1. [52 points] A travel agent is auctioning two one-week vacations for the first week of July. The first (call it prize A) is a vacation in Hawaii and the second (call it prize B) a vacation in Texas. There are three bidders, imaginatively called 1, 2 and 3. All bidders consider prize A more desirable than prize B and none of the bidders is interested in winning both prizes (they find it hard to be in two places at the same time). Let \( a_i \) be the value of prize A to bidder \( i \) and \( b_i \) the value of prize B to bidder \( i \). Thus \( a_i > b_i \) for all \( i \in \{1,2,3\} \). Having read about the good properties of Vickrey’s second-price auction, the travel agent has decided to organize the following auction: a sealed bid auction in which each bidder submits a non-negative price \( p_i \); the highest bidder wins prize A and pays the second-highest price; the second-highest bidder wins prize B and pays the lowest price; in case of ties, the player with the lower index prevails over the bidder(s) who have submitted the same bid as she did. For example, if \( p_1 = $10, p_2 = $12, p_3 = $10 \) then bidder 2 wins prize A and pays $10, bidder 1 wins prize B and pays $10 and bidder 3 pays nothing and wins nothing. [Note that the second-highest price is the higher of the two prices remaining after removing the bid of the highest bidder: for example if the bids are 15, 15 and 8 then the highest bid is 15, the second highest is also 15 and the lowest is 8; if all the players bid $23 then the highest, second highest and lowest are all the same and equal to 23.] Each player is selfish and greedy (that is, each player cares only about his/her own net gain).

(a) [10 points] For the case where \( a_1 = 100, b_1 = 70, a_2 = 90, b_2 = 60, a_3 = 75, b_3 = 40 \) and the possible bids are $100 and $50, write the strategic-form of the game.

From now on assume that bids can be any non-negative numbers and that all you know about the \( a_i \)s and \( b_i \)s is that \( a_i > b_i > 0 \) for all \( i \in \{1,2,3\} \).

(b) [8 points] Write the payoff function of Player 1.

(c) [10 points] Does Player 1 have a dominant strategy? Prove your claim.

(d) [2 points] Is there a dominant-strategy equilibrium?

(e) [10 points] Assume that \( a_1 > a_2 > a_3 \). What further restrictions on the parameters \( a_i \) and \( b_i \) (\( i \in \{1,2,3\} \)) are necessary and sufficient for \((a_1,a_2,a_3)\) to be a Nash equilibrium? Prove your claim (prove both sufficiency and necessity).

(f) [6 points] Suppose that \( a_1 = 180, b_1 = 95, a_2 = 120, b_2 = 88, a_3 = 80, b_3 = 68 \). Is \((180,120,80)\) a Nash equilibrium?

(g) [6 points] Suppose that \( a_1 = 180, b_1 = 95, a_2 = 120, b_2 = 78, a_3 = 80, b_3 = 68 \). Is \((180,120,80)\) a Nash equilibrium?
2. [36 points]. Consider the following game, where the payoffs are von Neumann-Morgenstern payoffs:

(a) [10 points] Find the best reply function of Player 1 (against every possible mixed strategy of Player 2).

(b) [8 points] Explain why there are no pure-strategy weak sequential equilibria.

(c) [10 points] Find a weak sequential equilibrium. For the strategy profile $\sigma$ that you suggest find all the assessments that contain $\sigma$ and are weak sequential equilibria.

(d) [8 points] Find a sequential equilibrium (one is enough; prove your claim).

3. [12 points] Consider the following game:

(a) [3 points] Are there values of $x$ for which Player 3 has a strictly dominant strategy? Explain your answer.

(b) [2 points] Does Player 1 have a weakly dominated strategy? (If your answer is Yes, name the strategy; if your answer is No prove your claim.)

(c) [4 points] What strategy profiles are Nash equilibria irrespective of the value of $x$?

(d) [3 points] Find all the backward induction solutions for the cases where $x = 0$ and $x = 10$. 