokey-Samuelson Theorem is the element of conflict it establishes between the real factor rewards of the two factors: Given the conditions (1A.5) relating to relative factor intensities, any change in relative commodity prices works against one and in favor of the other. From Theorems 1A.4 and 1A.5(d) we now see that, if a generalized version of (1A.5) is satisfied, the element of conflict is preserved when $n \leq 4$. When $n = 4$ not all factor rewards need unambiguously increase or decrease; we can be sure, however, that at least one will increase and at least two decrease.

The Dual Samuelson-Ryczewski Theorem

Already, in Eq. (1.24) of the text, we have noted the duality of the factor reward-commodity price relationship and the commodity output-factor endowment relationship. It follows from Eq. (1.24) that Theorems (1A.1) - (1A.5), which are overtly concerned with the factor reward-commodity price relationship $\partial w_i/\partial p_k$, apply equally to the commodity output-factor endowment relationship $\partial X_{kj}/\partial V_i$. 

The diagonal elements in (1A.23) are all positive and, from the inequalities (1A.10), the last two diagonal elements are greater than one. If $\alpha_{22} \leq \alpha_{12}$, the cofactor of the (2, 2)th element in (1A.23) is clearly positive. If, alternatively, $\alpha_{23} > \alpha_{14}$, the same is true; for, from (1A.19) and (1A.10),

$$\begin{vmatrix} \alpha_{23} & \alpha_{23} & \alpha_{34} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

It follows that

$$\begin{vmatrix} \alpha_{22} - \alpha_{12} & \alpha_{23} & \alpha_{34} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$(1A.24)$$

If $\alpha_{22} \leq \alpha_{12}$, the coefficient of the (3, 3)th element in (1A.24) is clearly positive. If, alternatively, $\alpha_{23} > \alpha_{14}$, the same is true; for from (1A.19) and (1A.10)

$$\begin{vmatrix} \alpha_{23} & \alpha_{23} & \alpha_{34} \\ \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \end{vmatrix}$$

It follows finally that

$$A_{11} + A_{21} = \alpha_{33}\alpha_{43}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Q.E.D.
A Simple Closed Economy

A3. UNEQUAL NUMBERS OF FACTORS AND PRODUCTS

In Sec. A2 we have examined the possibility of extending the Stolper-Samuelson and Samuelson-Rybczynski Theorems to cover situations with arbitrary but equal numbers of factors and products. We now consider the implications of relaxing the requirement that factors and products be equal in number.

More Products than Factors

We consider first a world of just three products and two factors. This is the simplest context in which the problems to be discussed can be adequately formulated. It must be supposed that, initially, commodity prices and factor rewards are consistent with the production of all three goods; otherwise we are back to the two-by-two case, or worse. Of course, it is not necessary that all three commodities be produced in positive amounts; in the case under consideration there is an inevitable and unavoidable element of indeterminacy in production. But it is necessary that, within limits to be described presently, it be a matter of indifference to producers whether they produce all three goods, or just two, or (a singular possibility) just one.

In the world we have just described it is obviously impossible to associate one factor with each product. The most we can contrive is to restrict the matrix \( S \) so that each factor is associated with a product. Thus, after a suitable renumbering of products, the inequalities (1A.10) must be specialized to

\[
\begin{align*}
\alpha_{is} & \geq \alpha_{jt} \\
\alpha_{is} & \geq \alpha_{js} 
\end{align*}
\]

where \( s = 1, 2, 3 \) and \( i \neq j \) or \( i \neq s \).

Figure 1A.1 illustrates an initial equilibrium. The three one-dollar iso-revenue curves possess a common tangent line on the slope of which represents the equilibrium ratio of factor rewards. The ray \( OP_i \) indicates the profit-maximizing ratio of factors in the \( i \)th industry and will be referred to as the \( i \)th "factor ray." The relative slopes of the three factor rays in Fig. 1A.1 are implied by (1A.27). Provided the ratio of factor endowments can be represented by a ray (the "endowment ray") of gentler slope than that of \( OP_s \) and steeper slope than that of \( OP_i \), something of all three commodities can be produced, though the actual production mix will be indeterminate. If the endowment ray lies between the first and third factor rays, something of all three commodities may be produced, or something of just the first and second, or something of just the first and third. If the endowment ray lies between the second and third factor rays, something of all three goods may

\[ i \neq j, \]
Suppose next that the price of the third product changes. (It is unnecessary to separately consider a change in the second price. It is clear from Fig. 1A.1 that the implications of such a change are symmetrical with those of a change in the first price, and they may in fact be obtained from the preceding paragraph by simply interchanging "first" and "second.") If the price falls, production of the third commodity ceases, relative factor rewards remain unchanged and both factors benefit in terms of their increased purchasing power over the third commodity. This is so, whatever the endowment ratio.

If, however, the price rises, the range of possible outcomes is wider. Thus if the endowment ray lies in the interior of the cone $P_1OP_3$, or in the interior of the cone $P_2OP_3$, the real reward of one factor must rise and that of the other factor fall. But if the endowment ray coincides with the common edge $OP_3$ of the two cones, the first and second industries both prove to be unprofitable in the new equilibrium, relative factor rewards remain unchanged, and the real rewards of both factors increase in terms of the first and second commodities.

We dwell briefly on a second special case before attempting to generalize. Suppose that at initial prices and factor rewards four commodities could be produced and that each industry requires positive amounts of each of three factors of production. The appropriate specialization of the inequalities...
(1A.10) is now

$$\frac{\alpha_{ei}}{\alpha_{e}} > \frac{\alpha_{sj}}{\alpha_{s}} \quad i, j = 1, 2, 3 \quad i \neq j \quad s = 1, 2, 3, 4 \quad s \neq t \quad (1A.28)$$

In an initial equilibrium each of four one-dollar isorevenue surfaces is tangential to a plane the partial slopes of which indicate ratios of factor rewards. In Fig. 1A.2, which depicts a special case, the four points of tangency are marked by $P_1, P_2, P_3, P_4$. ($P'_1, \ldots, P''_4$ mark alternative positions for the fourth tangency.) In accordance with the assumption that positive amounts of every commodity can be produced, the factor endowment ray must be supposed to lie in the interior of the finite cone generated by the four factor rays $OP_1, \ldots, OP_4$. In accordance with conditions (1A.28) each of the first three factor rays is an edge of that cone; the fourth factor ray may or may not be an edge.

Suppose that the equilibrium is disturbed by a slight fall in the price of the first commodity, so that the one-dollar isorevenue surface of that commodity shifts out from the origin. What happens to outputs and factor rewards depends on the precise orientation of the endowment ray. If $P_1, P_2,$ and $P_4$ are not collinear and if the endowment ray lies in the cone formed by the second, third, and fourth factor rays, the first industry must prove to be unprofitable in the new equilibrium. Relative factor rewards will remain unchanged and all factors will benefit in terms of their command over the first commodity. If, on the other hand, the endowment ray lies outside that cone (as it must if $P_1, P_2,$ and $P_4$ are collinear), the first commodity must be produced in the new equilibrium. Either it continues to be possible to produce all four commodities (as when $P_2, P_3,$ and $P_4$ are collinear), or one of the other three commodities proves to be unprofitable, or two of the other three prove to be unprofitable (as when the endowment ray cuts the line $P_1P_4$ in Fig. 1A.2). If it remains possible to produce three (or even four) goods, we again find ourselves governed by the results of Sec. A2; in particular, it follows from (1A.28) and Theorem 1A.3 that if the fourth industry proves unprofitable the first factor will suffer and the others benefit from the price change. Otherwise, we are in a situation with more factors than products. Suppose, alternatively, that the price of the first commodity rises slightly. Then the first isorevenue surface shifts towards the origin and, whatever the position of the endowment ray, something of the first good must be produced in the new equilibrium. Either it continues to be possible to produce all four goods (as when $P_1, P_2,$ and $P_4$ are collinear), or one of the other three goods proves to be unprofitable, or two of the other three prove to be unprofitable (as when $OP_3$ is the fourth factor ray and the endowment ray cuts the line $P_2P_4$ south of the line $P_1P_3$). In any but the last of these cases we find ourselves effectively in a three-by-three world and subject to the conclusions of Sec. A2.\(^{131}\)

The implications of a change in the price either of the second or of the third product are symmetrical with those of a change in the first product and may be obtained from the preceding paragraph by interchanging "first" and "second" or "third." Nor, if the fourth factor ray is an edge of the cone generated by the four factor rays, does a change in the fourth price furnish any new possibilities. A change in the fourth price is worthy of special attention only if the fourth factor ray lies in or on the cone generated by the first three factor rays alone (as in Fig. 1A.2). Suppose that this is the case and that the price of the fourth product declines. Then the fourth isorevenue surface shifts out from the origin, the fourth industry proves to be unprofitable in the new equilibrium, relative factor rewards remain unchanged, and all real rewards increase in terms of the fourth commodity. If, however, the price rises, the possibilities are more numerous. The fourth isorevenue surface shifts towards the origin, the fourth commodity is necessarily producible in the new equilibrium, and at least one of the first three industries must reveal itself to be unprofitable. In fact two of the three may turn out to be unprofitable (as when $P_4$ is in the interior of the triangle $P_1P_2P_3$ and the endowment ray cuts one of the lines $P_1P_3, P_2P_3, P_1P_4, P_2P_4$, or even all three (as when the endowment ray passes through $P_4$).

\(^{131}\) If the endowment ray lies in or on the cone generated by the second third and fourth factor rays, it is possible for the real reward of the first factor to increase, whatever the direction of change of $p_i$. Cf. footnote 22.
There are elements of asymmetry in the above conclusions. There is, however, a common thread running through them. Moreover, the same common features can be discerned in the general case of \( n \) products and \( r \) factors \((n > r)\). If the factor endowment is such that \( r \) commodities (including the \( i \)th) must be produced in positive amounts to ensure full employment, any change in the \( i \)th price will render unprofitable not more than \( n - r \) industries (exactly \( n - r \) if no \( r \) initial production points lie in an \((r - 1)\)-hyperplane) and excluding the \( i \)th. To any \( r \) of those commodities (including the \( i \)th) which remain profitable after the price change, the results of Sec. A2 remain applicable. Otherwise, either all real rewards rise together, or the results of Sec. A2 apply, or fewer goods are produced in the new equilibrium than there are factors, depending on the orientations of the \( n \) factor rays and of the endowment ray. (Figures 1A.1 and 1A.2 provide illustrations.) The \( i \)th product must be produced if and only if the \( i \)th factor ray provides an edge both for every \( r \)-edged cone containing the endowment ray and for the cone generated by the \( n \) initial factor rays. The inequalities (1A.10), or an appropriate primed specialization of them, ensures that the \( i \)th factor ray is an edge of the cone generated by the \( n \) initial factor rays.\( ^{(28)} \)

All of the preceding discussion has rested on the supposition that price changes are autonomous and arbitrary. It is possible, however, to take a broader point of view and to treat the country under consideration as a small number of a trading community. From this point of view one can say a little more about the distributional implications of changes in commodity prices, for then the vector of price changes must be treated as endogeneous and its elements as related. Thus suppose for the time being that the endowment rays of the several countries coincide that the countries have a common technology. Then if all commodities are produced it will in each country be not unprofitable to produce all of them. (It is immaterial if in fact a country produces less than the entire range.) Moreover, unless world demand functions are quite odd, small price changes must be consistent with the continued world production of positive amounts of all goods. But if price changes are so restricted\( ^{(28)} \) we are back in a simple and familiar world to which Theorems 1A.4 and 1A.5 apply. Figure 1A.3 illustrates for the three-by-two case.

To achieve such simplicity it is not necessary to assume that the two endowment rays are identical. It is enough that any cone formed by \( r \) initial factor rays which contains one country's endowment ray also contains the other countries' endowment rays. If this condition is not met a much wider range of price changes is consistent with positive world production of all commodities, for it is no longer necessary that each country should find it possible to produce all goods.

### More Factors than Products

If there are three factors and two products (the Classical case\( ^{(27)} \)) one cannot determine factor rewards from the marginal conditions alone. To complete the system one must add equations requiring that all factors be fully employed. Unfortunately these additional conditions contribute complicated (factor) substitution terms to our derived relations between product prices and factor rewards. To obtain clear-cut results it becomes necessary to impose restrictions on the matrix of substitution terms, a possibility we do not explore here.

\( ^{(27)} \) The allusion is to the classical discussion of the distributional implications of the relaxation of the Corn Laws. Three factors were recognized (labor, land, and capital) and two groups of products (agricultural products, represented by corn, and manufactured goods).