Dialogues, Logics and Other Strange Things
Essays in Honour of Shahid Rahman

CONTENTS

A Sound and Complete Temporal Logic for Belief Revision

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ABSTRACT. Branching-time temporal logic offers a natural setting for a theory of belief change, since belief revision deals with the interaction of belief and information over time. We propose a temporal logic that, besides the next-time operator, contains a belief operator and an information operator. It is shown that this logic is sound and complete with respect to the class of branching-time frames augmented, for each instant t, with a set of states and two binary relations on it, representing beliefs and information, respectively.

1 Introduction

Belief revision deals with the interaction between initial beliefs and new evidence. As new information is acquired over time, beliefs are correspondingly changed to accommodate that information. Temporal logic provides a natural framework for a theory of belief revision. We propose a basic logic for belief revision which, besides the next-time operator $\bigcirc$, contains a belief operator B and an information operator I. The information operator is not a normal operator and is formally similar to the "all I know" operator introduced by Levesque [9]. On the semantic side we consider branching-time frames to represent different possible evolutions of beliefs. For every date t, beliefs and information are represented by binary relations $\mathcal{B}_t$ and $\mathcal{I}_t$ on a set of states Ω_t. As usual, the link between syntax and semantics is provided by the notion of valuation and model. The truth of a formula in a model is defined at a state-instant pair (ω, t). We prove soundness and completeness of this basic logic with respect to the class of frames considered. Extensions of this basic logic are studied elsewhere (Bonanno [3], [4]). In particular, it is shown in [3] that a suitable extension of the basic logic considered in this paper provides an axiomatic characterization of the AGM theory of belief revision (Alchourrón et al. [1]).

2 Syntax

We consider a propositional language with five modal operators: the next-time operator $\bigcirc$ and its inverse $\bigcirc^{-1}$, the belief operator B, the information operator I and the "all state" operator A. The intended interpretation is as follows:

$\bigcirc\varphi$:	"at every next instant it will be the case that φ"
$\bigcirc^{-1}\varphi$:	"at every previous instant it was the case that φ"
$B\varphi$:	"the agent believes that φ"
$I\varphi$:	"the agent is informed that φ"
$A\varphi$:	"it is true at every state that φ".

The "all state" operator A is needed in order to capture the non-normality of the information operator I (see below). For a thorough discussion of the "all state" operator see Goranko and Passy [7].

The formal language is built in the usual way (see Blackburn et al. [2]) from a countable set of atomic propositions, the connectives $\neg$ and $\vee$ (from which the connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators $\bigcirc$, $\bigcirc^{-1}$, B, I and A. Let $\Diamond\varphi \overset{def}{=} \neg\bigcirc\neg\varphi$, and $\Diamond^{-1}\varphi \overset{def}{=} \neg\bigcirc^{-1}\neg\varphi$. Thus the interpretation of $\Diamond\varphi$ is "at *some* next instant it will be the case that φ " while the interpretation of $\Diamond^{-1}\varphi$ is "at some previous instant it was the case that φ".

We denote by $\mathbb{L}_0$ the basic logic of belief revision defined by the following axioms and rules of inference.

AXIOMS:

1. All propositional tautologies.
2. Axiom K for $\bigcirc$, $\bigcirc^{-1}$, B and A:
$$(\Box\varphi \wedge \Box(\varphi \rightarrow \psi)) \rightarrow \Box\psi \quad \text{for } \Box \in \{\bigcirc, \bigcirc^{-1}, B, A\} \quad \text{(K)}$$
3. Temporal axioms relating $\bigcirc$ and $\bigcirc^{-1}$:
$$\varphi \rightarrow \bigcirc\Diamond^{-1}\varphi \quad (\mathrm{O}_1)$$
$$\varphi \rightarrow \bigcirc^{-1}\Diamond\varphi \quad (\mathrm{O}_2)$$
4. Backward Uniqueness axiom:
$$\Diamond^{-1}\varphi \rightarrow \bigcirc^{-1}\varphi \quad \text{(BU)}$$
5. S5 axioms for A:
$$A\varphi \rightarrow \varphi \quad (\mathrm{T}_A)$$
$$\neg A\varphi \rightarrow A\neg A\varphi \quad (5_A)$$
6. Inclusion axiom for B (note the absence of an analogous axiom for I):
$$A\varphi \rightarrow B\varphi \quad (\mathrm{Incl}_B)$$
7. Axioms to capture the non-standard semantics for I:

$$(I\varphi \wedge I\psi) \rightarrow A(\varphi \leftrightarrow \psi) \qquad (\mathrm{I}_1)$$
$$A(\varphi \leftrightarrow \psi) \rightarrow (I\varphi \leftrightarrow I\psi) \qquad (\mathrm{I}_2)$$

RULES OF INFERENCE:

1. Modus Ponens: $\frac{\varphi,\ \varphi\rightarrow\psi}{\psi}$ (*MP*)
2. Necessitation for A, $\bigcirc$ and $\bigcirc^{-1}$: $\frac{\varphi}{\Box\varphi}$ for $\Box \in \{\bigcirc, \bigcirc^{-1}, A\}$ (*Nec*).

Note that from *MP*, $Incl_B$ and Necessitation for A one can derive necessitation for B ($\frac{\varphi}{B\varphi}$). On the other hand, necessitation for I is *not* a rule of inference of this logic (indeed it is not validity preserving).

3 Semantics

On the semantic side we consider branching-time structures with the addition of a set of states, a belief relation and an information relation for every instant t.

DEFINITION 1. A *next-time branching frame* is a pair $\langle T, \rightarrowtail \rangle$ where T is a (possibly infinite) set of instants or dates and $\rightarrowtail$ is a binary relation on T satisfying the following properties: $\forall t_1, t_2, t_3 \in T$,

(1)	uniqueness	if $t_1 \rightarrowtail t_3$ and $t_2 \rightarrowtail t_3$ then $t_1 = t_2$,
(2)	acyclicity	if $\langle t_1, ..., t_n \rangle$ is a sequence with $t_i \rightarrowtail t_{i+1}$ for every $i = 1, ..., n-1$, then $t_n \neq t_1$.

The interpretation of $t_1 \rightarrowtail t_2$ is that t_2 is an *immediate successor* of t_1 or t_1 is the *immediate predecessor* of t_2: every instant has at most one unique immediate predecessor but can have several immediate successors.

Given a next-time branching frame $\langle T, \rightarrowtail \rangle$, we denote by $\prec$ the transitive closure of $\rightarrowtail$. Thus, for $t, t' \in T$, $t \prec t'$ if and only if there is a sequence $\langle t_1, ..., t_n \rangle$ in T such that $t_1 = t$, $t_n = t'$ and $t_i \rightarrowtail t_{i+1}$ for all $i = 1, ..., n-1$. The interpretation of $t \prec t'$ is that t is a *predecessor of* t' or t' is a *successor* of t.

REMARK 2. (Backward linearity of $\prec$). It is straightforward to show that if $t_0, t_1, t_2 \in T$ are such that $t_0 \prec t_2$ and $t_1 \prec t_2$ then either $t_0 = t_1$ or $t_0 \prec t_1$ or $t_1 \prec t_0$.

DEFINITION 3. A *general temporal belief revision frame* is a tuple $\langle T, \rightarrowtail, \Omega, \{\Omega_t, \mathcal{B}_t, \mathcal{I}_t\}_{t\in T}\rangle$, where $\langle T, \rightarrowtail \rangle$ is a next-time branching frame; Ω is a set of states; for every $t \in T$, $\varnothing \neq \Omega_t \subseteq \Omega$; and $\mathcal{B}_t$ and $\mathcal{I}_t$ are binary relations on Ω_t.

The interpretation of $\omega \mathcal{I}_t \omega'$ is that at state ω and time t according to the information received it is possible that the true state is ω'. On the other hand, the interpretation of $\omega \mathcal{B}_t \omega'$ is that at state ω and time t in light of the information received (if any) the individual considers state ω' possible (an alternative expression is "ω' is a doxastic alternative to ω at time t"). We shall use the following notation:

$$\mathcal{B}_t(\omega) = \{\omega' \in \Omega_t : \omega \mathcal{B}_t \omega'\} \text{ and, similarly, } \mathcal{I}_t(\omega) = \{\omega' \in \Omega_t : \omega \mathcal{I}_t \omega'\}.$$

Thus $\mathcal{B}_t(\omega)$ is the set of states that are reachable from ω according to the relation $\mathcal{B}_t$ and similarly for $\mathcal{I}_t(\omega)$.

General temporal belief revision frames can be used to describe either a situation where the objective facts describing the world do not change − so that only the beliefs of the agent change over time − or a situation where both the facts and the doxastic state of the agent change. In the computer science literature the first situation is called belief revision, while the latter is called belief update (Katsuno and Mendelzon [8]). In this paper we restrict attention to belief revision.

DEFINITION 4. Given a general temporal belief revision frame, define the binary relation $\hookrightarrow$ on $\Omega \times T$ as follows: $(\omega, t) \hookrightarrow (\omega', t')$ if and only if (1) $\omega = \omega'$, (2) $\omega \in \Omega_t \cap \Omega_{t'}$ and either (3a) $t \rightarrowtail t'$ or (3b) $t \prec t'$ and, for every $x \in T$ if $t \prec x$ and $x \prec t'$ then $\omega \notin \Omega_x$.

The interpretation of $(\omega, t) \hookrightarrow (\omega, t')$ is that, from the point of view of state ω, instant t is the immediate predecessor of t'. Thus the immediate predecessor of an instant can be different at different states.[1]

Given a general temporal belief revision frame $\langle T, \rightarrowtail, \Omega, \{\Omega_t, \mathcal{B}_t, \mathcal{I}_t\}_{t \in T}\rangle$ one obtains a *model based on it* by adding a function $V : S \to 2^{\Omega}$ (where S is the set of atomic propositions and 2^{Ω} denotes the set of subsets of Ω) that associates with every atomic proposition q the set of states at which q is true. Note that defining a valuation this way is what frames the problem as one of belief revision, since the truth value of an atomic proposition depends only on the state and not on the date.[2] Given a model, a formula φ, an instant t and a state ω such that $\omega \in \Omega_t$, we write $(\omega, t) \models \varphi$ to denote that φ is true at state ω and time t. Let $\|\varphi\|$ denote the truth set of φ, that is, $\|\varphi\| = \{(\omega, t) \in \Omega \times T : \omega \in \Omega_t \text{ and } (\omega, t) \models \varphi\}$ and let $\lceil \varphi \rceil_t \subseteq \Omega_t$ denote the set of states at which φ is true *at time* t, that is, $\lceil \varphi \rceil_t = \{\omega \in \Omega_t : (\omega, t) \models \varphi\}$. Truth at a pair (ω, t) is defined recursively as follows:

[1] A special class of general temporal belief revision frames is the class that satisfies the resctriction that, for every $t \in T$, $\Omega_t = \Omega$. It is straightforward to show that, within this class, $(\omega, t) \hookrightarrow (\omega', t')$ if and only if $\omega = \omega'$ and $t \rightarrowtail t'$, so that the immediate predecessor of an instant t is the same at every state. This is the class of frames called 'temporal belief revision frames' in [3]. Hence the addition of the adjective 'general' in Definition 3.

[2] Belief update would require a valuation to be defined as a function $V : S \to 2^X$ where $X = \{(\omega, t) \in \Omega \times T : \omega \in \Omega_t\}$.

if $q \in S$,	$(\omega, t) \models q$ if and only if $\omega \in \Omega_t$ and $\omega \in V(q)$.
$(\omega, t) \models \neg\varphi$	if and only if $\omega \in \Omega_t$ and $(\omega, t) \nvDash \varphi$.
$(\omega, t) \models \varphi \vee \psi$	if and only if either $(\omega, t) \models \varphi$ or $(\omega, t) \models \psi$ (or both).
$(\omega, t) \models \bigcirc\varphi$	if and only if, for all $t' \in T$, if $(\omega, t) \hookrightarrow (\omega, t')$ then $(\omega, t') \models \varphi$.
$(\omega, t) \models \bigcirc^{-1}\varphi$	if and only if, for all $t'' \in T$, if $(\omega, t'') \hookrightarrow (\omega, t)$ then $(\omega, t'') \models \varphi$.
$(\omega, t) \models B\varphi$	if and only if $\mathcal{B}_t(\omega) \subseteq \lceil\varphi\rceil_t$, that is, if $(\omega', t) \models \varphi$ for all $\omega' \in \mathcal{B}_t(\omega)$.
$(\omega, t) \models I\varphi$	if and only if $\mathcal{I}_t(\omega) = \lceil\varphi\rceil_t$, that is, if (1) $(\omega', t) \models \varphi$ for all $\omega' \in \mathcal{I}_t(\omega)$, and (2) if $(\omega', t) \models \varphi$ then $\omega' \in \mathcal{I}_t(\omega)$.
$(\omega, t) \models A\varphi$	if and only if $\lceil\varphi\rceil_t = \Omega_t$, that is, if $(\omega', t) \models \varphi$ for all $\omega' \in \Omega_t$.

Note that, while the truth condition for the operator B is the standard one, the truth condition for the operator I is non-standard: instead of simply requiring that $\mathcal{I}_t(\omega) \subseteq \lceil\varphi\rceil_t$ we require equality: $\mathcal{I}_t(\omega) = \lceil\varphi\rceil_t$. Thus our information operator is formally similar to the "all I know" operator introduced by Levesque [9], although the interpretation is different.

A formula φ is *valid in a model* if $(\omega, t) \models \varphi$ for every $(\omega, t) \in \Omega \times T$ with $\omega \in \Omega_t$. A formula φ is *valid in a frame* if it is valid in every model based on it.

4 Soundness and completeness

PROPOSITION 5. *Logic $\mathbb{L}_0$ is sound with respect to the class of general temporal belief revision frames, that is, every theorem of $\mathbb{L}_0$ is valid in every general temporal belief revision frame.*

Proof. We need to show that (1) the rules of inference are validity preserving and (2) the axioms of $\mathbb{L}_0$ are valid in an arbitrary general temporal belief revision frame.

The proof of (1) is entirely standard and is omitted. The proof of validity of axiom K for $\bigcirc$, $\bigcirc^{-1}$, B and A and for the temporal axioms (O_1) and (O_2) is also standard and is omitted.

Validity of the backward uniqueness axiom $\Diamond^{-1}\varphi \rightarrow \bigcirc^{-1}\varphi$. Let (ω, t) be such that $(\omega, t) \models \Diamond^{-1}\varphi$. Then there exists a $t' \in T$ such that $(\omega, t') \hookrightarrow (\omega, t)$ and $(\omega, t') \models \varphi$. By Definition 4,

$$\text{(1)} \qquad \omega \in \Omega_{t'},\ t' \prec t \text{ and either } t' \rightarrowtail t \text{ or, for every } x \in T \text{ such that } t' \prec x \text{ and } x \prec t,\ \omega \notin \Omega_x.$$

Fix an arbitrary $t_0 \in T$ and suppose that $(\omega, t_0) \hookrightarrow (\omega, t)$. Then, by Definition 4,

$$\text{(2)} \qquad \omega \in \Omega_{t_0},\ t_0 \prec t \text{ and either } t_0 \rightarrowtail t \text{ or, for every } x \in T \text{ such that } t_0 \prec x \text{ and } x \prec t,\ \omega \notin \Omega_x.$$

We want to show that $t_0 = t'$, so that $(\omega, t_0) \models \varphi$ and, therefore, $(\omega, t) \models \bigcirc^{-1}\varphi$. Since $t' \prec t$ and $t_0 \prec t$, by backward linearity of $\prec$ (see Remark 2), either $t_0 = t'$ or $t_0 \prec t'$ or $t' \prec t_0$. The case $t_0 \prec t'$ contradicts (2) since, by (1), $t' \prec t$ and $\omega \in \Omega_{t'}$ (note that by definition of branching-time frame - see Definition 1 - if $t_0 \rightarrowtail t$ then there is no x such that $t_0 \prec x$ and $x \prec t$). Similarly, the case $t' \prec t_0$ contradicts (1), since, by (2), $t_0 \prec t$ and $\omega \in \Omega_{t_0}$. Thus $t_0 = t'$.

Validity of the S5 axioms for A. Suppose that $(\omega, t) \models A\varphi$. Then $(\omega', t) \models \varphi$ for every $\omega' \in \Omega_t$, thus in particular for $\omega' = \omega$. Similarly, if $(\omega, t) \models \neg A\varphi$ then there exists an $\omega' \in \Omega_t$ such that $(\omega', t) \models \neg\varphi$. Hence $(\omega'', t) \models \neg A\varphi$ for every $\omega'' \in \Omega_t$ and, therefore, $(\omega, t) \models A\neg A\varphi$.

The proof of validity of the inclusion axiom for B (Incl_B) is straightforward and is omitted.

Validity of axiom I_1: $I\varphi \wedge I\psi \rightarrow A(\varphi \leftrightarrow \psi)$. Suppose that $(\omega, t) \models I\varphi \wedge I\psi$. Then $\mathcal{I}_t(\omega) = \lceil\varphi\rceil_t$ and $\mathcal{I}_t(\omega) = \lceil\psi\rceil_t$. Thus $\lceil\varphi\rceil_t = \lceil\psi\rceil_t$, so that $\lceil\varphi \leftrightarrow \psi\rceil_t = \Omega_t$, yielding $(\omega, t) \models A(\varphi \leftrightarrow \psi)$.

Validity of axiom I_2: $A(\varphi \leftrightarrow \psi) \rightarrow (I\varphi \leftrightarrow I\psi)$. Suppose that $(\omega, t) \models A(\varphi \leftrightarrow \psi)$. Then $\lceil\varphi \leftrightarrow \psi\rceil_t = \Omega_t$ and, therefore, $\lceil\varphi\rceil_t = \lceil\psi\rceil_t$. Thus, $(\omega, t) \models I\varphi$ if and only if $\mathcal{I}_t(\omega) = \lceil\varphi\rceil_t$, if and only if $\mathcal{I}_t(\omega) = \lceil\psi\rceil_t$, if and only if $(\omega, t) \models I\psi$. Hence $(\omega, t) \models I\varphi \leftrightarrow I\psi$. ∎

PROPOSITION 6. *Logic* $\mathbb{L}_0$ *is complete with respect to the class of general temporal belief revision frames, that is, if* φ *is a formula which is valid in every general temporal belief revision frame then* φ *is a theorem of* $\mathbb{L}_0$.

To prove Proposition 6 we need to show that, for every consistent formula φ, there is a state-instant pair (ω, t) in a model based on a general temporal belief revision frame such that $(\omega, t) \models \varphi$. We follow the constructive approach of Burgess [5]: given a consistent formula φ_0, we construct a chronicle (see Definition 11 below) where φ_0 is true at some state-instant pair and then extend it to a perfect chronicle. First some preliminary definitions and lemmas.

Let $\mathbb{M}_{\mathbb{L}_0}$ denote the set of maximally consistent sets of formulas of logic $\mathbb{L}_0$.

DEFINITION 7. Define the binary relations $\mathcal{A}^c$, $\mathcal{B}^c$ and $\hookrightarrow^c$ on $\mathbb{M}_{\mathbb{L}_0}$ as follows:

1. $m\mathcal{A}^c m'$ if and only if $\{\varphi : A\varphi \in m\} \subseteq m'$, that is, if $A\varphi \in m$ implies $\varphi \in m'$;
2. $m\mathcal{B}^c m'$ if and only if $\{\varphi : B\varphi \in m\} \subseteq m'$, that is, if $B\varphi \in m$ implies $\varphi \in m'$;
3. $m \hookrightarrow^c m'$ if and only if $\{\varphi : \bigcirc\varphi \in m\} \subseteq m'$, that is, if $\bigcirc\varphi \in m$ implies $\varphi \in m'$.

REMARK 8. For every $\Box \in \{A, B, \bigcirc\}$ and for every $m, m' \in \mathbb{M}_{\mathbb{L}_0}$, $\{\varphi : \Box\varphi \in m\} \subseteq m'$ if and only if $\{\neg\Box\neg\varphi : \varphi \in m'\} \subseteq m$ (see Chellas [6] Theorem 4.30(1),

p. 158) . Furthermore, $\mathcal{A}^c$ is an equivalence relation because of axioms T_A and 5_A (Chellas [6] Theorem 5.13 (2) and (5), p. 175), $\mathcal{B}^c$ is a subrelation of $\mathcal{A}^c$ because of axiom Incl_B, and the relation $\hookrightarrow^c$ satisfies the following properties: (1) because of the temporal axioms O_1 and O_2, $m \hookrightarrow^c m'$ if and only if $\{\varphi : \bigcirc^{-1}\varphi \in m'\} \subseteq m$,[3] and (2) because of axiom BU, if $m_1 \hookrightarrow^c m$ and $m_2 \hookrightarrow^c m$ then $m_1 = m_2$.[4]

The following lemma is well-known (see Blackburn et al. [2], Lemma 4.20, p. 198).

LEMMA 9. *Let $m \in \mathbb{M}_{\mathbb{L}_0}$. Then: (1) if $\neg A \neg\varphi \in m$ then there exists an $m' \in \mathbb{M}_{\mathbb{L}_0}$ such that $m\mathcal{A}^c m'$ and $\varphi \in m'$, (2) if $\neg B \neg\varphi \in m$ then there exists an $m' \in \mathbb{M}_{\mathbb{L}_0}$ such that $m\mathcal{B}^c m'$ and $\varphi \in m'$, (3) if $\Diamond\varphi \in m$ then there exists an $m' \in \mathbb{M}_{\mathbb{L}_0}$ such that $m \hookrightarrow^c m'$ and $\varphi \in m'$, (4) if $\Diamond^{-1}\varphi \in m$ then there exists an $m' \in \mathbb{M}_{\mathbb{L}_0}$ such that $m' \hookrightarrow^c m$ and $\varphi \in m'$.*

LEMMA 10. *Let $m, m' \in \mathbb{M}_{\mathbb{L}_0}$ be such that $m\mathcal{A}^c m'$ and let φ be a formula such that $I\varphi \in m$ and $\varphi \in m'$. Then, for every formula ψ, if $I\psi \in m$ then $\psi \in m'$, that is, $\{\psi : I\psi \in m\} \subseteq m'$.*

Proof. Suppose that $m\mathcal{A}^c m'$, $I\varphi \in m$ and $\varphi \in m'$. Fix an arbitrary ψ such that $I\psi \in m$. Then $I\varphi \wedge I\psi \in m$. Since $(I\varphi \wedge I\psi) \to A(\varphi \leftrightarrow \psi)$ is a theorem, it belongs to every MCS, in particular to m. Hence $A(\varphi \leftrightarrow \psi) \in m$. Then, since $m\mathcal{A}^c m'$, $\varphi \leftrightarrow \psi \in m'$. Since $\varphi \in m'$, it follows that $\psi \in m'$. ■

DEFINITION 11. A *chronicle* is a general temporal belief revision frame together with a function $\mu : \{(\omega, t) \in \Omega \times T : \omega \in \Omega_t\} \to \mathbb{M}_{\mathbb{L}_0}$ that associates with every state-instant pair an MCS. A chronicle μ is *coherent* if it satisfies the following properties:

(1) if $\bigcirc\varphi \in \mu(\omega, t)$ and $(\omega, t) \hookrightarrow (\omega, t')$ then $\varphi \in \mu(\omega, t')$, that is, $\mu(\omega, t) \hookrightarrow^c \mu(\omega, t')$;

(2) if $A\varphi \in \mu(\omega, t)$ then, for all $\omega' \in \Omega_t$, $\varphi \in \mu(\omega', t)$, that is, if $\omega' \in \Omega_t$ implies $\mu(\omega, t)\mathcal{A}^c\mu(\omega', t)$;

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[3]Proof. Suppose that $m \hookrightarrow^c m'$ and $\bigcirc^{-1}\varphi \in m'$. Then $\Diamond \bigcirc^{-1} \varphi \in m$. Since the following is an instance of axiom O_1: $\neg\varphi \to \bigcirc\neg\bigcirc^{-1}\varphi$ and is propositionally equivalent to $\Diamond\bigcirc^{-1}\varphi \to \varphi$, it belongs to m. Thus $\varphi \in m$. Conversely, suppose that $\{\varphi : \bigcirc^{-1}\varphi \in m'\} \subseteq m$. Then (see Chellas [6] Theorem 4.30(1), p. 158) $\{\Diamond^{-1}\varphi : \varphi \in m\} \subseteq m'$. We want to show that $m \hookrightarrow^c m'$, that is, that if $\bigcirc\varphi \in m$ then $\varphi \in m'$. Fix an arbitrary φ such that $\bigcirc\varphi \in m$. Then $\Diamond^{-1} \bigcirc \varphi \in m'$. Since axiom O_2 is equivalent to $\Diamond^{-1} \bigcirc \varphi \hookrightarrow^c \varphi$, the latter belongs to m'. Thus $\varphi \in m'$.

[4]Proof. Suppose that $m_1 \hookrightarrow^c m$ and $m_2 \hookrightarrow^c m$ and $m_1 \neq m_2$. Then, by definition of maximally consistent set (MCS), there exists a formula φ such that $\varphi \in m_1$ and $\neg\varphi \in m_2$. It follows that $\Diamond^{-1}\varphi \in m$ and $\Diamond^{-1}\neg\varphi \in m$. By axiom BU, $(\Diamond^{-1}\varphi \to \bigcirc^{-1}\varphi) \in m$ and $(\Diamond^{-1}\neg\varphi \to \bigcirc^{-1}\neg\varphi) \in m$. Thus $(\bigcirc^{-1}\varphi \wedge \bigcirc^{-1}\neg\varphi) \in m$, which implies, since $m_1 \hookrightarrow^c m$, that $\varphi \wedge \neg\varphi \in m_1$, contradicting the definition of MCS.

(3) if $B\varphi \in \mu(\omega, t)$ and $\omega' \in \mathcal{B}_t(\omega)$ then $\varphi \in \mu(\omega', t)$, that is, if $\omega' \in \mathcal{B}_t(\omega)$ implies $\mu(\omega, t)\mathcal{B}^c\mu(\omega', t)$;
(4a) if $I\varphi \in \mu(\omega, t)$ and $\omega' \in \mathcal{I}_t(\omega)$ then $\varphi \in \mu(\omega', t)$;
(4b) if $I\varphi \in \mu(\omega, t)$ and $\omega' \in \Omega_t$ and $\varphi \in \mu(\omega', t)$ then $\omega' \in \mathcal{I}_t(\omega)$.

Let φ_0 be a consistent formula. Then by Lindenbaum's lemma there is an $m_0 \in \mathbb{M}_{\mathbb{L}_0}$ such that $\varphi_0 \in m_0$. Construct the following chronicle: $T = \{t\}$, $\rightarrowtail = \varnothing$, $\Omega = \Omega_t = \{\alpha\}$, $\mathcal{B}_t(\alpha) = \varnothing$,

$$\mathcal{I}_t(\alpha) = \begin{cases} \varnothing & \text{if, for every } \varphi, \varphi \notin m_0 \text{ whenever } I\varphi \in m_0 \\ \{\alpha\} & \text{if, for some } \varphi, I\varphi \in m_0 \text{ and } \varphi \in m_0 \end{cases}$$

and $\mu(\alpha, t) = m_0$.

LEMMA 12. *The above chronicle is coherent.*

Proof. Condition (1) of Definition 11 is satisfied trivially since $\rightarrowtail = \varnothing$. Condition (2) is satisfied because the relation $\mathcal{A}^c$ is reflexive. Condition (3) is satisfied trivially since $\mathcal{B}_t(\alpha) = \varnothing$. Now we turn to conditions (4a) and (4b). If there is no φ such that $I\varphi \in m_0$ and $\varphi \in m_0$, then (4a) is satisfied trivially because, by construction, $\mathcal{I}_t(\alpha) = \varnothing$, and (4b) is satisfied trivially because if $I\varphi \in m_0$ then $\varphi \notin m_0 = \mu(\alpha, t)$. Suppose therefore that, for some φ, $I\varphi \in m_0$ and $\varphi \in m_0$. Fix an arbitrary formula ψ and suppose that $I\psi \in m_0$. It follows from Lemma 10, using the fact that $m_0\mathcal{A}^c m_0$, that $\psi \in m_0$. Thus (4a) and (4b) are satisfied since, by construction, $\mathcal{I}_t(\alpha) = \{\alpha\}$. ∎

DEFINITION 13. Fix a chronicle $\langle \mathcal{R}, \mu \rangle$ where $\mathcal{R} = \langle T, \rightarrowtail, \Omega, \{\Omega_t, \mathcal{B}_t, \mathcal{I}_t\}_{t \in T} \rangle$. We say that the chronicle $\langle \mathcal{R}', \mu' \rangle$, with $\mathcal{R}' = \left\langle T', \rightarrowtail', \Omega', \{\Omega'_t, \mathcal{B}'_t, \mathcal{I}'_t\}_{t \in T'} \right\rangle$ is an *extension of* $\langle \mathcal{R}, \mu \rangle$ if
(1) $T \subseteq T'$,
(2) $\Omega \subseteq \Omega'$, and, for every $t \in T$, $\Omega_t \subseteq \Omega'_t$,
and, identifying relations and functions with sets of ordered pairs,
(3) $\rightarrowtail = \rightarrowtail' \cap (T \times T)$,
(4) for all $t \in T$, $\mathcal{B}_t = \mathcal{B}'_t \cap (\Omega_t \times \Omega_t)$,
(5) for all $t \in T$, $\mathcal{I}_t = \mathcal{I}'_t \cap (\Omega_t \times \Omega_t)$ and
(6) $\mu \subseteq \mu'$.

DEFINITION 14. Fix a chronicle and a pair $(\alpha, t) \in \Omega \times T$ with $\alpha \in \Omega_t$. We say that at (α, t) there is:

- an *A-defect* if there is a formula φ such that $\neg A \neg \varphi \in \mu(\alpha, t)$ and there is no $\omega \in \Omega_t$ such that $\varphi \in \mu(\omega, t)$,

- a B-*defect* if there is a formula φ such that $\neg B\neg\varphi \in \mu(\alpha, t)$ and there is no $\omega \in \mathcal{B}_t(\alpha)$ such that $\varphi \in \mu(\omega, t)$,
- a $\bigcirc^{-1}$-*defect* if there is a formula φ such that $\Diamond^{-1}\varphi \in \mu(\alpha, t)$ and there is no $t' \in T$ such that $(\alpha, t') \hookrightarrow (\alpha, t)$ and $\varphi \in \mu(\alpha, t')$,
- a $\bigcirc$-*defect* if there is a formula φ such that $\Diamond\varphi \in \mu(\alpha, t)$ and there is no $t' \in T$ such that $(\alpha, t) \hookrightarrow (\alpha, t')$ and $\varphi \in \mu(\alpha, t')$.

Note that there is no need to consider the possibility of an I-defect, since $(\alpha, t) \models \neg I\neg\varphi$ does *not* mean that there is an $\omega \in \mathcal{I}_t(\alpha)$ such that $(\omega, t) \models \varphi$ but rather that $\mathcal{I}_t(\alpha) \neq \lceil\neg\varphi\rceil_t$. For example, it could be that $\mathcal{I}_t(\alpha)$ is a proper subset of $\lceil\neg\varphi\rceil_t$.

LEMMA 15. *(Repair Lemma). Fix a coherent chronicle $\langle\mathcal{R}, \mu\rangle$ where T and Ω are finite sets. Suppose that there is a defect at (α, t). Then there exists a finite coherent extension $\langle\mathcal{R}', \mu'\rangle$ of $\langle\mathcal{R}, \mu\rangle$ where that defect at (α, t) is no longer present.*

Proof. Let D be a countably infinite set containing T and W a countably infinite set containing Ω.

Suppose first that there is an A-**defect** at (α, t_1), that is, there is a formula φ such that $\neg A\neg\varphi \in \mu(\alpha, t_1)$ and there is no $\omega \in \Omega_{t_1}$ such that $\varphi \in \mu(\omega, t_1)$. By Lemma 9 there is an $\hat{m} \in \mathbb{M}_{\mathbb{L}_0}$ such that $\mu(\alpha, t_1)\mathcal{A}^c\hat{m}$ and $\varphi \in \hat{m}$. Construct the following extension of $\langle\mathcal{R}, \mu\rangle$: $T' = T$; $\rightarrowtail' = \rightarrowtail$; let $\hat{\omega} \in W\backslash\Omega$ and define $\Omega' = \Omega \cup \{\hat{\omega}\}$; for every $t \in T\backslash\{t_1\}$, let $\Omega'_t = \Omega_t$, $\mathcal{B}'_t = \mathcal{B}_t$ and $\mathcal{I}'_t = \mathcal{I}_t$; let $\Omega'_{t_1} = \Omega_{t_1} \cup \{\hat{\omega}\}$ and $\mu'(\hat{\omega}, t_1) = \hat{m}$; for $\omega \in \Omega_{t_1}$, $\mathcal{B}'_{t_1}(\omega) = \mathcal{B}_{t_1}(\omega)$ and $\mathcal{B}'_{t_1}(\hat{\omega}) = \varnothing$; let $\mathcal{I}'_{t_1}$ be defined as follows:

(i) for every $\omega \in \Omega_{t_1}$, $\mathcal{I}'_{t_1}(\omega) = \begin{cases} \mathcal{I}_{t_1}(\omega) & \text{if there is no } \psi \text{ such that } I\psi \in \mu(\omega, t_1) \text{ and } \psi \in \hat{m} \\ \mathcal{I}_{t_1}(\omega) \cup \{\hat{\omega}\} & \text{if there is a } \psi \text{ such that } I\psi \in \mu(\omega, t_1) \text{ and } \psi \in \hat{m} \end{cases}$

and

(ii) for every $\omega \in \Omega_{t_1} \cup \{\hat{\omega}\}$, $\omega \in \mathcal{I}'_{t_1}(\hat{\omega})$ if and only if, for some ψ, $I\psi \in \hat{m}$ and $\psi \in \mu'(\omega, t_1)$.

Since $\hat{\omega} \in \Omega'_{t_1}$, $\mu'(\hat{\omega}, t_1) = \hat{m}$ and $\varphi \in \hat{m}$, the A-defect at (α, t_1) is no longer present in $\langle\mathcal{R}', \mu'\rangle$ so defined. We need to show that $\langle\mathcal{R}', \mu'\rangle$ is coherent. Since, by hypothesis, $\langle\mathcal{R}, \mu\rangle$ is coherent and $T' = T$, condition (1) of Definition 11 is satisfied (note that, since $\hat{\omega} \notin \Omega$, $\hat{\omega} \notin \Omega'_t = \Omega_t$ for every $t \neq t_1$). For condition (2) we need to show that if $t \in T' = T$ and $\omega, \omega' \in \Omega'_t$, then $\mu'(\omega, t)\mathcal{A}^c\mu'(\omega', t)$. If $\omega, \omega' \in \Omega$ then it follows from the hypothesis that $\langle\mathcal{R}, \mu\rangle$ is coherent. If $t = t_1$ and $\omega = \omega' = \hat{\omega}$ then it follows from the fact that $\mathcal{A}^c$ is reflexive. If $t = t_1$, $\omega \in \Omega_{t_1}$ and $\omega' = \hat{\omega}$ then it follows from the fact that (i) $\mu(\omega, t_1)\mathcal{A}^c\mu(\alpha, t_1)$, by the hypothesis that $\langle \mathcal{R}, \mu \rangle$ is coherent, (ii) $\mu(\alpha, t_1)\mathcal{A}^c \hat{m}$, by construction, and (iii) transitivity of $\mathcal{A}^c$. Finally, if $t = t_1$, $\omega = \hat{\omega}$ and $\omega' \in \Omega_{t_1}$ then

1.	$\mu(\alpha, t_1)\mathcal{A}^c \hat{m}$	by construction
2.	$\hat{m}\mathcal{A}^c \mu(\alpha, t_1)$	by 1 and symmetry of $\mathcal{A}^c$
3.	$\mu(\alpha, t_1)\mathcal{A}^c \mu(\omega', t_1)$	by coherence of $\langle \mathcal{R}, \mu \rangle$
4.	$\hat{m}\mathcal{A}^c \mu(\omega', t_1)$	by 2, 3 and transitivity of $\mathcal{A}^c$.

Condition (3) of Definition 11 is satisfied, since (i) for $t \in T$ and $\omega \in \Omega_t$, $\mathcal{B}'_t(\omega) = \mathcal{B}_t(\omega)$ and by hypothesis $\langle \mathcal{R}, \mu \rangle$ is coherent and (ii) $\mathcal{B}'_{t_1}(\hat{\omega}) = \varnothing$ and thus the condition holds trivially. Now we turn to condition (4a). Suppose that $I\psi \in \mu'(\omega, t)$ and $\omega' \in \mathcal{I}'_t(\omega)$. If $t \in T\backslash\{t_1\}$ or if $t = t_1$ and $\omega, \omega' \in \Omega_{t_1}$ then $\psi \in \mu'(\omega', t) = \mu(\omega', t)$ by coherence of $\langle \mathcal{R}, \mu \rangle$. If $t = t_1$, $\omega \in \Omega_{t_1}$ and $\omega' = \hat{\omega}$, then, by construction (since, by hypothesis, $\hat{\omega} \in \mathcal{I}'_{t_1}(\omega)$) there is a χ such that $I\chi \in \mu(\omega, t_1)$ and $\chi \in \hat{m}$; since, by hypothesis, $I\psi \in \mu'(\omega, t_1) = \mu(\omega, t_1)$ and, as shown above, $\mu(\omega, t_1)\mathcal{A}^c \hat{m}$, it follows from Lemma 10 that $\psi \in \hat{m}$. If $t = t_1$, $\omega = \hat{\omega}$ and $\omega' \in \Omega_{t_1}$ then, by construction, there is a χ such that $I\chi \in \hat{m}$ and $\chi \in \mu(\omega', t_1)$; since $\hat{m}\mathcal{A}^c \mu(\omega', t_1)$ (shown above) it follows from Lemma 10 that $\psi \in \mu(\omega', t_1)$. Finally, if $\omega = \omega' = \hat{\omega}$ then, by construction, there is a χ such that $I\chi \in \hat{m}$ and $\chi \in \hat{m}$. Since $\hat{m}\mathcal{A}^c \hat{m}$, it follows from Lemma 10 that $\psi \in \hat{m}$. Next we turn to condition (4b). Fix arbitrary $t \in T' = T$ and $\omega, \omega' \in \Omega'_t$ and suppose that $I\psi \in \mu'(\omega, t)$ and $\psi \in \mu'(\omega', t)$. We need to show that $\omega' \in \mathcal{I}'_t(\omega)$. If $t \in T\backslash\{t_1\}$ it follows from coherence $\langle \mathcal{R}, \mu \rangle$, since $\Omega'_t = \Omega_t$ and $\mathcal{I}'_t = \mathcal{I}_t$. Similarly if $t = t_1$ and $\omega, \omega' \in \Omega_{t_1}$. If $t = t_1$ and $\omega = \omega' = \hat{\omega}$ then, since, by hypothesis, $I\psi \in \hat{m}$ and $\psi \in \hat{m}$, it follows, by construction, that $\hat{\omega} \in \mathcal{I}'_{t_1}(\hat{\omega})$. Similarly for $t = t_1$, $\omega' \in \Omega_{t_1}$ and $\omega = \hat{\omega}$. Finally, if $t = t_1$, $\omega \in \Omega_{t_1}$ and $\omega' = \hat{\omega}$, then $I\psi \in \mu(\omega, t_1)$ and $\psi \in \hat{m}$ and, by construction, $\hat{\omega} \in \mathcal{I}'_{t_1}(\omega)$.

Suppose now that there is a B-**defect** at (α, t_1), that is, there is a formula φ such that $\neg B\neg\varphi \in \mu(\alpha, t_1)$ and there is no $\omega \in \mathcal{B}_{t_1}(\alpha)$ such that $\varphi \in \mu(\omega, t_1)$. By Lemma 9 there is an $\hat{m} \in \mathbb{M}_{\mathbb{L}_0}$ such that $\mu(\alpha, t_1)\mathcal{B}^c \hat{m}$ and $\varphi \in \hat{m}$. Construct the following extension of $\langle \mathcal{R}, \mu \rangle$: $T' = T$; $\rightarrowtail' = \rightarrowtail$; let $\hat{\omega} \in W\backslash\Omega$ and define $\Omega' = \Omega \cup \{\hat{\omega}\}$; for every $t \in T\backslash\{t_1\}$, let $\Omega'_t = \Omega_t$, $\mathcal{B}'_t = \mathcal{B}_t$ and $\mathcal{I}'_t = \mathcal{I}_t$; let $\Omega'_{t_1} = \Omega_{t_1} \cup \{\hat{\omega}\}$ and $\mu'(\hat{\omega}, t_1) = \hat{m}$; for $\omega \in \Omega_{t_1}\backslash\{\alpha\}$, let $\mathcal{B}'_{t_1}(\omega) = \mathcal{B}_{t_1}(\omega)$; let $\mathcal{B}'_{t_1}(\alpha) = \mathcal{B}_{t_1}(\alpha) \cup \{\hat{\omega}\}$ and $\mathcal{B}'_{t_1}(\hat{\omega}) = \varnothing$; let $\mathcal{I}'_{t_1}$ be defined as follows:

$$\text{(i) for every } \omega \in \Omega_{t_1},\ \mathcal{I}'_{t_1}(\omega) = \begin{cases} \mathcal{I}_{t_1}(\omega) & \text{if there is no } \psi \text{ such that } I\psi \in \mu(\omega, t_1) \text{ and } \psi \in \hat{m} \\ \mathcal{I}_{t_1}(\omega) \cup \{\hat{\omega}\} & \text{if there is a } \psi \text{ such that } I\psi \in \mu(\omega, t_1) \text{ and } \psi \in \hat{m} \end{cases}$$

and

(ii) for every $\omega \in \Omega_{t_1} \cup \{\hat{\omega}\}$, $\omega \in \mathcal{I}'_{t_1}(\hat{\omega})$ if and only if, for some ψ, $I\psi \in \hat{m}$ and $\psi \in \mu'(\omega, t_1)$.

Since $\hat{\omega} \in \mathcal{B}'_{t_1}(\alpha)$, $\mu'(\hat{\omega}, t_1) = \hat{m}$ and $\varphi \in \hat{m}$, the B-defect at (α, t_1) is no longer present in $\langle \mathcal{R}', \mu' \rangle$ so defined. Since $\mathcal{B}^c$ is a subrelation of $\mathcal{A}^c$ (see Remark 8), the proof that $\langle \mathcal{R}', \mu' \rangle$ is coherent is identical to the previous proof (condition (3) of Definition 11 is satisfied by construction).

Next we consider the case of a $\bigcirc^{-1}$**-defect** at (α, t_1), that is, there is a formula φ such that $\Diamond^{-1}\varphi \in \mu(\alpha, t_1)$ and there is no $t \in T$ such that $(\alpha, t) \hookrightarrow (\alpha, t_1)$ and $\varphi \in \mu(\alpha, t)$. Let T_1 be the set of predecessors of t_1 in T, that is, $T_1 = \{t \in T : t \prec t_1\}$. Suppose first that $T_1 = \varnothing$. By Lemma 9 there is an $\hat{m} \in \mathbb{M}_{\mathbb{L}_0}$ such that $\hat{m} \hookrightarrow^c \mu(\alpha, t_1)$ and $\varphi \in \hat{m}$. Construct the following extension of $\langle \mathcal{R}, \mu \rangle$: let $\hat{t} \in D \backslash T$ and define $T' = T \cup \{\hat{t}\}$; $\rightarrowtail' = \rightarrowtail \cup \{(\hat{t}, t_1)\}$; $\Omega' = \Omega$; for every $t \in T$, $\Omega'_t = \Omega_t$, $\mathcal{B}'_t = \mathcal{B}_t$ and $\mathcal{I}'_t = \mathcal{I}_t$; let $\Omega'_{\hat{t}} = \{\alpha\}$, $\mathcal{B}'_{\hat{t}}(\alpha) = \varnothing$ and

$$\mathcal{I}'_{\hat{t}}(\alpha) = \begin{cases} \varnothing & \text{if, for every } \psi,\ \psi \notin \hat{m} \text{ whenever } I\psi \in \hat{m} \\ \{\alpha\} & \text{if, for some } \psi,\ I\psi \in \hat{m} \text{ and } \psi \in \hat{m}. \end{cases}$$

Finally, let $\mu'(\alpha, \hat{t}) = \hat{m}$. Clearly the $\bigcirc^{-1}$-defect at (α, t_1) is no longer present in $\langle \mathcal{R}', \mu' \rangle$ so defined. For $t \in T$, coherence of $\langle \mathcal{R}', \mu' \rangle$ follows from coherence of $\langle \mathcal{R}, \mu \rangle$. Thus we only need to consider $t = \hat{t}$. Condition (1) of Definition 11 is satisfied trivially, since $\hat{t}$ has no predecessors in T'. Condition (2) follows from the fact that, by construction, $\Omega'_{\hat{t}} = \{\alpha\}$ and, by reflexivity of $\mathcal{A}^c$, $\hat{m}\mathcal{A}^c\hat{m}$. Condition (3) is satisfied trivially, since, by construction, $\mathcal{B}'_{\hat{t}}(\alpha) = \varnothing$. Now we turn to conditions (4a) and (4b). If, for every ψ, $I\psi \in \hat{m}$ implies $\psi \notin \hat{m}$, then (4a) is satisfied trivially because, by construction, $\mathcal{I}_t(\alpha) = \varnothing$ and (4b) is satisfied trivially because $\psi \notin \hat{m}$. Suppose therefore that, for some ψ, $I\psi \in \hat{m}$ and $\psi \in \hat{m}$. Fix an arbitrary formula χ and suppose that $I\chi \in \hat{m}$. It follows from Lemma 10, using the fact that $\hat{m}\mathcal{A}^c\hat{m}$, that $\chi \in \hat{m}$. Thus (4a) and (4b) are satisfied since, by construction, $\mathcal{I}_{\hat{t}}(\alpha) = \{\alpha\}$ and $\mu'(\alpha, \hat{t}) = \hat{m}$.

Consider now the case where $T_1 \neq \varnothing$. By hypothesis, for every $t \in T_1$, it is not the case that $(\alpha, t) \hookrightarrow (\alpha, t_1)$.[5] Thus, by Definition 4, it must be that,

$$\text{(3)} \qquad \text{for all } \ t \in T_1, \ \ \alpha \notin \Omega_t.$$

Let t_0 be the farthest predecessor of t_1 in T, that is, $t_0 \in T_1$ and, for every $t \in T_1$, either $t = t_0$ or $t_0 \prec t$ (such a t_0 exists because of backward linearity of $\prec$ and finiteness of T). By Lemma 9 there is an $\hat{m} \in \mathbb{M}_{\mathbb{L}_0}$ such that $\hat{m} \hookrightarrow^c \mu(\alpha, t_1)$ and $\varphi \in \hat{m}$. Construct the following extension of $\langle \mathcal{R}, \mu \rangle$: let $\hat{t} \in D \backslash T$ and define $T' = T \cup \{\hat{t}\}$; $\rightarrowtail' = \rightarrowtail \cup \{(\hat{t}, t_0)\}$; $\Omega' = \Omega$; for every $t \in T$, $\Omega'_t = \Omega_t$, $\mathcal{B}'_t = \mathcal{B}_t$ and $\mathcal{I}'_t = \mathcal{I}_t$; let $\Omega'_{\hat{t}} = \{\alpha\}$, $\mathcal{B}'_{\hat{t}}(\alpha) = \varnothing$ and

$$\mathcal{I}'_{\hat{t}}(\alpha) = \begin{cases} \varnothing & \text{if, for every } \psi,\ \psi \notin \hat{m} \text{ whenever } I\psi \in \hat{m} \\ \{\alpha\} & \text{if, for some } \psi,\ I\psi \in \hat{m} \text{ and } \psi \in \hat{m}. \end{cases}$$

Let $\mu'(\alpha, \hat{t}) = \hat{m}$. By (3) and Definition 4, $(\alpha, \hat{t}) \hookrightarrow' (\alpha, t_1)$ and therefore the $\bigcirc^{-1}$-defect at (α, t_1) is no longer present in $\langle \mathcal{R}', \mu' \rangle$ so defined. The proof that $\langle \mathcal{R}', \mu' \rangle$

[5] If there were a $t \in T_1$ such that $(\alpha, t) \hookrightarrow (\alpha, t_1)$ then, by coherence of $\langle \mathcal{R}, \mu \rangle$, we would have that $\varphi \in \mu(\alpha, t)$, since $\Diamond^{-1}\varphi \in \mu(\alpha, t_1)$, contradicting our hypothesis.

is coherent is the same as in the previous case.

Finally we consider the case of a $\bigcirc$**-defect** at (α, t_1), that is, there is a formula φ such that $\Diamond\varphi \in \mu(\alpha, t_1)$ and there is no $t \in T$ such that $(\alpha, t_1) \hookrightarrow (\alpha, t)$ and $\varphi \in \mu(\alpha, t)$. Let T_2 be the set of successors of t_1 in T, that is, $T_2 = \{t \in T : t_1 \prec t\}$. Suppose first that $T_2 = \varnothing$. By Lemma 9 there is an $\hat{m} \in \mathbb{M}_{\mathbb{L}_0}$ such that $\mu(\alpha, t_1) \hookrightarrow^c \hat{m}$ and $\varphi \in \hat{m}$. Construct the following extension of $\langle \mathcal{R}, \mu \rangle$: let $\hat{t} \in D \backslash T$ and define $T' = T \cup \{\hat{t}\}$; $\rightarrowtail' = \rightarrowtail \cup \{(t_1, \hat{t})\}$; $\Omega' = \Omega$; for every $t \in T$, $\Omega'_t = \Omega_t$, $\mathcal{B}'_t = \mathcal{B}_t$ and $\mathcal{I}'_t = \mathcal{I}_t$; let $\Omega'_{\hat{t}} = \{\alpha\}$, $\mathcal{B}'_{\hat{t}}(\alpha) = \varnothing$ and

$$\mathcal{I}'_{\hat{t}}(\alpha) = \begin{cases} \varnothing & \text{if, for every } \psi,\ \psi \notin \hat{m} \text{ whenever } I\psi \in \hat{m} \\ \{\alpha\} & \text{if, for some } \psi,\ I\psi \in \hat{m} \text{ and } \psi \in \hat{m}. \end{cases}$$

Finally, let $\mu'(\alpha, \hat{t}) = \hat{m}$. Clearly the $\bigcirc$-defect at (α, t_1) is no longer present in $\langle \mathcal{R}', \mu' \rangle$ so defined. For $t \in T$, coherence of $\langle \mathcal{R}', \mu' \rangle$ follows from coherence of $\langle \mathcal{R}, \mu \rangle$. Thus we only need to consider $t = \hat{t}$. Condition (1) of Definition 11 is satisfied trivially, since $\hat{t}$ has no successors in T'. Condition (2) follows from the fact that, by construction, $\Omega_{\hat{t}} = \{\alpha\}$ and, by reflexivity of $\mathcal{A}^c$, $\hat{m}\mathcal{A}^c\hat{m}$. Condition (3) is satisfied trivially, since, by construction, $\mathcal{B}'_{\hat{t}}(\alpha) = \varnothing$. Now we turn to conditions (4a) and (4b). If, for every ψ, $I\psi \in \hat{m}$ implies $\psi \notin \hat{m}$, then (4a) is satisfied trivially because, by construction, $\mathcal{I}_{\hat{t}}(\alpha) = \varnothing$ and (4b) is satisfied trivially because $\psi \notin \hat{m}$. Suppose therefore that, for some ψ, $I\psi \in \hat{m}$ and $\psi \in \hat{m}$. Fix an arbitrary formula χ and suppose that $I\chi \in \hat{m}$. It follows from Lemma 10, using the fact that $\hat{m}\mathcal{A}^c\hat{m}$, that $\chi \in \hat{m}$. Thus (4a) and (4b) are satisfied since, by construction, $\mathcal{I}_{\hat{t}}(\alpha) = \{\alpha\}$.

Consider now the case where $T_2 \neq \varnothing$. By hypothesis, for every $t \in T_2$, it is not the case that $(\alpha, t_1) \hookrightarrow (\alpha, t)$.[6] Thus, by Definition 4, it must be that,

$$\text{(4)} \qquad \text{for all } t \in T_2, \ \ \alpha \notin \Omega_t.$$

Let t_2 be any successor of t_1 in T with no successors of its own, that is, $t_2 \in T_2$ and, for every $t \in T$, $t_2 \not\prec t$ (such a t_2 exists because of finiteness of T). By Lemma 9 there is an $\hat{m} \in \mathbb{M}_{\mathbb{L}_0}$ such that $\mu(\alpha, t_1) \hookrightarrow^c \hat{m}$ and $\varphi \in \hat{m}$. Construct the following extension of $\langle \mathcal{R}, \mu \rangle$: let $\hat{t} \in D \backslash T$ and define $T' = T \cup \{\hat{t}\}$; $\rightarrowtail' = \rightarrowtail \cup \{(t_2, \hat{t})\}$; $\Omega' = \Omega$; for every $t \in T$, $\Omega'_t = \Omega_t$, $\mathcal{B}'_t = \mathcal{B}_t$ and $\mathcal{I}'_t = \mathcal{I}_t$; let $\Omega'_{\hat{t}} = \{\alpha\}$, $\mathcal{B}'_{\hat{t}}(\alpha) = \varnothing$ and

$$\mathcal{I}'_{\hat{t}}(\alpha) = \begin{cases} \varnothing & \text{if, for every } \psi,\ \psi \notin \hat{m} \text{ whenever } I\psi \in \hat{m} \\ \{\alpha\} & \text{if, for some } \psi,\ I\psi \in \hat{m} \text{ and } \psi \in \hat{m}. \end{cases}$$

Let $\mu'(\alpha, \hat{t}) = \hat{m}$. By (4) and Definition 4, $(\alpha, t_1) \hookrightarrow' (\alpha, \hat{t})$ and therefore the $\bigcirc$-defect at (α, t_1) is no longer present in $\langle \mathcal{R}', \mu' \rangle$ so defined. The proof that $\langle \mathcal{R}', \mu' \rangle$ is coherent is the same as in the previous case. ■

[6] If there were a $t \in T_2$ such that $(\alpha, t_1) \hookrightarrow (\alpha, t)$ then by coherence of $\langle \mathcal{R}, \mu \rangle$ we would have that $\varphi \in \mu(\alpha, t)$, since $\Diamond\varphi \in \mu(\alpha, t_1)$, contradicting our hypothesis.

The final step in the completeness proof (construction of a perfect chronicle by a countable application of Lemma 15) is entirely standard (see Burgess [5], p. 101) and is omitted.

5 Conclusion

Bonanno [3] considers an extension of logic $\mathbb{L}_0$ obtained by adding several axioms for belief revision[7] and shows that it provides an axiomatic characterization of the theory of belief revision due to Alchourrón et al. [1], known as the AGM theory. It is shown there that the proposed logic is sound with respect to the sub-class of general temporal belief revision frames that satisfy the restriction that $\Omega_t = \Omega$, for all $t \in T$ (see Footnote 1). An open question is whether the completeness result of the previous section can be proved with respect to this class of frames and whether it can be extended to the several logics (extensions of $\mathbb{L}_0$) proposed in [3] and [4].

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[7] For example, the axiom $I\varphi \to B\varphi$ which says that if the agent is informed that φ then she believes that φ.