1. (a) The strategic form is as follows:

Player 2

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</thead>
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<td>50</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>40</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Player 3 bids

Player 2

<table>
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<th>50</th>
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</tr>
</thead>
<tbody>
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<td>0</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>25</td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>−10</td>
</tr>
</tbody>
</table>

Player 3 bids

(b) The payoff function of player 1 is as follows:

\[
\pi_1 = \begin{cases} 
    a_i - \max \{p_2, p_3\} & \text{if } p_1 \geq \max \{p_2, p_3\} \\
    b_1 - p_3 & \text{if } p_1 < p_2 \text{ and } p_1 \geq p_3 \\
    b_1 - p_2 & \text{if } p_1 < p_3 \text{ and } p_1 \geq p_2 \\
    0 & \text{if } p_1 < p_2 \text{ and } p_1 < p_3 
\end{cases}
\]

(c) Player 1 does not have a dominant strategy.

Proof: suppose that bidding \( \hat{p} \) were a dominant strategy for player 1. It cannot be \( \hat{p} \geq a_1 \) because in the case where \( p_2 = \hat{p} \) and \( p_3 = 0 \) we have that \( \pi_i(\hat{p}, \hat{p}, 0) = a_1 - \hat{p} \leq 0 \) while \( \pi_i(b_1, \hat{p}, 0) = b_1 > 0 \). It cannot be \( b_1 \leq \hat{p} < a_1 \) because in the case where \( p_2 = \hat{p} + \epsilon < a_1 \) and \( p_3 = b_1 \) (where \( \epsilon > 0 \)) we have that \( \pi_i(\hat{p}, \hat{p} + \epsilon, b_1) = b_1 - b_1 = 0 \) while \( \pi_i(a_1, \hat{p} + \epsilon, b_1) = a_1 - (\hat{p} + \epsilon) > 0 \). Finally, it cannot be \( \hat{p} < b_1 \) because in the case where \( p_2 = p_3 = \hat{p} \) we have that \( \pi_i(\hat{p}, \hat{p}, b_1) = 0 \) while \( \pi_i(a_1, b_1, b_1) = a_1 - b_1 > 0 \).

(d) Since there is at least one player who does not have a dominant strategy, there is no dominant strategy equilibrium.

(e) The restrictions are: (1) \( b_1 - a_3 \leq a_1 - a_2 \) and (2) \( b_2 - a_3 \geq 0 \).

Proof that these restrictions are sufficient for \( (a_1, a_2, a_3) \) to be a Nash equilibrium:
(i) $\pi_i(a_1, a_2, a_3) = a_1 - a_2 > 0$ (since $a_1 > a_2$); for every $p_i \geq a_2$, $\pi_i(p_i, a_2, a_3) = a_1 - a_2$; for $p_i < a_2$, either $\pi_i(p_i, a_2, a_3) = b_1 - a_2$ (if $p_i \geq a_1$) or $\pi_i(p_i, a_2, a_3) = 0$ (if $p_i < a_1$), so that, by restriction (1), player 1 cannot increase his payoff.

(ii) $\pi_2(a_1, a_2, a_3) = b_2 - a_1$. For $p_2 > a_1$, $\pi_2(a_1, p_2, a_3) = a_2 - a_1 < 0$. For $a_3 \leq p_2 \leq a_1$, $\pi_2(a_1, p_2, a_3) = b_2 - a_1$. For $p_2 < a_1$, $\pi_2(a_1, p_2, a_3) = 0$. Thus player 2 cannot increase her payoff if constraint (2) is satisfied.

(iii) $\pi_3(a_1, a_2, a_3) = 0$. For $p_3 \leq a_2$, $\pi_3(a_1, a_2, p_3) = 0$. For $a_2 < p_3 \leq a_1$, $\pi_3(a_1, a_2, p_3) = b_3 - a_2$. For $p_3 > a_1$, $\pi_3(a_1, a_2, p_3) = a_3 - a_1$. Since $b_3 < a_3 < a_2 < a_1$, player 3 cannot increase his payoff.

Proof that they are necessary: (1) if $b_1 - a_1 > a_1 - a_2$ then player 1 can increase his payoff by switching to any $p_1 = a_3$; (2) if $b_2 - a_3 < 0$ then player 2 can increase her payoff by switching to $p_2 = 0$.

(f) Yes because $b_1 - a_1 = 15 < a_1 - a_2 = 60$ and $b_2 - a_3 = 8 > 0$.

(g) No, because $b_2 - a_3 = -2 < 0$.

2.

(a) Let $p$ be the probability with which Player 2 plays $D$. The following diagram shows the Player 1's expected payoff from each of her pure strategies:

![Diagram showing expected payoffs](image)

From the diagram it is clear that Player 1’s best reply function is as follows:

$$BR_i(p) = \begin{cases} 
B & \text{if } p < \frac{1}{3} \\
\left( \begin{array}{ccc}
A & B & C \\
0 & q & 1-q \\
\end{array} \right) & \text{with any } q \in [0,1] \\
C & \text{if } \frac{1}{3} < p < \frac{2}{3} \\
\left( \begin{array}{ccc}
A & B & C \\
q & 0 & 1-q \\
\end{array} \right) & \text{with any } q \in [0,1] \\
A & \text{if } \frac{2}{3} < p \leq 1 
\end{cases}$$
(b) Since (the strategy component of) a weak sequential equilibrium is a Nash equilibrium, it is sufficient to show that there are no pure-strategy Nash equilibria.

- \((A,D)\) is not a Nash equilibrium because \(D\) is not a best reply to \(A\).
- \((A,E)\) is not a Nash equilibrium because \(A\) is not a best reply to \(E\).
- \((B,D)\) is not a Nash equilibrium because \(B\) is not a best reply to \(D\).
- \((B,E)\) is not a Nash equilibrium because \(E\) is not a best reply to \(B\).
- \((C,D)\) is not a Nash equilibrium because \(C\) is not a best reply to \(D\).
- \((C,E)\) is not a Nash equilibrium because \(C\) is not a best reply to \(E\).

(c) Let \(x\) denote the left node of Player 2's information set and \(y\) the right node. Consider the assessment \((\sigma, \mu), \sigma = \left[ \begin{array}{cc} A & B & C \\ 0 & 0 & 1 \end{array} \right], \mu = \left[ \begin{array}{cc} x & y \\ q & 1-q \end{array} \right]\). For \(C\) to be sequentially rational it is necessary and sufficient that \(\frac{1}{3} \leq p \leq \frac{2}{3}\). For \(\left[ \begin{array}{cc} D & E \\ p & 1-p \end{array} \right]\) to be sequentially rational with \(0 < p < 1\) it must be that Player 2 is indifferent between playing \(D\) and \(E\); a necessary and sufficient condition for this to be the case is that \(q = \frac{1}{2}\). Thus any assessment of the form \(\sigma = \left[ \begin{array}{cc} A & B & C \\ 0 & 0 & 1 \end{array} \right], \mu = \left[ \begin{array}{cc} x & y \\ q & 1-q \end{array} \right]\), \(\mu = \left[ \begin{array}{cc} x & y \\ \frac{1}{2} & \frac{1}{2} \end{array} \right]\) with \(\frac{1}{3} \leq p \leq \frac{2}{3}\) is a weak sequential equilibrium.

(d) \(\sigma = \left[ \begin{array}{cc} A & B \end{array} \right], \left[ \begin{array}{cc} C \\ 0 & 0 \end{array} \right], \left[ \begin{array}{cc} D & E \\ \frac{1}{2} & \frac{1}{2} \end{array} \right], \mu = \left[ \begin{array}{cc} x & y \\ \frac{1}{2} & \frac{1}{2} \end{array} \right]\) is a sequential equilibrium. To see this, consider the completely mixed strategy profile \(\sigma_n = \left[ \begin{array}{cc} A & B & C \\ \frac{1}{n} & \frac{1}{n} & 1-\frac{2}{n} \end{array} \right], \left[ \begin{array}{cc} D & E \\ \frac{1}{2} & \frac{1}{2} \end{array} \right]\). Then \(\sigma_n \rightarrow \sigma\) as \(n \rightarrow \infty\). The beliefs obtained from \(\sigma_n\) using Bayes’ rule are \(\mu_n = \left[ \begin{array}{cc} x & y \\ \frac{1}{2} & \frac{1}{2} \end{array} \right]\), so convergence to \(\mu\) is trivial. Sequential rationality is satisfied because the given assessment is a weak sequential equilibrium.

3.

(a) For no values of \(x\) does Player 3 have a strictly dominant strategy: if Player 1 plays \(A\) then Player 3’s two strategies yield the same payoff, namely 1.

(b) Yes: \(B\) is weakly dominated by \(A\).

(c) The following 4 strategy profiles: \((A,D,G,H), (A,E,G,H), (A,D,G,L)\) and \((A,E,G,L)\).

(d) When \(x = 0\) there are two: \((A,D,G,L)\) and \((B,D,G,L)\). When \(x = 10\) there is only one: \((A,E,G,H)\).