1. The Variance of $\hat{\alpha}$ and $\hat{\beta}$ in the Linear Regression Model

1.1. Two useful results

- **The Law of Iterated Expectations:**

\[
E_x(X) = E_y(E(X|Y))
\]  
(1.1)

- **The Decomposition of the Variance:**

\[
V(Y) = V_x(E(Y|X)) + E_x(V(Y|X))
\]  
(1.2)

1.2. The variance of $\hat{\beta}$

Recall:

\[
\hat{\beta} = \beta_0 + \frac{\sum_{t=1}^{T} (x_t - \bar{x})(\varepsilon_t - \bar{\varepsilon})}{\sum_{t=1}^{T} (x_t - \bar{x})^2}
\]

Note:

\[
V(\hat{\beta}|X) = V\left(\beta_0 + \frac{\sum_{t=1}^{T} (x_t - \bar{x})(\varepsilon_t - \bar{\varepsilon})}{\sum_{t=1}^{T} (x_t - \bar{x})^2} \middle| X\right) = \frac{1}{\left(\sum_{t=1}^{T} (x_t - \bar{x})^2\right)^2} V\left(\sum_{i=1}^{T} (x_t - \bar{x})(\varepsilon_t - \bar{\varepsilon}) \middle| X\right)
\]
This step is possible due to two properties. The first is that $\beta_0$ is a constant and therefore its variance is zero. The second is that by conditioning on $X$, we can pull outside the variance term the denominator of this expression. Expanding the variance term explicitly we obtain,

$$
\frac{1}{\left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^2} \left\{ (x_1 - \bar{x})^2 V(\varepsilon_1 - \bar{\varepsilon}|X) + \ldots + (x_T - \bar{x})^2 V(\varepsilon_T - \bar{\varepsilon}|X) \right\} =
$$

$$
\frac{1}{\left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^2} \left\{ \sigma^2 \left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right) \right\} =
$$

$$
\frac{\sigma^2}{\left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^2}
$$

Next, we use this result and the decomposition of the variance to obtain,

$$
V(\bar{x}) = V_x(E(\bar{x}|X)) + E_x(V(\bar{x}|X)) =
$$

$$
V_x(\beta_0) + \sigma^2 E_x \left( \frac{1}{\left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^2} \right) = \sigma^2 E_x \left( \frac{1}{\left(\sum_{t=1}^{T}(x_t - \bar{x})^2\right)^2} \right) \quad \text{(Q.E.D.)}
$$
1.3. The variance of $\hat{\alpha}$

Using similar techniques, one can show that:

$$V(\hat{\alpha}) = \sigma^2 E_x \left( \frac{\sum_{t=1}^{T} x_t^2}{T \left( \sum_{t=1}^{T} (x_t - \bar{x})^2 \right)^2} \right)$$

However, proper derivation of this result is more involved.

1.4. The covariance between $\hat{\alpha}$ and $\hat{\beta}$

This result is a bit easier to derive. By definition of the covariance

$$COV(\hat{\alpha}, \hat{\beta}) = E(\hat{\alpha} - E(\hat{\alpha}))(\hat{\beta} - E(\hat{\beta}))$$

Note:

$$E(\hat{\alpha}) = \alpha_0$$

$$E(\hat{\beta}) = \beta_0$$

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$$

Substituting these results into the formula for the covariance,

$$E(\hat{\alpha} - E(\hat{\alpha}))(\hat{\beta} - E(\hat{\beta})) = E \left\{ (\bar{Y} - \hat{\beta}\bar{X} - \alpha_0)(\hat{\beta} - \beta_0) \right\} =$$
\[ E \left\{ (\alpha_0 + \beta_0 X + \varepsilon - \hat{\beta} X - \alpha_0)(\hat{\beta} - \beta_0) \right\} = E \left\{ (X(\hat{\beta} - \beta_0)^2 \right\} \]

where we have used the fact that \( E(\varepsilon) = 0 \). By the Law of Iterated Expectations:

\[ E \left\{ (-X(\hat{\beta} - \beta_0)^2 \right\} = E_x \left\{ E \left\{ (-X(\hat{\beta} - \beta_0)^2 | X \right\} \right\} \]

\[ E \left\{ (-X(\hat{\beta} - \beta_0)^2 | X \right\} = -X E \left\{ (\hat{\beta} - \beta_0)^2 | X \right\} = -XV(\hat{\beta} | X) \]

Therefore,

\[ E_x \left( \frac{-X \sigma^2}{\sum_{t=1}^{T}(x_t - \bar{X})^2} \right) = -\sigma^2 E_x \left( \frac{-X}{\sum_{t=1}^{T}(x_t - \bar{X})^2} \right) \]  \( \text{(Q.E.D.)} \)