GIACOMO BONANNO

VERTICAL DIFFERENTIATION WITH COURNOT COMPETITION

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Vertical Differentiation with Cournot Competition *

1. *Introduction*

In this paper we reconsider the problem of whether two firms in the same market would choose to produce a homogeneous product or differentiated products. This problem was first studied by Hotelling (1929) in a model where the type of product differentiation is that which Lancaster (1979) labelled « horizontal differentiation ». In Hotelling's model consumers are uniformly distributed on a line segment and there are two firms which produce a homogeneous product and can locate at any point on that segment. Consumers face a transportation cost which is a linear function of the distance travelled and each consumer buys exactly one unit of the good from that firm which quotes the least delivered price, namely mill price plus transportation cost. The notion of equilibrium used by Hotelling is that which today would be called subgame perfect equilibrium of a two-stage game in which firms first choose their location and then compete in prices. Hotelling's conclusion was that both sellers would locate at the centre of the market. D'Aspremont, Gabszewicz and Thisse (1979) recently showed that the above « principle of minimum differentiation » could not in fact be proved in Hotelling's model, since - due to the linearity of the transportation cost - when firms are located sufficiently close to each other the second-stage game has no Nash equilibria in prices. They also showed that if one replaces the hypothesis of linear transportation cost with that of quadratic transportation cost, then the model actually yields a unique perfect equilibrium at which the two firms decide to locate as far as possible from each other, that is, at the two extremes of the market [1].

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[1] In a more recent paper, Gabszewicz and Thisse (1986) showed that the problem of non-existence of a Nash equilibrium in prices in Hotelling's model (when firms are located sufficiently close to each other) is a serious one, since it is present even when the transportation cost function is quadratic with a non-zero linear term. In this case the demand functions are continuous and the non-existence of equilibria is due to the fact that the profit functions are not quasi-concave.

The problem was recently reconsidered in a number of papers which concentrated on a different type of product differentiation, namely that which Lancaster labelled « vertical differentiation » (Gabszewicz and Thisse (1979, 1980), Shaked and Sutton (1982)). In these models products differ in quality and if two distinct products are offered at the same price, then all consumers will agree in choosing the same (higher quality) alternative. The paper which is most relevant in this context is the one by Shaked and Sutton (1982). In their model firms play a three-stage game in which they first choose whether or not to enter the market, then the quality of their products and finally their prices. Costs of production are assumed to be zero. Shaked and Sutton show that for some values of the parameters the game has a unique subgame perfect equilibrium at which only two firms enter the market; furthermore, the two firms choose to produce differentiated products and earn positive profits at equilibrium. The intuition behind this result is that firms resort to product differentiation in order to relax price competition.

The purpose of this paper is to investigate further the implications of Shaked and Sutton's result and its robustness relative to the solution concept (or type of competition) which is adopted for the post-entry game. We shall therefore remain within the case of vertical differentiation.

We consider the case of two firms which play a two-stage game in which they first choose the quality of their products and then their prices or outputs. We study the subgame perfect equilibria (Selten (1975)) of this game. The results we obtain are the following:

(i) The two firms may decide to produce a homogeneous product or differentiated products, depending on the solution concept adopted for the marketing stage of the game. If this is Bertrand-Nash then products will be differentiated; if Cournot-Nash, then a homogeneous product will result (sections 3 and 4).

(ii) In the Bertrand-Nash case, utility functions exist such that the amount of differentiation compatible with perfect equilibrium is arbitrarily small (section 5).

(iii) If costs of production are introduced into the analysis — so that higher quality invokes higher cost — then Cournot-Nash may give rise to a unique perfect equilibrium with maximum product differentiation (an example is given: section 6).

Cournot competition in the case of vertical differentiation has recently been investigated by Gal-Or in a number of papers (Gal-Or (1983a, 1983b, 1984)). However, the problems studied by the author differ from the one considered here. An interesting interpretation of the Cournot-Nash solution concept was recently given by Kreps and Sheinkman (1983).

2. *The Model*

The model of consumer behaviour which we use was first introduced by Gabszewicz and Thisse (1979) and subsequently used in a number of papers (Gabszewicz and Thisse (1980), Shaked and Sutton (1982)).

Let $k \varepsilon [c,d]$ (with $c < d$), be a physical characteristic (e.g. location) of a given product. Higher k means higher quality and all consumers agree on this [2]. There is a continuum of consumers, represented by the unit interval [0,1]. Consumers have identical preferences but different incomes. The income of consumer $t \varepsilon [0,1]$ is given by

$$E(t) = E\,t,\ E > 0. \tag{1}$$

Consumers are assumed to make indivisible and mutually exclusive purchases and every consumer buys at most one unit of the good. Thus if consumer t is faced with the alternative of buying a product of quality k_1 or a product of quality k_2 (or nothing) and decides to buy the first, then he buys that product only and a single unit of it. All consumers

[2] As an example, consider the following case, in the same spirit as Hotelling's model. A homogeneous product can be sold at any point of a line segment, which we take to be the unit interval [0, 1]. There are many consumers, all located at the right extreme of the line segment. Consumers have different incomes but identical preferences, in particular they all dislike travelling. The utility a consumer derives from not consuming the good and keeping his/her income e is given by (3). On the other hand, if (s)he consumes one unit of the good and an income e and has to travel a distance s to obtain the good, his/her utility is $V(1, e, s) = U_1 ef(s)$ where $-f(s)$ is disutility of travelling and therefore $f'(s) < 0$. In this example, $s = 1 - k$, where $k \varepsilon [0, 1]$ denotes the location of the shop. We can write $V(k, e) = U_1 ef(1 - k)$ and define $U(k) = U_1 f(1 - k)$. Clearly, $U'(k) > 0$. The parameter k (location) can now be interpreted as an index of quality and, provided $U_1 f(1) > U_0$, we have an example of the general model of section 2. Alternatively (Gabszewicz and Thisse (1986)), one could imagine the consumers being spread out over the segment (rather than bunched at one extreme) while the firms are established on the same side of the segment, outside of the market.

Later on in the paper (section 5) we shall consider alternative utility functions $U(k)$. In the example above this would mean looking at different 'economies' where consumers, while they still dislike travelling, differ in their marginal disutility of travelling. These differences may result in different location patterns for the two firms.

have the same utility function $V(k, e)$, which denotes the utility of having one unit of a product of quality k and income e. We shall assume that

$$V(k, e) = U(k)\, e. \tag{2}$$

When the consumer does not purchase the product, his/her utility is given by

$$V(0, e) = U_o\, e, \text{ with } U_o > 0. \tag{3}$$

We shall assume that $U(k)$ is continuously differentiable, that consumers like the good, that is,

$$U(k) > U_o \text{ for all } k \,\varepsilon\, [c, d] \tag{4}$$

and that higher k means higher quality, that is, that $U(k)$ is increasing on $[c, d]$:

$$U'(k) > 0 \text{ for all } k \,\varepsilon\, [c, d]. \tag{5}$$

Let there be two firms ,1 and 2, and let $k_i \,\varepsilon\, [c, d]$ be the quality of firm i's product ($i = 1,2$). Without loss of generality we can assume that

$$k_2 \leqslant k_1 \tag{6}$$

(otherwise it would be sufficient to renumber the firms).
We now derive the demand functions faced by the firms for any given choice of qualities k_1 and k_2.

Suppose the two firms choose the same quality k. Then a consumer $t \,\varepsilon\, [0,1]$ is indifferent between purchasing the commodity at price p and not purchasing it if

$$V(0, E(t)) = V(k, E(t) - p) \tag{7}$$

and, using (1)-(2) and solving for t, we obtain that the indifferent consumer is given by

$$t' = \frac{U(k)}{E\,(U(k) - U_o)}\, p \tag{8}$$

Consumers richer than t' will prefer to buy the commodity and consumers poorer than t' will prefer not to buy it. Therefore total demand is given by $1 - t'$ or

$$D(p) = 1 - \frac{U(k)}{E\,(U(k) - U_o)}\,p \tag{9}$$

Now suppose the two firms choose different qualities, $k_1 > k_2$. Let p_i be the price of firm i ($i = 1,2$) and let

$$x = U(k_1) \text{ and } y = U(k_2) \tag{10}$$

(thus $x > y$). Consumer t will be indifferent between quality k_1 and quality k_2 if and only if

$$V\,(k_1,\ E(t) - p_1) = V\,(k_2,\ E(t) - p_2) \tag{11}$$

and, using (1)-(3) and solving for t, we obtain that the indifferent consumer is given by

$$\bar{t} = \frac{x}{E\,(x - y)}\,p_1 - \frac{y}{E\,(x - y)}\,p_2 \tag{12}$$

(notice that $\bar{t} > 0$ requires $p_2 < p_1$, since $x > y$). All $t < \bar{t}$ will prefer the low quality k_2 and all consumers $t > \bar{t}$ will prefer the high quality k_1.

Next, define t_o to be the consumer who is indifferent between buying nothing and buying the low-quality good: solving

$$V\,(k_2,\ E(t) - p_2) = V\,(0,\ E(t)) \tag{13}$$

for t we obtain

$$t_o = \frac{y}{E\,(y - U_o)}\,p_2 \tag{14}$$

Then, over the relevant range [3], demand for the high-quality good and the low-quality good, denoted by $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$, respectively, is given by

$$\begin{cases} D_1(p_1, p_2) = 1 - \bar{t} = 1 - \dfrac{x}{E(x-y)} p_1 + \dfrac{y}{E(x-y)} p_2 \\[2ex] D_2(p_1, p_2) = \bar{t} - t_o = \dfrac{x}{E(x-y)} p_1 - \dfrac{y(x-U_o)}{E(x-y)(y-U_o)} p_2 \end{cases} \tag{15}$$

Let

$$K = \{(k_1, k_2) \, \varepsilon \, [c, d]^2 / k_2 \leqslant k_1\} \tag{16}$$

The two firms play a two-stage game as explained in the introduction. We shall investigate the subgame perfect equilibria of this game.

Therefore, we have to work backwards from the second-stage game, the solution of which transforms the first-stage game in a two-person game with payoff (profit) functions defined on K. These payoff functions will depend on the solution concept which is adopted for the second-stage game. In the next two sections we consider two alternative solution concepts.

3. *The Post-entry Game as Bertrand-Nash*

In this and the following section we shall make the common assumption of zero production costs [4]. We first consider the case in

[3] More precisely, the demand functions are given as follows. By (12) $D_2 = 0$ if $p_2 \geqslant p_1$. Also, $D_2 = 0$ if $p_2 \geqslant (y - U_0)/y$, where the *RHS* is the reservation price of the richest consumer for the low-quality good. Thus (15) requires $p_2 < \min\{p_1, (y - U_0)/y\}$.

Similarly, $D_1 = 0$ if $p_1 \geqslant (x - U_0)/x$, where the *RHS* is the reservation price of the richest consumer for the high-quality good. Also, $D_1 = 0$ if $p_2 < p_1$ but $\bar{t} > 1$, where $\bar{t}$ is given by (12); this is equivalent to $p_1 \geqslant (x - y)/x + y\, p_2/x$. If $p_2 \geqslant p_1$ and $p_1 < (x - U_0)/x$, then $D_1 = 1 - x\, p_1/(x - U_0)$. Thus (15) requires $p_2 < p_1$ and $p_1 < \min\{(x-y)/x + y\, p_2/x, (x - U_0)/x\}$. It can be shown that there is no loss of generality in restricting oneself to the price range for which (15) is satisfied.

[4] The assumption of zero production costs is justified as follows in the literature. If at equilibrium one firm refrains from increasing the quality of its product, even though the higher quality could be produced at zero cost, then *a fortiori* it will refrain from increasing

which the second-stage game is Bertrand-Nash, that is, each firm — having chosen the quality of its own product in the previous stage and having observed the quality chosen by the other firm — now selects its *price* in order to maximise its own profits, taking as given the price of the other firm. The following proposition is a simple extension of Shaked and Sutton's (1982) result to the case where the range of incomes extends to zero (or, equivalently, to the case where the unit cost of production is so high that at least one consumer cannot afford to buy the good at any price exceeding unit variable cost).

Lemma 1. For every choice of quality k_1 and k_2 by firms 1 and 2 respectively (with $k_2 \leqslant k_1$), there exists a unique Bertrand-Nash equilibrium of the second-stage game at which the profits (revenues) of firms 1 and 2 are given by

$$R_1{}^* = \frac{4\,E\,(x - U_o)^2\,(x - y)}{x\,(4x - y - 3\,U_o)^2} \tag{17a}$$

and

$$R_2{}^* = \frac{E\,(y - U_o)\,(x - y)\,(x - U_o)}{y\,(4x - y - 3\,U_o)^2} \tag{17b}$$

respectively, where $x = U(k_1)$ and $y = U(k_2)$.

Lemma 2 is easily proved, by considering that the profit (revenue) function of firm i ($i = 1, 2$) is given by $R_i(p_1, p_2) = p_i\,D_i(p_1, p_2)$, where $D_i(p_1, p_2)$ is given by (15). R_i is strictly concave in p_i and taking first-order conditions we obtain (17a) and (17b) (note that when $x = y$, that is, when the products are homogeneous, then both (17a) and (17b) become zero, in accordance with Bertrand's theorem).

Proposition 1. Assume that the costs of production are zero. Then if the second-stage game is Bertrand-Nash, the set of perfect equilibria of the two-stage game is non-empty. Each perfect equilibrium satisfies

the quality of its product if the higher quality is more expensive to produce. All the essential features are present in the zero-cost case and nothing important is lost by not considering production costs.

the following properties: (i) the two firms enter the market with differentiated products, (ii) one firm chooses the highest-quality product, (iii) both firms make positive profits.

Proposition 1 states the intuitive « anti-Hotelling » result first proved by Shaked and Sutton (1982): if the firms produce a homogeneous product, their profits will be zero by Bertrand's theorem; thus firms will try to relax price competition by introducing some degree of product differentiation. A proof of proposition 2 can be found in Bonanno (1985) and is based on the following facts: (a) both R_1^* and R_2^* are strictly increasing in k_1, (b) R_1^* is strictly greater than R_2^* for all (k_1, k_2) with $k_1 > k_2$, (c) R_2^* is positive if and only if $k_2 < k_1$.

We shall discuss the implications of the result of proposition 1 in section 5.

4. *The Post-entry Game as Cournot-Nash*

In this section we consider the case in which the post-entry game is Cournot-Nash, that is, each firm — having chosen the quality of its own product in the previous stage and having observed the quality chosen by the other firm — now selects its *output* in order to maximise its own profits, taking as given the output of the other firm.

The following lemma and proposition are the parallel of the results of the previous section.

Lemma 2. For every choice of quality k_1 and k_2 by firms 1 and 2 respectively (with $k_2 \leqslant k_1$), there exists a unique Cournot-Nash equilibrium of the second-stage game at which the profits (revenues) of firms 1 and 2 are given by

$$\hat{R}_1 = \frac{E\,(x - U_o)\,(2x - y - U_o)^2}{x\,(4x - y - 3\,U_o)^2} \tag{18}$$

and

$$\hat{R}_2 = \frac{E\,(y - U_o)\,(x - U_o)^2}{y\,(4x - y - 3\,U_o)^2} \tag{19}$$

respectively, where $x = U(k_1)$ and $y = U(k_2)$.

Proof. See Appendix.

The proof of lemma 2 is straightforward and along the lines of the proof of lemma 1 (we first invert the demand functions (15) and then solve the first-order conditions involving profit functions in which now outputs rather than prices are the decision variables).

The proof of the following proposition — given in the Appendix — is based on the following properties of $\hat{R}_1$ and $\hat{R}_2$. For every k_2, $\hat{R}_1(k_1, k_2)$ is a strictly increasing function of k_1 and, similarly, for every k_1, $\hat{R}_2(k_1, k_2)$ is a strictly increasing function of k_2. That is, it never pays one firm to choose a quality which is lower than that chosen by the other firm (since $\partial \hat{R}_2/\partial k_2 > 0$) and, on the other hand, either firm can always do better by increasing the quality of its own product (since $\partial \hat{R}_1/\partial k_1 > 0$). Therefore, since $\hat{R}_1$ and $\hat{R}_2$ are always positive, there can be only one perfect equilibrium, where both firms choose the highest quality.

Proposition 2. Assume that the costs of production are zero. Then if the post-entry game is Cournot-Nash there is a unique perfect equilibrium of the two-stage game where both firms enter with the highest-quality product and make positive profits.

Proof. See Appendix.

We can summarize the results of propositions 1 and 2 as follows. In the case of vertical differentiation firms have a tendency to increase the quality of their products, since consumers are willing to pay higher prices for products of higher quality. However, in the Bertrand-Nash case, price competition becomes fiercer the more similar the products of the two firms. Thus in the Bertrand-Nash case there is a counteracting tendency and the net result is that firms choose to maintain some degree of product differentiation. In the Cournot-Nash case, on the other hand, this counteracting tendency is absent and therefore the two firms will both choose to locate at the upper extreme of the quality space. Thus, in the Cournot-Nash case with zero production costs, Hotelling's principle of minimum differentiation holds.

We shall investigate further the Cournot-Nash case in section 6.

5. *Discussion of the Bertrand-Nash Case*

Proposition 1 (section 3) is essentially a corollary to Bertrand's theorem, which says that if two firms produce (at zero costs) a homogeneous product and compete in prices, then the only Nash equilibrium is one where both firms charge zero price and therefore make zero profits. If the two firms can make positive profits by producing differentiated products, it is obvious that they would do so.

D'Aspremont, Gabszewicz and Thisse (1983) have tried « to go a step further » and argue

> « that under mild assumptions, the 'Principle of Minimum Differentiation' *never* holds, so that, given the product specification of one of the sellers, it can never become advantageous to the other one to choose his own specification arbitrarily close to it, if prices adapt themselves at a non-cooperative equilibrium » (p. 20).

The argument put forward by the authors is very general (it applies both to vertical and horizontal differentiation) and goes as follows. If on the diagonal of the quality space (where the two qualities coincide) firms' profits are zero (by Bertrand's theorem), then — by continuity — they will be almost zero at points close to the diagonal and therefore the two fims will always want to be sufficiently away from the diagonal.

In this section we show that the above argument — although correct — does *not* imply (at least in the case of vertical differentiation) that there is a lower bound to the degree of product differentiation observed in equilibrium.

Let U be the set of utility functions defined on the quality space $[c, d]$ and taking on the same values at c and d. That is, let α and β be any two numbers such that $U_o < \alpha < \beta$ and let

$$U = \{U : [c, d] \to \mathfrak{R} / U \text{ is } C^1 \text{, increasing, } U(c) = \alpha \text{ and } U(d) = \beta\}. \quad (20)$$

For the purpose of the proof of the following proposition we require α to be sufficiently close to U_o.

Proposition 3. Assume that the second stage game is Bertrand-Nash and that the costs of production are zero. Then for every real number $\varepsilon > 0$, there exists a utility function U in U such that the set of perfect equilibria of the corresponding two-stage game is non-empty and all perfect equilibria are characterized by the fact that the two firms enter with products whose qualities differ by *less than* ε and make positive profits (one firm always chooses the highest-quality product).

Proof. See Appendix.

The smaller ε, the more convex the utility function or, in other words, utility increases very slowly with quality at first and then the function becomes very steep near d. As a consequence, firms will tend to cluster at the upper bound of the quality space.

6. *Discussion of the Cournot-Nash Case*

In this section we investigate the robustness of the « principle of minimum differentiation » proved in section 4. In particular, we want to see if that result holds also when costs of production are introduced. Clearly, if production costs are independent of quality, then the result of proposition 2 is not affected. Typically, however, higher quality is coupled with higher costs. We shall consider the following cost function (which we assume to be continuously differentiable):

$$C(q,\ k) = C(k)\ , \quad \text{with} \quad C(c) \geqslant 0 \quad \text{and} \quad C'(k) > 0 \qquad (21)$$

where q is output and k is quality. That is, there is a fixed set-up cost which increases with quality and zero marginal cost (the assumption of zero marginal cost is not important and we make it only in order to simplify the analysis). It is possible to think of situations in which marginal cost does not vary with quality, while the set-up cost increases with quality. For example, if consumers are concentrated in one area (at the end of a line, say: cf. footnote 2), higher quality could mean a location which is closer to that area, and the set-up cost could be rent, which is higher the closer the firm locates to the « residential » area.

The profit (payoff) functions of the first-stage game (calculated at the Cournot-Nash equilibrium of the second-stage game) are now given by

$$\hat{\pi}_1(k_1,\ k_2) = \hat{R}_1(k_1,\ k_2) - C(k_1) \qquad (22)$$

and

$$\hat{\pi}_2 (k_1, k_2) = \hat{R}_2 (k_1, k_2) - C (k_2) \tag{23}$$

where $\hat{R}_1$ and $\hat{R}_2$ are given by (18) and (19) respectively. Since the proof of proposition 2 depended on the fact that the payoff function of each player was strictly increasing in the player's decision variable (that is, $\partial \hat{R}_1/\partial k_1 > 0$ and $\partial \hat{R}_2/\partial k_2 > 0$), the result of proposition 2 can still be true when the payoff functions are given by (22) and (23). In fact let

$$m_1 = min \quad \text{of} \quad (\partial \hat{R}_1/\partial k_1) \quad \text{on} \quad K \tag{24}$$

$$m_2 = min \quad \text{of} \quad (\partial \hat{R}_2/\partial k_2) \quad \text{on} \quad K \tag{25}$$

$$m_3 = max \quad \text{of} \quad C' \quad \text{on} \quad [c, d] \; . \tag{26}$$

Then a sufficient condition for the existence of a unique perfect equilibrium of the two-stage game at which both firms enter with the highest-quality product is

$$m_3 < min \{m_1, m_2\} \; . \tag{27}$$

In other words, if cost does not increase too much with quality, we would still observe no product differentiation in equilibrium.

We shall now construct an example, however, where the cost function is a special case of (21) and the two-stage game has a unique perfect equilibrium at which one firm enters with the lowest-quality product, the other firm with the highest-quality product and both firms make positive profits.

We shall first establish two properties of the partial derivatives $\partial \hat{R}_1/\partial k_1$ and $\partial \hat{R}_2/\partial k_2$.

The first property is that in the set K (defined by (16)) $\partial \hat{R}_1/\partial k_1$ has a unique global minimum at (d, c):

Lemma 3. For every $(k_1, k_2) \in K$ with $(k_1, k_2) \neq (d, c)$

$$\frac{\partial \hat{R}_1}{\partial k_1} (k_1, k_2) > \frac{\partial \hat{R}_1}{\partial k_1} (d, c) \tag{28}$$

Proof. It is shown in the Appendix that in the set

$$Q^* = \{(x, y)/U_o < y \leqslant x \leqslant U(d)\} \tag{29}$$

$\partial \hat{R}_1/\partial x$ decreases along the directions illustrated in Figure 1.

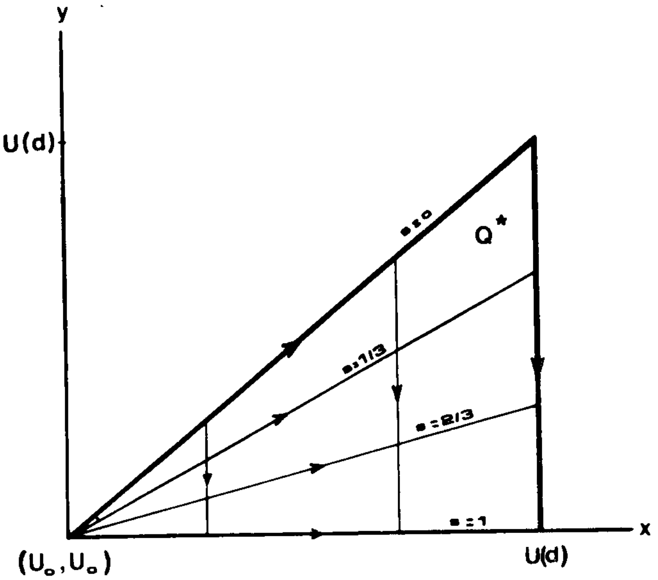

Fig. 1 - The set Q^* and the directions along which the function $\partial \hat{R}_1/\partial x$ decreases.

Established this, the proof of lemma 3 is immediate. In fact, since $(\partial \hat{R}_1/\partial k_1) = (\partial \hat{R}_1/\partial x)\, U'(k_1)$ and U' is positive, it is enough to show that for each $(x, y) \varepsilon Q$, where Q is defined by

$$Q = \{(x, y)/U(c) \leqslant y \leqslant x \leqslant U(d)\}\ , \tag{30}$$

with $(x, y) \neq (U(d), U(c))$ we have

$$\frac{\partial \hat{R}_1}{\partial x}(x, y) > \frac{\partial \hat{R}_1}{\partial x}(U(d), U(c)) \tag{31}$$

This is easily proved as follows. Fix an arbitrary $(x, y) \varepsilon Q$ with $(x, y) \neq (U(d), U(c))$. Let $A = (x, y)$ and let B be the point where the ray through (U_o, U_o) and A meets the line $x = U(d)$ (see Figure 2). Then, by the property illustrated in Figure 1,

$$\frac{\partial \hat{R}_1}{\partial x}(A) \geqslant \frac{\partial \hat{R}_1}{\partial x}(B) > \frac{\partial \hat{R}_1}{\partial x}(U(d),\ U(c)) \tag{32}$$

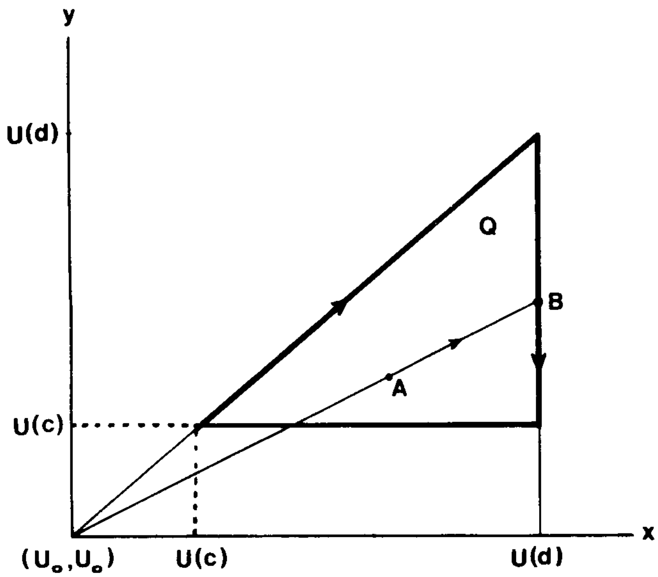

Fig. 2 - The arrows denote directions along which $\partial \hat{R}_1/\partial x$ decreases in the set $Q \subset Q^*$.

The second property we want to establish is that $\partial \hat{R}_2/\partial k_2$ has a unique global maximum at (c, c):

Lemma 4. For every $(k_1, k_2) \varepsilon K$ with $(k_1, k_2) \neq (c, c)$

$$\frac{\partial \hat{R}_2}{\partial k_2}(k_1,\ k_2) < \frac{\partial \hat{R}_2}{\partial k_2}(c,\ c) \tag{33}$$

Proof. See Appendix.

We are now ready to construct our example. Let the utility function be given by

$$U(k) = a\,k\ , \qquad \text{with} \qquad a > 0 \qquad \text{and} \qquad ac > U_o \tag{34}$$

and let the cost function be given by the following special case of (21):

$$C(q, k) = b(k - c), \qquad b > 0 \tag{35}$$

The profit functions of firms 1 and 2 are therefore given by

$$\hat{\pi}_1(k_1, k_2) = \hat{R}_1(k_1, k_2) - b(k_1 - c) \tag{36}$$

and

$$\hat{\pi}_2(k_1, k_2) = \hat{R}_2(k_1, k_2) - b(k_2 - c) \tag{37}$$

Now, using (18) and (34) we obtain (cf. (xiii) in the Appendix)

$$\frac{\partial \hat{R}_1}{\partial k_1}(d, c) = \frac{E\,U_o(2ad - ac - U_o)^2}{ad^2(4ad - ac - 3U_o)^2} +$$

$$+ \frac{4E(ad - U_o)(ac - U_o)(2ad - ac - U_o)}{d(4ad - ac - 3U_o)^3} \equiv M \tag{38}$$

and using (19) and (34) we obtain (cf. (xvi) in the Appendix)

$$\frac{\partial \hat{R}_2}{\partial k_2}(c, c) = \frac{E\,U_o}{9ac^2} + \frac{2E}{27c} \equiv L \tag{39}$$

Then by (28) and (33) if b satisfies the following inequality (recall that b is the marginal cost of quality: cf. (35))

$$L < b < M \tag{40}$$

(where L is the RHS of (39) and M the RHS of (38)) we have that for all $(k_1, k_2) \varepsilon K$ with $(k_1, k_2) \neq (d, c)$

$$\frac{\partial \hat{\pi}_1}{\partial k_1}(k_1, k_2) = \frac{\partial \hat{R}_1}{\partial k_1}(k_1, k_2) - b > M - b > 0 \tag{41}$$

and for all $(k_1, k_2) \varepsilon K$ with $(k_1, k_2) \neq (c, c)$

$$\frac{\partial \hat{\pi}_2}{\partial k_2}(k_1, k_2) = \frac{\partial \hat{R}_2}{\partial k_2}(k_1, k_2) - b < L - b < 0 \tag{42}$$

Thus since $\hat{\pi}_1$ is strictly increasing in k_1 and $\hat{\pi}_2$ is strictly decreasing in k_2, we have proved the following

Proposition 4. Let the parameters a, b, c, d, U_o, E satisfy inequality (40) (for example, $E = 1$, $U_o = 0{,}99$, $a = c = 1$, $b = 0.19$, $d = 1.1$), then the two-stage game has a ***unique*** perfect equilibrium at which one firm enters with the lowest-quality product ($k_2 = c$), the other firm enters with the highest-quality product ($k_1 = d$) and both firms make positive profits.

To complete the proof of proposition 4 we only need to show that at equilibrium both firms make positive profits. This is done in the Appendix.

As concluding remark we observe that, since in this example equilibrium in unique and characterized by the maximum degree of product differentiation, we can also deduce a negative answer to the question whether, in general, Cournot competition leads to less product differentiation than Bertrand competition.

GIACOMO BONANNO

Nuffield College, Oxford, U.K.

APPENDIX

Proof of *lemma* 2. If $k_1 = k_2 = k$, the demand function for the homogeneous product is given by (9). The inverse demand function is therefore given by

$$p = G(q) = \frac{E(U(k) - U_o)}{U(k)} (1 - q) \tag{i}$$

where q denotes total output. Let q_i be the output of firm i ($i = 1, 2$). Then the revenue function of firm i is given by

$$R_i(q_1, q_2) = p\, q_i = \frac{E(U(k) - U_o)}{U(k)} (1 - q_1 - q_2)\, q_i \tag{ii}$$

R_i is strictly concave in q_i and solving the first-order conditions we obtain a unique Cournot-Nash equilibrium given by

$$\hat{q}_1(k) = \hat{q}_2(k) = 1/3 \tag{iii}$$

The corresponding equilibrium revenues are given by

$$\hat{R}_1(k) = \hat{R}_2(k) = \frac{E(U(k) - U_o)}{9\, U(k)} \tag{iv}$$

If $k_2 < k_1$, then the demand functions are given by (15) (where $x = U(k_1)$ and $y = U(k_2)$ and thus $y < x$). The inverse demand functions are therefore given by

$$\left[\begin{array}{l} p_1 = G_1(q_1, q_2) = -\dfrac{E(x - U_o)}{x} q_1 - \dfrac{E(y - U_o)}{x} q_2 + \dfrac{E(x - U_o)}{x} \\[2ex] p_2 = G_2(q_1, q_2) = -\dfrac{E(y - U_o)}{y} q_1 - \dfrac{E(y - U_o)}{y} q_2 + \dfrac{E(y - U_o)}{y} \end{array}\right. \tag{v}$$

The revenue function of firm i ($i = 1, 2$) is therefore given by

$$R_i(q_1, q_2) = q_i\, G_i(q_1, q_2) \tag{vi}$$

For every (x, y) (with $y < x$), R_i is strictly concave in q_i. Therefore solving the first-order conditions we obtain a unique Cournot-Nash equilibrium given by

$$\hat{q}_1(x, y) = \frac{2x - y - U_o}{4x - y - 3U_o} \tag{vii}$$

and

$$\hat{q}_2(x, y) = \frac{x - U_o}{4x - y - 3U_o} \tag{viii}$$

The corresponding equilibrium revenues are given by (18) and (19) respectively and are obtained as follows. From $R_i = q_i G_i$ we obtain $\partial R_i/\partial q_i = G_i + q_i(\partial G_i/\partial q_i)$ and therefore

$$R_i = q_i G_i = q_i \frac{\partial R_i}{\partial q_i} - q_i^2 \frac{\partial G_i}{\partial q_i}$$

Since by definition of $(\hat{q}_1, \hat{q}_2)$

$$\frac{\partial R_i}{\partial q_i}(\hat{q}_1, \hat{q}_2) = 0$$

we finally obtain

$$\hat{R}_i = R_i(\hat{q}_1, \hat{q}_2) = -\hat{q}_i^2 \frac{\partial G_i}{\partial q_i}(\hat{q}_1, \hat{q}_2)$$

Finally, note that when $x = y$ (and hence $k_1 = k_2$), (18) and (19) coincide with (iv).

Proof of proposition 2. Let

$$W = U([c, d]) \tag{ix}$$

and let

$$Q = \{(x, y) \varepsilon W^2 / y \leqslant x\} \tag{x}$$

Since

$$\partial \hat{R}_1/\partial k_1 = (\partial \hat{R}_1/\partial x)\, U'(k_1) \tag{xi}$$

and

$$\partial \hat{R}_2/\partial k_2 = (\partial \hat{R}_2/\partial y)\, U'(k_2) \tag{xii}$$

and U' is always positive, it is sufficient to prove that $\partial \hat{R}_1/\partial x$ and $\partial \hat{R}_2/\partial y$ are positive on Q. In fact we have

$$\frac{\partial \hat{R}_1}{\partial x} =$$

$$= E \left\{ \frac{U_o (2x - y - U_o)^2}{x^2 (4x - y - 3U_o)^2} + \frac{2(x - U_o)(2x - y - U_o)(2y - 2U_o)}{x(4x - y - 3U_o)^3} \right\} =$$

$$= f_1(x, y) f_2(x, y) \tag{xiii}$$

where

$$f_1(x, y) = \frac{E(2x - y - U_o)}{x^2 (4x - y - 3U_o)^3} \tag{xiv}$$

and

$$f_2(x, y) = U_o (4x - y - 3U_o)(2x - y - U_o) + \\ + 4x(x - U_o)(y - U_o)\,. \tag{xv}$$

Since $(x, y) \varepsilon Q$ implies $x \geq y > U_o$, both f_1 and f_2 are positive on Q. Thus $\partial \hat{R}_1/\partial x$ is positive. On the other hand,

$$\frac{\partial \hat{R}_2}{\partial y} = \frac{U_o E (x - U_o)^2}{y^2 (4x - y - 3U_o)^2} + \frac{2E(y - U_o)(x - U_o)^2}{y(4x - y - 3U_o)^3} \tag{xvi}$$

which is positive on Q.

Proof of proposition 3. Let α and β be any two numbers such that

$$\beta > \alpha > U_o > 0 \tag{xvii}$$

and let U be the set of utility functions given by (20).

Fix an arbitrary $U \varepsilon U$. Then, by proposition 1, the set of perfect equilibria of the corresponding two-stage game is non-empty and consists of all the points $(d, k_2^*) \varepsilon K$ such that k_2^* is a global maximum of the function $R_2^*(d, k_2)$ on $[c, d]$. Consider the function R_2^* given by (17b) with $x = U(d) = \beta$. This function is zero when $y = U_o$ and greater than zero for $y > U_o$. Hence if $a = U(c)$ is sufficiently close to U_o, it must be

$$\frac{\partial R_2^*}{\partial y}(\beta, a) > 0 \tag{xviii}$$

From now on we shall assume that a is sufficiently close to U_o for (xviii) to be satisfied. It then follows that all the global maxima k_2^* of the function R_2^* (d, k_2) on $[c, d]$ are such that

$$k_2^* > c \tag{xix}$$

Let k^* be the smallest of these global maxima (the set of global maxima does indeed have a smallest element, because it is a compact subset of $[c, d]$).

Now fix an arbitrary $\varepsilon > 0$ and let $\hat{k} \varepsilon (c, d)$ be an arbitrary point such that

$$\hat{k} < d - \varepsilon \tag{xx}$$

We want to show that there is a function $V \varepsilon U$ such that all the perfect equilibria of the corresponding two-stage game are of the form (d, k_2) with $k_2 \geq \hat{k}$, that is, at a perfect equilibrium the two firms choose products whose qualities differ by less than ε and, furthermore, both firms make positive profits (note that, once we have shown the existence of such a utility function $V \varepsilon U$ then the existence of perfect equilibria at which both firms make positive profits follows directly from proposition 1). Let

$$r^* = (k^* - c)/(d - c) \tag{xxi}$$

and

$$\hat{r} = (\hat{k} - c)/(d - c) \tag{xxii}$$

and

$$r = \log r^*/\log \hat{r} \tag{xxiii}$$

Define the function $h: [c, d] \rightarrow [c, d]$ by

$$h(k) = c + (d - c)\left(\frac{k - c}{d - c}\right)^r \tag{xxiv}$$

Finally, let $V: [c, d] \rightarrow \mathcal{R}$ be given by

$$V(k) = U(h(k)) \tag{xxv}$$

Then $V(c) = U(c) = \alpha$, $V(d) = U(d) = \beta$ and V is C^1 and strictly increasing on $[c, d]$. Thus $V \varepsilon U$ (intuitively, V is obtained from U by applying a change of coordinates $h^{-1}:[c, d] \rightarrow [c, d]$ which takes k^* to $\hat{k}$).

The payoff functions of the second-stage game are still given by (17) where now

$$x = V(k_1) \tag{xxvi}$$

and

$$y = V(k_2) \tag{xxvii}$$

A perfect equilibrium of the two-stage game is now given by $k_1 = d$ and $k_2 = \hat{k}$. If the original game had several equilibria, the new game will have the same number of equilibria, but they will all be of the form (d, k_2) with $k_2 \geq \hat{k}$ and thus $d - k_2 < \varepsilon$ (because of the way we chose k^*).

Proof of Lemma 3. First of all, we want to show that

$$\frac{\partial^2 \hat{R}_1}{\partial y \, \partial x}(x, y) > 0 , \qquad \text{for all} \qquad (x, y) \, \varepsilon \, Q \tag{xxviii}$$

where the set Q is defined by (30). Differentiating (xiii) with respect to y we obtain

$$\frac{\partial^2 \hat{R}_1}{\partial y \, \partial x} = \frac{\partial f_1}{\partial y} f_2 + f_1 \frac{\partial f_2}{\partial y} \tag{xxix}$$

where f_1 and f_2 are given by (xiv) and (xv) respectively. Since f_1 and f_2 are positive on Q, it will be enough to show that also $(\partial f_1/\partial y)$ and $(\partial f_2/\partial y)$ are non-negative on Q and at least one is positive. In fact we have

$$\frac{\partial f_1}{\partial y} = \frac{2 E (x - y)}{x^2 (4 x - y - 3 U_o)^3} \tag{xxx}$$

which is non-negative for all (x, y) such that $x \geq y > U_o$, and

$$(\partial f_2/\partial y) = 4 x^2 - 10x \, U_o + (2 y \, U_o + 6 \, U_o^2) = g_y(x) \, . \tag{xxxi}$$

For each y, the function $g_y(x)$ is strictly convex in x and reaches its minimum value at $x^o = 5 \, U_o/4$. Since

$$g_y(x^o) = - (1/4) \, U_o^2 + 2 y \, U_o > (7/4) \, U_o^2 > 0$$

(since $y > U_o$), it follows that $g_y(x) > 0$ for all y and x and therefore $\partial f_2/\partial y$ is positive on Q.

Now consider the extension of the function $\partial \hat{R}_1/\partial x$ to the set Q^* defined by (29), which contains the set Q. In Q^* along rays of the form (x, y) with

$$y = (1 - s)\, x + s\, U_o\,, \quad \text{with constant} \quad s \,\varepsilon\, [0, 1] \tag{xxxii}$$

we have (substituting (xxxii) for y in (xiii))

$$\left.\frac{\partial \hat{R}_1}{\partial x}\right|_{y = (1-s)\,x + s\,U_o} = \frac{U_o\, E\,(1+s)^2}{x^2\,(3+s)^2} + \frac{4\,E\,(1-s^2)}{x\,(3+s)^3} \tag{xxxiii}$$

which is strictly decreasing in x for every $s \,\varepsilon\, [0, 1]$. Figure 1 shows the set Q^* and the directions along which $\partial \hat{R}_1/\partial x$ decreases (obtained from (xxviii) and (xxxiii)).

The proof of lemma 3 can now be completed using the argument given in the text after lemma 3.

Proof of lemma 4. We first show that

$$\frac{\partial^2 \hat{R}_2}{\partial x\, \partial y}(x, y) < 0 \qquad \text{for all} \qquad (x, y) \,\varepsilon\, Q \tag{xxxiv}$$

Differentiating (xvi) with respect to x we obtain

$$\frac{\partial^2 \hat{R}_2}{\partial x\, \partial y} = -\frac{2\,E\,U_o\,(x - U_o)\,(y - U_o)}{y^2\,(4\,x - y - 3\,U_o)^3} - \frac{4\,E\,(x - U_o)\,(y - U_o)\,(2\,x + y - 3\,U_o)}{y\,(4\,x - y - 3\,U_o)^4} \tag{xxxv}$$

which is negative for all (x, y) such that $x \geq y > U_o$.

We can now show that for each $(x, y) \,\varepsilon\, Q$ with $(x, y) \neq (U(c), U(c))$

$$\frac{\partial \hat{R}_2}{\partial y}(x, y) < \frac{\partial \hat{R}_2}{\partial y}(U(c), U(c)) \tag{xxxvi}$$

First of all, setting $x = y$ in (xvi) we can see that along the diagonal $x = y$ in Q we have

$$\left.\frac{\partial \hat{R}_2}{\partial y}\right|_{x = y} = \frac{U_o\,E}{9\,x^2} + \frac{2\,E}{27\,x} \tag{xxxvii}$$

which is strictly decreasing in x.

Now fix an arbitrary $(x, y) \varepsilon Q$ with $(x, y) \neq (U(c), U(c))$. If $y = U(c)$ then (xxxvi) follows directly from (xxxiv). If $y > U(c)$ then by (xxxvii) we have

$$\frac{\partial \hat{R}_2}{\partial y}(y, y) < \frac{\partial \hat{R}_2}{\partial y}(U(c), U(c)) \qquad \text{(xxxviii)}$$

and by (xxxiv)

$$\frac{\partial \hat{R}_2}{\partial y}(y, y) \geq \frac{\partial \hat{R}_2}{\partial y}(x, y) \qquad \text{(xxxix)}$$

Since $(\partial \hat{R}_2/\partial k_2) = (\partial \hat{R}_2/\partial y)\, U'(k_2)$ and U' is positive, lemma 4 follows from (xxxvi).

Proof of proposition 4. It only remains to prove that at equilibrium both firms make positive profits. Now,

$$\hat{\pi}_2(d, c) = \hat{R}_2(d, c) > 0$$

and by (41)

$$\hat{\pi}_1(d, c) > \hat{\pi}_1(c, c) = \hat{R}_1(c, c) > 0.$$

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