Economics 105

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Calculus Review

1 Differentiation of one variable

Definition: The process of finding the derivative of a function is called differentiation.
Notation: For function $y = f(x)$, the first derivative is generally denoted as $f'(x)$ or $\frac{dy}{dx}$.
The derivative of f at x is given by:

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x} \tag{1}$$

1.1 Derivative as the slope of the tangent line

The slope of a strait line (linear function $y = f(x)$) is :

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x + \triangle x) - f(x)}{\triangle x}, \text{ if } x_2 - x_1 = \triangle x \tag{2}$$

If now we are working with a non-linear function such as $f(x) = x^2$, the slope of the curve changes at each point of the curve. To find a general formula of the slope, we use the derivative of the function $f(x)$. In our example, the slope of the curve x^2 is given by its derivative which is $2x$.

1.2 Derivative as the rate of change

Coming back to the linear function f, its slope measures how much $f(x)$ increases for each unit increase in x. Thus, it measures the *rate of change* of the function f. In equation 2, $\triangle x$ measures the change in x and $f(x + \triangle x) - f(x)$ the change in y.

1.3 Differentiation Rules

Suppose that k is an arbitrary constant and that f, g are differentiable functions at $x = x_1$.

1.3.1 The Constant Rule

The derivative of a constant k is zero.

$$\frac{d}{dx}[k] = 0 \tag{3}$$

1.3.2 The Power Rule

If n is a rational number, a simple power rule is

$$\frac{d}{dx}[x^n] = nx^{n-1} \tag{4}$$

A more general rule is:

$$\frac{d}{dx}[f(x)^n] = n(f(x)^{n-1})f'(x) \tag{5}$$

Example: If $f(x) = 3x^3$, $f'(x) = 9x^2$
If $g(x) = (2x + 4)^2$, $g'(x) = 2(2x + 4) \cdot 2 = 4(2x + 4) = 8x + 16$

1.3.3 The Product Rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + f'(x)g(x) \tag{6}$$

Example: If $f(x) = 3x^3$ and $l(x) = (x+4)$, then $(f(x) \cdot l(x))' = 9x^2(x+4) + 3x^3 = 12x^3 + 36x^2$

1.3.4 The Quotient Rule

$$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)^2]}, \text{ with g(x)} \neq 0 \tag{7}$$

Example: If $f(x) = 3x^3$ and $l(x) = (x+4)$, then $(\frac{f(x)}{l(x)})' = \frac{9x^2(x+4)-3x^3 \cdot 1}{(x+4)^2} = \frac{6x^3+36x^2}{(x+4)^2}$

1.3.5 The Sum and Difference Rules

$$(f \pm g)'(x) = f'(x) \pm g'(x) \tag{8}$$

Example: If $n(x) = 3x^4 + x^5$, $n'(x) = 12x^3 + 5x^4$

1.3.6 The Chain Rule

Let's define the function h as $h(x) = g(f(x))$ and i as $i(x) = f(g(x))$
Example: if $f(x) = x + 4$ and $g(x) = x^2$, then $h(x) = g(f(x)) = (x+4)^2$ and $i(x) = f(g(x)) = x^2 + 4$.

$$\frac{d}{dx}[g(f(x))] = g'(f(x))f'(x) \tag{9}$$

Example: $h'(x) = 2(x+4) \times 1 = 2x + 8$

1.3.7 The Derivative of the Log Function

A simple version of this rule is:

$$\frac{d}{dx}[\ln x] = \frac{1}{x} \tag{10}$$

The more general rule is:

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)} \tag{11}$$

Example: If $f(x) = 3x^3$ then $(\ln(3x^3))' = \frac{9x^2}{3x^3} = \frac{3}{x}$

2 Partial Derivative

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x}[f(x,y)] = f_x(x,y) = \frac{\partial z}{\partial x} \tag{12}$$

and

$$\frac{\partial}{\partial y}[f(x,y)] = f_y(x,y) = \frac{\partial z}{\partial y} \tag{13}$$

Example: Let's define $f(x,y) = 3x - x^2y^2 + 2x^3y$, then, $f_x(x,y) = 3 - 2xy^2 + 6x^2y$ and $f_y(x,y) = -2x^2y + 2x^3$