A Search-Theoretic Model of the Term Premium

Chicago Fed Workshop on Money, Banking, Payments, and Finance

Athanasios Geromichalos, Lucas Herrenbrueck, and Kevin Salyer
UC Davis

August 07, 2013
Motivation and Goal of this Paper

- One of the most consistent features of bond yields: Longer term bonds receive, on average, higher returns.

- Inconsistent with expectations hypothesis.

- Available explanations:
  - Segmented Markets
  - Risk Premium theories based on the CAPM

- Popular explanation in the empirical finance literature: **Liquidity Premium**

- For instance, Cochrane (1999) casually states:
  The small increase in returns for long term bonds, equivalent to an upward slope in the yield curve, is usually excused as a “liquidity premium”.

- **Goal** of this paper: provide a **theoretical foundation** for this explanation.
Our Story

- An agent can work now and store her wealth;
  In three months, a consumption opportunity might arise

- The agent has 3 ways to store wealth:
  - Hold money: great in terms of liquidity, not a good store of value
  - Hold a three-month bond: good store of value and as liquid as money;
    Best of both worlds
  - Hold a six-month bond: also good store of value but not liquid

- If agent holds six-month bond and needs to consume, she has to liquidate;
  Often, such liquidation is associated with certain frictions

- Question: “Will the agent hold a six-month bond”?
  - Segmented markets theory says: No!
  - Our theory says: Yes, if the price is right!
  - Agent has to be compensated for the relative illiquidity of long-term asset
How we do it

- Use a model in tradition of modern monetary theory (Lagos-Wright 2005)

- Hence a medium of exchange is necessary

- Short term assets mature in time to take advantage of consumption opportunities (closer substitutes to money)

- Long term assets are not (direct) substitutes to money

- Agents who hold them can sell them for liquid assets in an OTC market; Follows Duffie-Gârleanu-Pedersen (2005)
Main Findings

- All assets carry liquidity premia:
  - Short term assets because they are substitutes to money (*direct*)
  - Long term assets because they help agents bypass the cost of holding liquid assets (*indirect*)

- **Main Result:** An upward sloping Yield Curve;
  Long term assets sell at a discount reflecting their relative illiquidity

- **Other empirically supported results:**
  - Slope of Yield Curve is steeper for assets with less liquid secondary markets
  - Newly issued assets sell at higher price than previously issued assets that mature on nearby dates
Related Literature

Conceptually related paper:

- Vayanos and Vila (2009)
  - Micro-foundation of a story offered in the Money and Banking textbook
  - They focus on the preferred habitat story

- This paper
  - Micro-foundation of the 2nd main explanation offered in the textbook
  - The Liquidity Premium explanation
Related Literature (Cont’ed)

Vast literature on the term premium:

- Backus, Gregory, and Zin (1989)
- Fama (1990)
- Duffie and Kan (1996)
- Bansal and Yaron (2004)
- Piazzesi and Schneider (2007)
- Cochrane and Piazzesi (2008)
- Singleton (2009)
- Gürkaynak and Wright (2012)
Liquidity properties of assets other than fiat money have been explored by (among others):

- Lagos and Rocheteau (2008)
- Lester, Postlewaite, and Wright (2008)
- Rocheteau (2011)
- Lagos (2011)
- Jacquet and Tan (2010)
- Andolfatto and Martin (2012)
- Venkateswaran and Wright (2012)

Assets carry liquidity premia because they allow agents to rebalance money holdings, after a consumption opportunity arises:

- Kocherlakota (2003)
- Boel and Camera (2006)
- Berentsen, Camera, and Waller (2007)
- Berentsen and Waller (2011)
- Berentsen, Huber, and Marchesiani (2011)
We believe that the term structure of interest rates is influenced by:

- Risk factors...
- Market segmentation, and...
- Liquidity

We see our contribution as providing a theoretical basis of the last factor.

Hence the model focuses on liquidity.
Towards a Model of Liquidity

• But which liquidity?

• In monetary theory: liquidity is an attribute of an asset: how easily it can be transformed into consumption

• In finance, liquidity is an attribute of a market: how easily an investor can find a counterparty for trade and at what cost

• This paper combines both of these notions of liquidity

• Builds on Geromichalos and Herrenbrueck (2012)
The Model

- Infinite horizon, discrete time, discount factor is $\beta \in (0, 1)$ between periods
- Period divided in three sub-periods:
  - Secondary asset market (OTC)
  - Decentralized goods market (LW)
  - Centralized market (CM)
- Two types of agents depending on their role in LW market
- Buyers with measure 1 and preferences: $U(X) - H + u(q)$
- Sellers with preferences: $U(X) - H - q$
- $X$ is consumption in CM, $H$ is work in CM, and $q$ is quantity of special good consumed and produced in LW
Unique Feature of the Model

- After CM a measure $\ell < 1$ of buyers learns that they will consume in LW; Refer to them as the C-types

- The remaining $1 - \ell$ buyers (N-types) will not consume in LW

- Since a medium of exchange is necessary in LW
  - C-types visit OTC to sell illiquid assets for liquid assets (to be defined)
  - N-types are the providers of liquidity

- OTC is strategically placed before LW...
  but after resolution of uncertainty regarding consumption
Third Subperiod: Centralized Market (CM)

- Agents consume and produce a general good (fruit)
  - Comes from two sources: labor and dividend of assets maturing that period
  - Access to technology that turns one unit of labor into one unit of fruit

- Each period $t$, a new set of trees are born:
  - They deliver 1 unit of fruit in period $t + i, i \in \{1, \ldots, N\}$
  - Agents can purchase any amount at the ongoing price $\psi_{i,t}$
  - Supply of trees that mature in $i$ periods is $A_i$, fixed over time; includes newly issued and older assets that mature on that date

- Fruit is delivered before the LW market opens
  - Can be stored at no cost between 2nd and 3rd subperiods
  - Perishable between periods

- Supply of money follows $M_{t+1} = (1 + \mu)M_t$; Market price is $\varphi_t$
Second Subperiod: Decentralized Goods Market (LW)

- A standard LW decentralized market
- C-type buyers meet bilaterally with sellers;
  Buyers make take-it-or-leave-it offers
- For simplicity all $\ell$ buyers match;
  and all sellers match (i.e pick sellers’ measure to be $\ell$)
- MOE: money and fruit (or promises to fruit) that has been delivered;
  Claims to trees that mature in future period cannot serve as MOE
- This assumption:
  - Captures the idea that assets that have already matured
    are as good (liquid) as money
  - ...and at the same time allows us to work with real assets
First Subperiod: Secondary Asset Market (OTC)

- C-types may not have enough liquid assets; They may want to rebalance their liquidity

- Buyers are ex ante identical, so N-types may have liquid assets; They will not use them in the current period

- Liquid assets worth more in hands of C-types (surplus to be exploited)

- A CRS function $f(\ell, 1 - \ell) \leq \min\{\ell, 1 - \ell\}$ brings two sides together

- Proportional bargaining; $\lambda \in (0, 1)$ is C-type’s bargaining power
Figure: Timing of events in a model with two maturities.
Discussion of Asset Trade

- All assets are issued in Walrasian markets; Thereafter, they only trade in OTC fashion.

- Periodical access to Walrasian markets (and quasi-linear preferences) is a methodological innovation that gives rise to degenerate asset distributions.

- This setup is convenient and (to some extent) realistic:
  - Often, issue price of assets is indeed determined in a competitive setting.
  - Assets are then traded in OTC markets, as documented by Duffie et al.

- Example: US T-Bills are issued through single-priced auctions; “to minimize the government’s costs... by promoting broad, competitive bidding” (Garbade and Ingber (2005)).
In this talk...

- Focus on the case $N = 2$

- Study equilibrium asset prices

- Explore the properties of asset prices in order to show:
  - The main result: upward sloping yield curve
  - Two interesting additional results, also empirically supported

- Then, show that the main result goes through for any $N > 2$

- Statements that involve interest rates (vs prices) exploit the formula
  \[ \psi_i = \frac{1}{(1 + r_i)^i}, \quad \text{for all } i = 1, \ldots, N. \]
CM Value Function for Buyers

\[ W(m, d, a_2) = \max_{X, H, \hat{m}, \hat{a}_1, \hat{a}_2} \left\{ U(X) - H + \beta \mathbb{E} \left\{ \Omega^i(\hat{m}, \hat{a}_1, \hat{a}_2) \right\} \right\} \]

s.t. \[ X + \varphi \hat{m} + \psi_1(\hat{a}_1 - a_2) + \psi_2 \hat{a}_2 = H + \varphi (m + \mu M) + d, \]

and \( \hat{a}_1 - a_2 \geq 0 \)

- \( \Omega^i \) is the OTC value function for type \( i \in \{C, N\} \)

- Three observations about value function:
  - At optimum, \( X = X^* \), where \( U'(X^*) = 1 \)
  - Choice of (\( \hat{m}, \hat{a}_1, \hat{a}_2 \)) does not depend on the state (with a caveat)
  - \( W^B \) is linear

One can write:

\[ W(z, a_2) = \Lambda + z + \psi_1 a_2, \]

where \( z \equiv \varphi m + d \), the buyers’ total real balances
CM Value Function for Sellers

- Sellers never leave the CM with any assets

- But they enter the CM with some liquid assets obtained in preceding LW

- Sellers’ CM value function is:

\[
W^S(z) = \max_{X,H} \left\{ U(X) - H + \beta V^S \right\}
\]

\[
s.t. \quad X = H + z,
\]

- Hence, we can write

\[
W^S(z) = U(X^*) - X^* + z + V^S \equiv \Lambda^S + z.
\]

- \(V^S\) is seller’s value function in the LW market
LW Value Functions

In the LW market, the value functions are as follows:

- For a buyer

\[ V(z, a_2) = u(q) + W(z - \pi, a_2), \]

where \( q, \pi \) are the solutions to the LW bargaining problem.

- For a seller

\[ V^S = -q + W^S(\pi). \]

- \( \pi \) is the real value of money and fruit that change hands in LW trade.
- \( q \) is quantity of special good exchanged.
- The objects \((\pi, q)\) are described in the appendix.
OTC Value Functions

In the OTC market, the value functions are given by

\[ \Omega^C(m, a_1, a_2) = \alpha_c \ V(z + \zeta, a_2 - \chi) + (1 - \alpha_c) \ V(z, a_2), \]
\[ \Omega^N(m, a_1, a_2) = \alpha_N \ W(z - \zeta, a_2 + \chi) + (1 - \alpha_N) \ W(z, a_2). \]

where

- \( \chi \) is the amount of long term assets sold by the C-type
- \( \zeta \) the real value of liquid assets received by the C-type
- \( \chi \) and \( \zeta \) are determined through bargaining (in the appendix)
- Finally,

\[ a_c = \frac{f(\ell, 1 - \ell)}{\ell}, \quad a_N = \frac{f(\ell, 1 - \ell)}{1 - \ell} \]
Replace bargaining solutions into the CM value function to obtain objective function of the typical buyer:

\[ J(\hat{m}, \hat{a}_1, \hat{a}_2) = -\varphi \hat{m} - \psi_1 \hat{a}_1 - \psi_2 \hat{a}_2 \]

matched C-type \( \rightarrow \) \( + f(\ell, 1 - \ell) \left[ u(\hat{z} + \zeta) + \hat{\psi}_1 (\hat{a}_2 - \chi) \right] \)

unmatched C-type \( \rightarrow \) \( + [\ell - f(\ell, 1 - \ell)] \left[ u(\hat{z}) + \hat{\psi}_1 \hat{a}_2 \right] \)

matched N-type \( \rightarrow \) \( + f(\ell, 1 - \ell) \left[ \hat{z} - \tilde{\zeta} + \hat{\psi}_1 (\hat{a}_2 + \tilde{\chi}) \right] \)

unmatched N-type \( \rightarrow \) \( + [1 - \ell - f(\ell, 1 - \ell)] \left( \hat{z} + \hat{\psi}_1 \hat{a}_2 \right) \)

It is understood that

- \( \chi = \chi(\hat{z}, \tilde{z}, \hat{a}_2), \ \zeta = \zeta(\hat{z}, \tilde{z}, \hat{a}_2) \)
- \( \tilde{\chi} = \chi(\tilde{z}, \hat{z}, \tilde{a}_2), \ \tilde{\zeta} = \zeta(\tilde{z}, \hat{z}, \tilde{a}_2) \)
- Agent chooses \( \hat{z} \equiv \hat{\varphi} \hat{m} + \hat{a}_1 \) (\( \hat{m} \) and \( \hat{a}_1 \) are perfect substitutes)
- (\( \hat{z}, \hat{a}_2 \)) are choice variables and (\( \tilde{z}, \tilde{a}_2 \)) are expectations
Optimal Choice of the Agent

The domain of the objective function can be divided into 5 regions, arising from three questions. Given prices and my beliefs about other agents’ holdings:

- When the C-type and the N-type pool their real balances in the OTC market, can they achieve the first-best in the LW market? (They would want to do that since the inflation cost is sunk at this point)

- If I am a C-type, do I carry enough assets to compensate the N-type?

- If I am an N-type, do I expect a C-type to carry enough assets to compensate me?
Figure: Regions of individual choice, with expectations $\tilde{z} = 0.35q^*$ and $\tilde{a}_2 = 0.5$. 
Figure: Regions of individual choice, with expectations $\bar{z} = 0.35q^*$ and $\bar{a}_2 = 0.5$. Sliced at $\hat{a}_2 = 0.3$. 

**Optimal Choice of the Agent (Cont’d)**
Figure: Demand for liquid assets, with expectations $\tilde{z} = 0.35 q^*$ and $\tilde{a}_2 = 0.5$. 
From Agent’s Optimality to (symmetric) Equilibrium

Figure: Aggregate regions of equilibrium, in terms of real balances.
Steady State Equilibrium

**DEFINITION:** A symmetric steady-state equilibrium is a list 
\[ \{ \varphi, \psi_1, \psi_2, \chi, \zeta, Z, q_1, q_2 \} \], where 
\[ Z = \varphi M + A_1 \] represents the real balances, and 
\[ q_2 (q_1) \] is the good exchanged in LW when the buyer was matched (not matched) in OTC. The equilibrium objects satisfy:

- Representative agent behaves optimally under prices \( \psi_1, \psi_2, \varphi, \) and 
  \[ \psi_1 = \hat{\psi}_1 = \varphi / \beta \hat{\phi} = 1 + \mu \text{ if } \hat{\phi} > 0 \]
- \( q_1 = Z, \) and \( q_2 \) is given by:
  \[ q_2(Z) = \begin{cases} 
  q^*, & \text{in Region 1,} \\
  \tilde{q}(Z), & \text{in Region 5,} 
\end{cases} \]
  where \( \tilde{q} \) solves
  \[ (1 - \lambda) [u(\tilde{q}) - u(Z)] + \lambda (\tilde{q} - Z) = \psi_1 A_2 \]
- The terms \( (\chi, \zeta) \) satisfy the solution to OTC bargaining problem, evaluated at the aggregate quantities \( Z \) and \( A_2 \)
- Markets clear at symmetric choices, and expectations are rational:
  \[ \hat{m} = (1 + \mu)M, \hat{z} = \tilde{z} = Z, \hat{a}_1 = A_1, \text{ and } \hat{a}_2 = \tilde{a}_2 = A_2 \]
Regions of Equilibrium

Figure: Aggregate regions of equilibrium in terms of inflation, for given $A_1 \in (\bar{A}_1, q^*)$. 

30
Characterization of Equilibrium

**PROPOSITION 1:** If $A_1 \geq q^*$, then the following are true:

- The equilibrium is always non-monetary regardless of $\mu$
- In any LW meeting, the first-best $q^*$ is produced
- No trade occurs in the OTC market
- Asset prices always equal their fundamentals: $\psi_i = \beta^i$ for $i = 1, 2$; This implies that $r_i = \frac{1}{\beta} - 1$ for $i = 1, 2$
- Hence, when the liquidity channel is shut down, the Yield Curve is flat
PROPOSITION 2: If $A_1 < q^*$, then:

- $\psi_1 = \min\{1 + \mu, 1 + \bar{\mu}(A_1, A_2)\}$
- $\psi_2$ depends on the value of $A_2$. We have two cases:

  **Case 1:** If $A_2 \geq \bar{A}_2(A_1)$, then $\psi_2 = \beta \psi_1$.

  **Case 2:** If $A_2 < \bar{A}_2(A_1)$, then there exists a cutoff $\bar{\mu}(A_2)$ such that:
  a) For all $\mu \in (\beta - 1, \bar{\mu}(A_2))$, we have $\psi_2 = \beta \psi_1$;
  b) For all $\mu \in (\bar{\mu}(A_2), \bar{\mu}(A_1, A_2))$, we have
      $\psi_2 = \beta \rho(\mu, A_2) \psi_1$, where $\rho(\mu, A_2) \in (1, (1 + \mu)/\beta)$,
      $\partial \rho(\mu, A_2)/\partial \mu > 0$, and $\partial \rho(\mu, A_2)/\partial A_2 < 0$.
  c) For all $\mu \geq \bar{\mu}(A_1, A_2)$, we have $\psi_2 = \beta \rho(\bar{\mu}, A_2) \psi_1$.

The term $\rho$ captures an *indirect* liquidity premium and is given by

$$\rho(\mu, A_2) = 1 + \lambda f \frac{u'(Z + \zeta) - 1}{(1 - \lambda)u'(Z + \zeta) + \lambda}$$
Discussion of Proposition 2

• Focus on monetary equilibria: $\psi_1 = 1 + \mu$

• Short term assets are perfect substitutes to money. $\partial \psi_1 / \partial \mu > 0$ as in Geromichalos, Licari, and Suarez or Lester, Postlewaite, and Wright

• $\psi_2 = \beta \rho(\mu, A_2) (1 + \mu)$, where $\rho(\mu, A_2) \geq 1$

• Long term assets carry two types of liquidity premia:
  ▶ They will become short term assets tomorrow (term $1 + \mu$)
  ▶ Help agents avoid the cost of carrying liquid assets (term $\rho(\mu, A_2)$)

• $\partial \rho(\mu, A_2) / \partial \mu > 0$ because this service is more valuable when $\mu$ is higher

• Equilibrium quantities in the appendix
Figure: Equilibrium prices as functions of inflation.
Result 1: Upward sloping Yield Curve

- \( r_2 > r_1 \iff \left( \frac{1}{\beta \rho \psi_1} \right)^{\frac{1}{2}} > \frac{1}{\psi_1} \iff \psi_1 > \beta \rho \)

- Recall that \( \psi_1 = 1 + \mu \) and \( \rho < (1 + \mu)/\beta \)

- We conclude that as long as \( A_1 < q^* \), Yield Curve is upward sloping
  - Long term assets have a higher yield to compensate agents for their (relative) lack of liquidity
  - Alternatively one can show that term premium is positive

- Only one way to obtain \( r_2 = r_1 \): if \( \ell = f \) (all C-types match) and \( \lambda = 1 \) (C-types have all the bargaining power)

- Finally, one can show that \( \frac{\partial (r_2 - r_1)}{\partial \mu} > 0 \)
Result 2: The Effect of Secondary market Liquidity on Assets Returns

- Consider an extension of the model with a new set of assets
  Identical to originals except once purchased, have to be held to maturity

- In the “interesting equilibria”, we have the following:
  \[ \psi_1 = p_1, \psi_2 = \beta \rho(\mu, A_2) \psi_1, \rho_2 = \beta \rho_1 \Rightarrow \psi_2 > p_2 \]
  The indirect liquidity premium \( \rho(\mu, A_2) \) is decreasing in \( A_2 \).

- Krishnamurthy and Vissing-Jorgensen (2012) show that:
  - Over 1984-2008 period the spread between 6-month FDIC-insured CDs and 6-month T-bills was 2.3 percentage points on average
  - The spread was negatively related to the supply of T-bills

- In general, our model predicts that \( \partial r_2 / \partial f < 0 \)
  - Seems to be consistent with empirical observation
  - Gürkaynak, Sack, and Wright (2010) demonstrate that the TIPS yields have fallen as market liquidity in the TIPS market has increased
Result 3: On-the-run and Off-the-run Price comparison

- N-types who buy assets in OTC (issued at $t-1$ and maturing at $t+1$) Could obtain identical assets (maturing at $t+1$) in the forthcoming CM

- One can compare the price of newly issued short term assets with the price of older assets (off-the-run) which mature on the same date

- We show that in any equilibrium $\psi_{OTC} < \psi_1$

- Here the key is that C-types are desperate for liquidity Hence, willing to sell at cheaper prices

- Like in Vayanos and Weill (2008), search frictions in OTC are crucial in order to be consistent with no-arbitrage
The Model with $N$ maturities

- Agents must be able to obtain the representative portfolio in CM without selling off-the-run assets
  
  - A simple sufficient condition: $A_1 \geq 2A_2 \geq \ldots \geq 2^{N-1}A_N$

- With $N > 2$, there are many combinations of long term asset portfolios that C-type can sell for additional liquidity in OTC
  
  - We do not place any restrictions on which assets can be traded for liquidity

- Even though $N > 2$, the interesting distinction is still between:
  
  - Assets that mature now (direct substitutes to money), and...
  
  - Assets that do not mature now (but can be traded for liquid assets in OTC)

- The spirit of the analysis of $N = 2$ case does not change
PROPOSITION 4: Assume that $2^{N-1}A_N \leq \ldots \leq 2A_2 \leq A_1 < q^*$, define $A_L \equiv \beta^{N-2}A_N + \ldots + A_2$, and focus on monetary equilibria, i.e. $\mu < \bar{\mu}(A_1, A_L)$.

- The equilibrium price of one-period assets is $\psi_1 = 1 + \mu$

- The equilibrium price $\psi_i, i \geq 2$ depends on $A_L$. We have two cases:

  **Case 1:** If $A_L \geq \bar{A}_2(A_1)$, then $\psi_i = \beta^{i-1}\psi_1$.

  **Case 2:** If $A_L < \bar{A}_2(A_1)$, then there exists a cutoff $\tilde{\mu}(A_L)$ such that:
  a) For all $\mu \in (\beta - 1, \tilde{\mu}(A_L)]$, we have $\psi_i = \beta^{i-1}\psi_1$;
  b) For all $\mu \in (\tilde{\mu}(A_L), \tilde{\mu}(A_1, A_L))$, we have $\psi_i = (\beta \rho^L)^{i-1}\psi_1$, for all $i \geq 2$.

The term $\rho^L \in (1, (1 + \mu)/\beta)$ is increasing in $\mu$ and decreasing in any $A_i, i \geq 2$. 
The Upward Sloping Yield Curve for $N > 2$

- We have already seen that $r_2 > r_1$

- For $i \in \{2, \ldots, N - 1\}$, we have

$$r_{i+1} > r_i \iff \left[ \frac{1}{\beta^i(1 + \mu)} \right]^\frac{1}{i+1} > \left[ \frac{1}{\beta^{i-1}(1 + \mu)} \right]^\frac{1}{i} \iff \left( \frac{1}{\beta} \right)^\frac{1}{i(i+1)} > \left( \frac{1}{1 + \mu} \right)^\frac{1}{i(i+1)}$$

- Always true, since $\mu > \beta - 1$

- Model delivers an upward sloping YC throughout the domain $i = 1, \ldots, N$
  
  - Even though assets with lifetime $i, j \geq 2$ are qualitatively similar

- Assets with maturity $i$ are still more liquid than those with maturity $i + 1$ because the former become one-period assets earlier
Conclusion

- Liquidity preference is often proposed as a resolution to the empirical failures of the expectations hypothesis of the term structure.

- This paper provides a theoretical basis for this preference. A positive liquidity premium emerges in the presence of 3 ingredients:
  - Agents are subject to stochastic consumption expenditures
  - The markets in which these expenditures take place are decentralized;
    A medium of exchange (i.e. a liquid asset) is necessary
  - Secondary assets markets where agents can sell assets to acquire liquidity are characterized by search and bargaining frictions.

- Our model helps understand the effect of secondary asset market liquidity on assets’ equilibrium prices (yields).

- Our model suggests a simple explanation of the on-the-run phenomenon.
Discussion of Assumptions

- MOE is either money or fruit that has been delivered; Claims to trees that mature in future period cannot serve as MOE

- This assumption:
  - Captures the idea that assets that have already matured are as good (liquid) as money
  - ...and at the same time allows us to work with real assets

- Rocheteau (2011) and Lester, Postlewaite, and Wright (2012) show that if there is asymmetric information regarding the future returns of assets...

- Then money (or, in our case, assets that have already matured) will arise endogenously as a superior medium of exchange
Consider a meeting in LW between a seller and a buyer with holdings $z, a_2$

- The bargaining problem is

$$\max_{\pi, q} \{ u(q) + W(z - \pi, a_2) - W(z, a_2) \},$$

$$s.t. \quad -q + W^S(\pi) - W^S(0) = 0$$

and $\pi \leq z$

- Using the linearity of the $W$'s, we obtain

$$\max_{\pi, q} \{ u(q) - \pi \},$$

$$s.t. \quad q = \pi, \text{ and } \pi \leq z$$

- The solution to this problem is

$$q(z) = \pi(z) = \min \{ q^*, z \},$$

where $q^* \equiv \arg \max_q \{ u(q) - q \}$
Bargaining Problem in OTC

A meeting between C and N-type with holdings \((z, a_2), (\bar{z}, \bar{a}_2)\), respectively

The bargaining problem is

\[
\max_{\chi, \zeta} S^C
\]

\[
\text{s.t. } \frac{S^C}{S^N} = \frac{\lambda}{1-\lambda},
\]

and \(\chi \leq a_2, \zeta \leq \bar{z}\),

where

\[
S^C \equiv V(z + \zeta, a_2 - \chi) - V(z, a_2),
\]

\[
S^N \equiv W(\bar{z} - \zeta, \bar{a}_2 + \chi) - W(\bar{z}, \bar{a}_2).
\]

After some manipulations:

\[
\max_{\chi, \zeta} \lambda \{u(z + \zeta) - u(z) - \zeta\},
\]

\[
\text{s.t. } \psi_1 \chi = \zeta + (1 - \lambda) [u(z + \zeta) - u(z) - \zeta].
\]
Bargaining Solution in OTC

The bargaining solution depends only on \((z, \tilde{z}, a_2)\). Define the cutoff point

\[
\bar{a}(z, \tilde{z}) \equiv \frac{1}{\psi_1} \left\{ (1 - \lambda) \left[ u(\min\{z + \tilde{z}, q^*\}) - u(z) \right] + \lambda \min\{q^* - z, \tilde{z}\} \right\}.
\]

Then, the solution is:

\[
\chi(z, \tilde{z}, a_2) = \begin{cases} 
\bar{a}(z, \tilde{z}), & \text{if } a_2 \geq \bar{a}(z, \tilde{z}), \\
\zeta(z, a_2), & \text{if } a_2 < \bar{a}(z, \tilde{z}).
\end{cases}
\]

\[
\zeta(z, \tilde{z}, a_2) = \begin{cases} 
\min\{q^* - z, \tilde{z}\}, & \text{if } a_2 \geq \bar{a}(z, \tilde{z}), \\
\zeta^a(z, a_2), & \text{if } a_2 < \bar{a}(z, \tilde{z}),
\end{cases}
\]

where \(\zeta^a(z, a_2)\) is implicitly defined by

\[
\zeta^a(z, a_2) \equiv \left\{ \zeta : (1 - \lambda) \left[ u(z + \zeta) - u(z) \right] + \lambda \zeta = \psi_1 a_2 \right\}.
\]
Alternative Definitions: The Term Premium

- In models with aggregate uncertainty, the term premium is often defined as
  \[ \tilde{\tau}_{1,2} = \mathbb{E}_t[(1 + r_{2,t})^2/(1 + r_{1,t+1})] - (1 + r_{1,t}) \]
  i.e. the excess return from selling a two-period bond after one period

- With no uncertainty and constant interest rates, \( r_2 > r_1 \iff \tilde{\tau}_{1,2} > 0 \);
  i.e. here a positive term premium is equivalent to an upward sloping YC

- Similarly, in \( N > 2 \) case, we show that \( r_{i+1} > r_i \), for all \( i \in \{1, ..., N - 1\} \)

- Alternatively, one can show this result in terms of positive term premia
  - For instance, \( \tilde{\tau}_{2,3} = \psi_2/\psi_3 - 1/\psi_1 = 1/\beta - 1/(1 + \mu) > 0 \)
  - OR one can show that term premia defined as the return differential of
    buying a \( k \)-period bond and holding to maturity relative to buying a
    sequence of \( k \) one-period bonds are also positive
**Equilibrium Quantities**

**PROPOSITION 3:** If $A_1 < q^*$, then:

- $q_1 = Z$. When $\mu < \bar{\mu}(A_1, A_2)$, then $\partial q_1 / \partial \mu < 0$, and when $\mu > \bar{\mu}(A_1, A_2)$, then $\partial q_1 / \partial \mu = 0$.

- Regarding $q_2$, we have 2 cases:

  **Case 1:** If $A_2 \geq \bar{A}_2(A_1)$, then $q_2 = q^*$ for any $\mu > \beta - 1$.

  **Case 2:** If $A_2 < \bar{A}_2(A_1)$, then for the same cutoff $\bar{\mu}(A_2)$ as in Proposition 2:

  a) For all $\mu \in (\beta - 1, \bar{\mu}(A_2)]$, $q_2 = q^*$;

  b) For all $\mu \in (\bar{\mu}(A_2), \bar{\mu}(A_1, A_2))$, $q_2 = Z + \zeta^M < q^*$ and $q_2$ is a strictly decreasing function of $\mu$;

  c) For all $\mu \geq \bar{\mu}(A_1, A_2)$, $q_2 = A_1 + \zeta^N < q^*$ which does not depend on $\mu$. 

47
**Figure:** Equilibrium LW quantities as functions of inflation.