SPRING 2017

Bob:

Ann:

(a) \( x^7 = \begin{pmatrix}
    1 \\
    1 \\
    1 \\
    0 \\
    1 \\
    1 \\
    ...
\end{pmatrix} \)

(x) Ann sends first message

Bob receives first message

Bob sends acknowledgment of first message

Ann receives Bob’s acknowledgment

Ann sends acknowledgment of Bob’s acknowledgment

Bob receives Ann’s acknowledgment

Bob does not send an acknowledgment of Ann’s acknowledgment

no more messages sent or received

(b) Ann has control over whether or not she sends an e-mail, which is encoded in components \( x_1, x_2, x_3, \text{ etc.} \) that is, \( x_{4n-3} \) for \( n \geq 1 \) (same as \( x_{1+4n} \) for \( n \geq 0 \)). Thus she knows the value of these. She also knows whether or not she receives an e-mail from Bob, which is encoded in components \( x_4, x_8, \text{ etc.} \), that is, \( x_4 \) for \( n \geq 1 \). Thus two states \( x \) and \( y \) are in the same cell of Ann’s partition if and only if, for all \( n \geq 1 \), \( x_{4n-3} = y_{4n-3} \) and \( x_4 = y_4 \).

Bob has control over whether or not he sends an e-mail, which is encoded in components \( x_5, x_7, x_{11}, \text{ etc.} \) that is, \( x_{4n-1} \) for \( n \geq 1 \). Thus he knows the value of these. He also knows whether or not he receives an e-mail from Ann, which is encoded in components \( x_2, x_6, \text{ etc.} \), that is, \( x_{4n-2} \) for \( n \geq 1 \). Thus two states \( x \) and \( y \) are in the same cell of Bob’s partition if and only if, for all \( n \geq 1 \), \( x_{4n-1} = y_{4n-1} \) and \( x_4 = y_4 \).

Thus the partitions are as follows:

Ann:

\[
\begin{array}{cccccccccccccccccccc}
\end{array}
\]

Bob:

\[
\begin{array}{cccccccccccccccccccc}
\end{array}
\]

Thus Ann’s partition is the collection of the following sets: for all \( n \geq 0 \), \( \{ x_{4n} \} \) and for all \( n \geq 1 \), \( \{ x_{4n-2}, x_{4n-1}, x_{4n} \} \), while Bob’s partition is the collection of the following sets: \( \{ x_1 \} \) and, for all \( n \geq 1 \), \( \{ x_{4n-1} \} \) and \( \{ x_{4n}, x_{4n+1}, x_{4n+2} \} \).

(c) Let \( E = \{ x^4, x^5, \ldots, x^{12} \} \). Then \( K_A E = \{ x^5, x^6, \ldots, x^{12} \} \), \( K_B E = \{ x^4, x^5, \ldots, x^{11} \} \), \( K_A K_B E = \{ x^5, x^6, \ldots, x^9 \} \), \( K_B K_A E = \{ x^7, x^8, \ldots, x^{11} \} \), \( K_A K_B K_A E = \{ x^9 \} \) and \( K_B K_A K_B E = \{ x^7 \} \). Thus \( E \) satisfies all of the required conditions. The interpretation of \( E \) is as follows: Ann sent at most the original message and two acknowledgments, Bob sent at most three acknowledgments, the last of which, if sent, got lost.
(d) The common knowledge partition is the trivial partition consisting of the entire set of states. Thus the only event which is common knowledge at every state (including state $x^0$) is the trivial event consisting of all the states.

(e) Since state $x^0$, where Ann does not send the information to Bob, belongs to the only event which is common knowledge at any state, namely the set of all states, common knowledge of the information contained in Ann’s first e-mail cannot be achieved, no matter how many e-mails are sent and acknowledged.

(f.1) Ann’s partition is: $0 1 2 3 4 5 6 7 8 9 10 11 12 \ldots$

   Bob’s partition is: $0 1 2 3 4 5 6 7 8 9 10 11 \ldots$

(f.2) Once again, the common knowledge partition is $\{N\}$ where $N$ is the set of all states (the set of natural numbers).

(f.3) The set of all states (and this is true not only at state 1024 but at every state).

(f.4)

<table>
<thead>
<tr>
<th>Total messages sent</th>
<th>Messages received by Bob</th>
<th>Messages received by Ann</th>
<th>Ex ante probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$(1-p)$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$p\varepsilon$</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>$p(1-\varepsilon)\varepsilon$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>$p(1-\varepsilon)^2\varepsilon$</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>$p(1-\varepsilon)^4\varepsilon$</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
<tr>
<td>$2n$</td>
<td>$n$</td>
<td>$n-1$</td>
<td>$p(1-\varepsilon)^{2n-1}\varepsilon$</td>
</tr>
<tr>
<td>$2n+1$</td>
<td>$n$</td>
<td>$n$</td>
<td>$p(1-\varepsilon)^{2n}\varepsilon$</td>
</tr>
<tr>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

(f.5) First Ann. If $n \geq 2$ is even, then Ann’s cell is $\{n-1, n\}$. The ex ante probabilities of these two states are $p(1-\varepsilon)^{n-2}\varepsilon$ and $p(1-\varepsilon)^{n-1}\varepsilon$, respectively. Thus using Bayes’ rule the probabilities obtained conditioning on $\{n-1, n\}$ are $\frac{1}{2-\varepsilon}$ and $\frac{1-\varepsilon}{2-\varepsilon}$, respectively. If $n \geq 2$ is odd, then Ann’s cell is $\{n, n+1\}$. The ex ante probabilities of these two states are $p(1-\varepsilon)^{n-1}\varepsilon$ and $p(1-\varepsilon)^n\varepsilon$, respectively. Thus using Bayes’ rule the probabilities obtained conditioning on $\{n, n+1\}$ are again $\frac{1}{2-\varepsilon}$ and $\frac{1-\varepsilon}{2-\varepsilon}$, respectively.

Now Bob. If $n \geq 2$ is even, then Bob’s cell is $\{n, n+1\}$. The ex ante probabilities of these two states are $p(1-\varepsilon)^{n-2}\varepsilon$ and $p(1-\varepsilon)^n\varepsilon$, respectively. Thus using Bayes’ rule the probabilities obtained conditioning on $\{n, n+1\}$ are $\frac{1}{2-\varepsilon}$ and $\frac{1-\varepsilon}{2-\varepsilon}$, respectively. If $n \geq 2$ is odd, then Bob’s cell is $\{n-1, n\}$. The ex ante probabilities of these two states are $p(1-\varepsilon)^{n-2}\varepsilon$ and $p(1-\varepsilon)^{n-1}\varepsilon$, respectively. Thus using Bayes’ rule the probabilities obtained conditioning on $\{n-1, n\}$ are again $\frac{1}{2-\varepsilon}$ and $\frac{1-\varepsilon}{2-\varepsilon}$, respectively.