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Modeling Production with Petri Nets

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The purpose of this paper is to bring to the attention of economists a tool of analysis, known as Petri nets, which was developed in computer science literature. Although, from a purely formal point of view, Petri nets are not a new tool, they do seem to provide a new perspective on models of production. First of all, the graph-theoretic representation of Petri nets makes it possible to see things that would be hard to detect from a purely algebraic formulation of the same problem. Secondly, the formal definition of a Petri net allows one to introduce a wedge between the notions of input and output (to a production process) and the notion of commodity. Among the inputs to (and outputs of) a production process one can include states of nature, logical conditions, etc. This enables us to show that one of the assumptions which is usually considered to be inherent to linear models of production, namely the absence of external economies and diseconomies among processes, can be dispensed with. We also show that Petri nets do not *require another assumption normally associated with activity analysis, namely that of constant returns to scale. Finally, Petri nets allow a simple analysis of the problem of what commodity vectors can be obtained from a given vector of initial resources.*

Introduction

The purpose of this paper is to bring to the attention of economists a tool of analysis, known as *Petri nets*, which was developed in computer science literature [1]. We shall attempt to demonstrate that Petri nets can be a useful

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[1] The concept of Petri nets has its origin in Petri's (1962) dissertation. Most of the Petri-net related papers written in English before 1980 are listed in the annotated bibliography of Peterson (1981), which is the first book on the subject. More recent papers up until 1984 are annotated in the appendix of Reisig (1985). A good introduction to Petri nets can also be found in Murata (1989). More recent books on Petri nets are Reutenauer (1990) and Reisig (1992).

tool for modeling production, at any level (a single firm, a group of firms, the entire economy). From a purely formal point of view, Petri nets cannot be considered a new tool, since – as we shall see – the notion of Petri nets is equivalent to the notion of an input-output system with integer coefficients (for a definition of input-output system see Appendix C). Thus Petri nets would fall within the category of linear production models or activity analysis. However, there are several points of view from which Petri nets may be a superior modeling tool to traditional linear production models. First of all, the graph-theoretic representation of Petri nets makes it possible to see things that would be hard to detect from a purely algebraic formulation of the same problem (the example of Figure 5.1, Section 5, is an illustration of this). Secondly, the formal definition of a Petri net allows one to introduce a wedge between the notions of input and output (to a production process) and the notion of commodity. Among the inputs to (and outputs of) a production process one can include states of nature, logical conditions, etc. This will enable us to show (Section 5) that one of the assumptions which is usually considered to be inherent to linear models of production, namely the absence of external economies and diseconomies among processes, can be dispensed with: Petri nets *can* incorporate external economies and diseconomies. We also show (Section 5) that Petri nets do *not* require another assumption normally associated with activity analysis, namely that of constant returns to scale. Finally, Petri nets allow a simple analysis of a problem, which so far has received little attention in general input-output analysis, namely what commodity vectors can be obtained from a given vector of initial resources (the so-called reachability and coverability problems).

The paper is organized as follows. In Sections 1-4 Petri nets are introduced and both their graph-theoretic and algebraic representations are illustrated. Two important concepts associated with Petri nets – execution and reachability – are explained and given an economic interpretation. In Section 5 we elaborate on the economic interpretation of Petri nets and show that the assumption of absence of external economies and diseconomies among processes and the assumption of constant returns to scale are not inherent to Petri nets. In Section 6 the production possibilities associated with a Petri net model of production with specified initial resources are discussed. Finally, in Section 7 the notion of commodity "augmentability" or "producibility" is introduced and a simple test for augmentability is proved. There are also three appendices where some of the issues are dealt with in greater detail, such as the definition of returns to scale appropriate to a model with integer constraints and the relationship between Petri nets and activity analysis or input-output systems.

1. *Petri Nets: Graph-Theoretic Representation and Economic Interpretation*

Definition 1. A *Petri net* is a quadruple ⟨P, T, A, ν⟩, where
(i) $P = \{p_1, ..., p_n\}$ is a set of *places* (thus n is the number of places);
(ii) $T = \{t_1, ..., t_m\}$ is a set of *transitions* (thus m is the number of transitions);
(iii) $P \cap T = \varnothing$;
(iv) $A \subseteq (P \times T) \cup (T \times P)$ is a set of *arcs*; if $(p_i, t_j) \in A$, we say that place p_i is an *input* to transition t_j; if $(t_j, p_i) \in A$, we say that place p_i is an *output* of transition t_j;
(v) $\nu : (P \times T) \cup (T \times P) \to \mathbb{N}$ (where $\mathbb{N}$ is the set of non-negative integers) is such that $\nu(a) = 0$ if and only if $a \notin A$; if $a \in A$, we call $\nu(a)$ the *multiplicity of arc a*.

Example 1.1. The following is a Petri net: $P = \{p_1, p_2\}$, $T = \{t_1, t_2\}$, $A = \{(p_1, t_1), (p_1, t_2), (p_2, t_1), (t_1, p_1), (t_2, p_2)\}$, $\nu(p_1, t_1) = 1$, $\nu(p_1, t_2) = 2$, $\nu(p_2, t_1) = 3$, $\nu(t_1, p_1) = 4$, $\nu(t_2, p_2) = 3$.

Graphically, each place p_i is represented by a circle and each transition t_j is represented by a rectangle. We draw an arrow from p_i to t_j if and only if $(p_i, t_j) \in A$ and we draw an arrow from t_j to p_i if and only if $(t_j, p_i) \in A$. Next to each arrow we write the multiplicity of the corresponding arc.

For instance, the Petri net of Example 1.1 can be represented as shown in Figure 1.1.

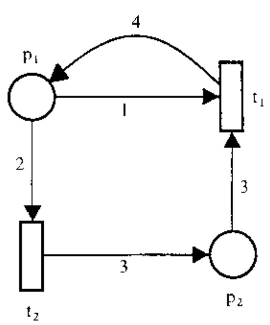


Figure 1.1

Economic interpretation. The most obvious economic interpretation of Petri nets is as follows: each transition represents a *production process* and each place represents a *commodity*. According to this interpretation, transition t_1 in Figure 1.1 represents a production process that requires 1 unit of commodity p_1 and 3 units of commodity p_2 to produce 4 units of commodity p_1, while transition t_2 represents a production process that uses 2 units of commodity p_1 as input and delivers 3 units of commodity p_2 as output.

We shall maintain throughout the original terminology of Petri nets and indicate separately the suggested interpretation. There are two reasons why we prefer not to depart from the original terminology: (1) it will be easier for the reader to refer to the computer science literature on the subject, and (2) there may be other useful economic interpretations of Petri nets beyond the one suggested in this paper.

Definition 1.2. A *marking* for a Petri net is a function $\mu : P \rightarrow \mathbb{N}$ (thus a marking can be thought of as a vector $\mu \in \mathbb{N}^n$: recall that n is the cardinality of the set P of places). A *marked Petri net* is a Petri net together with an initial marking μ.

Graphically, we can represent a marking in one of two ways: if the numbers $\mu_i \equiv \mu(p_i)$ $(i = 1, ..., n)$ are small, we draw, inside each place p_i, μ_i dots • called *tokens*; if the numbers are large, we write the number μ_i inside each place p_i. For example, the marking $\mu(p_1) = 4$, $\mu(p_2) = 3$ for the Petri net of Figure 1.1 can be represented in one of the two ways shown in Figure 1.2.

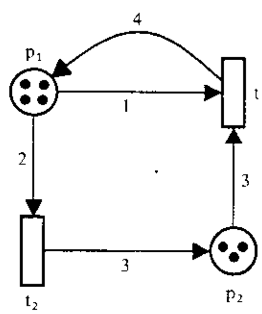


Figure 1.2a

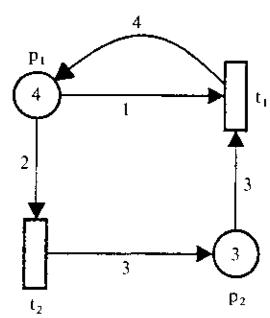


Figure 1.2b

Economic interpretation. A marking can be thought of as a vector of *available resources.* Thus Figure 1.2 represents a situation where the resources that are initially available are 4 units of commodity p_1 and 3 units of commodity p_2.

2. *Execution Rules for Petri Nets*

Definition 2.1. A transition of a marked Petri net is *enabled at marking* μ if each of its input places has at least as many tokens in it as the multiplicity of the arc from it to the transition. That is, transition t_j is enabled at μ if

$$(p_i, t_j) \in A \Rightarrow \mu(p_i) \geq \nu(p_i, t_j).$$

For example, in the marked Petri net of Figure 1.2, both transitions are enabled (since $\mu(p_1) = 4 > \nu(p_1, t_1) = 1$, $\mu(p_1) = 4 > \nu(p_1, t_2) = 2$, $\mu(p_2) = \nu(p_2, t_1) = 3$).

In the economic interpretation suggested above, to say that transition t_j is enabled at marking μ is to say that the production process represented by t_j can be activated (or operated) given the available resources. Thus in the example of Figure 1.2 production process t_2 can be activated, since it requires, for its operation, 2 units of commodity p_1, and (at least) 2 units of commodity p_1 are indeed available. Similarly, production process t_1 can be activated, given the available resources.

Definition 2.2. A transition can *fire at* μ only if it is enabled. When an enabled transition t_j fires, $\nu(p_i, t_j)$ tokens are removed from each input place p_i of t_j and $\nu(t_j, p_i)$ tokens are added in each output place p_i of t_j. Thus the firing of a transition at marking μ leads to a new marking μ' defined as follows [recall that if $(p_i, t_j) \notin A$, then $\nu(p_i, t_j) = 0$ by definition; similarly, if $(t_j, p_i) \notin A$, then, by definition, $\nu(t_j, p_i) = 0$]:

$$\mu'(p_i) = \mu(p_i) - \nu(p_i, t_j) + \nu(t_j, p_i) \quad (i = 1, ..., n).$$

Note that, since only enabled transitions may fire, *the number of tokens in each place always remains non-negative when a transition is fired.* Transition firings can continue as long as there is at least one enabled transition. When there are no enabled transitions, the execution *halts.*

Example 2.1. Consider the marked Petri net of Figure 1.2. Firing transition t_2 (operating production process t_2) leads to the marked Petri net of Figure 2.1.

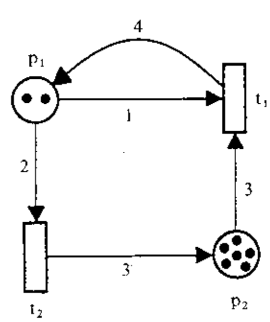


Figure 2.1a

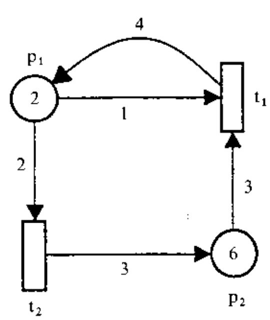


Figure 2.1b

In the marked Petri net of Figure 2.1, again both transitions are enabled. Firing transition t_2 (operating production process t_2 again) leads to the marked Petri net of Figure 2.2.

In the marked Petri net of Figure 2.2 none of the transitions is enabled: we have reached a *deadlock.* Thus if Figure 1.2 represents an economy at a

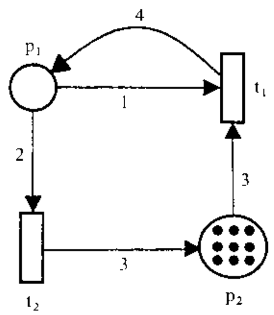


Figure 2.2a

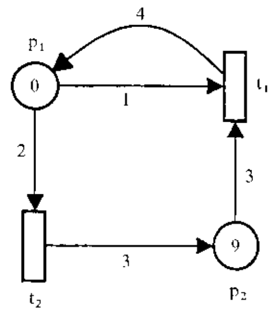


Figure 2.2b

given instant in time, operating only production process t_2 leads to a situation where commodity p_1 is depleted and no more production can take place (in this example a deadlock can be avoided by a suitable activation of production process t_1). In the next section we show how to represent the production possibilities that are associated with a given marked Petri net.

3. *The Reachability Digraph of a Petri Net*

Given a Petri net ⟨P, T, A, v⟩, the associated *reachability digraph* is the following arc-labeled infinite digraph. The set of vertices is $\mathbb{N}^n$ (the set of possible markings for the Petri net). Given two markings μ and μ' in $\mathbb{N}^n$, there is a directed arc from μ to μ' if and only if there is a set $S = \{t_{j_1}, ..., t_{j_r}\}$ of transitions all of which are enabled at μ and whose *simultaneous* firing is possible and leads from μ to μ'. For every such set S of transitions we draw an arc from μ to μ' and label it with S.

Example 3.1. Consider the Petri net of Figure 1.1. Figure 3.1 shows part of its reachability digraph. The path ⟨(4,3), t_2, (2,6), t_2, (0,9)⟩ is the one illustrated above in the sequence of Figures 1.2, 2.1 and 2.2 Note that both transitions t_1 and t_2 are enabled at marking (4,3) and also at marking (2,6). However, the *simultaneous* firing of the two transitions is only possible at (4,3) and not at (2,6) (firing both t_1 and t_2 at the same time requires at least 3 units of the commodity represented by place p_1). Thus there is no arc labeled $\{t_1, t_2\}$ out of node (2,6)[2].

Definition 3.1. A marking μ' is *reachable from* μ if either $\mu' = \mu$ or there is a

[2] Thus, in general, even if one can go from μ to μ' in two steps, by firing transitions t_j and t_k (with $k \neq j$) in any order, it may not be possible to go directly from μ to μ' by firing t_j and t_k *simultaneously*. The reason – as pointed out above – is that, although both t_j and t_k are enabled at μ (so that each can be fired in isolation), there may not be enough resources to fire both at the same time.

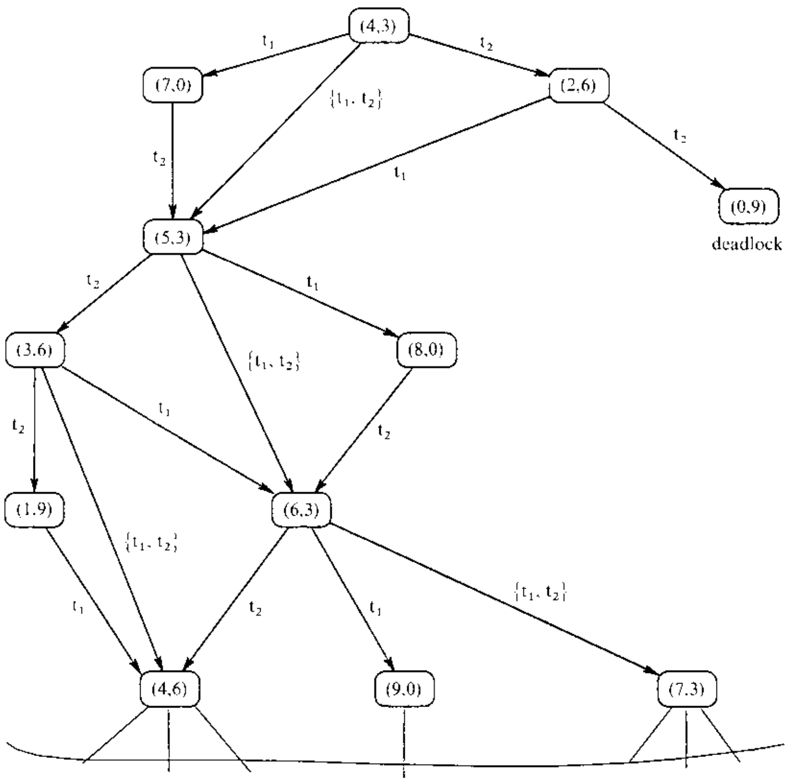


Figure 3.1

path from μ to μ' in the reachability digraph. The *reachability set of a marking* μ, denoted by $R(\mu)$, is the set of markings that are reachable from μ.

According to the economic interpretation suggested above, the reachability set $R(\mu)$ represents the *production possibilities associated with a given vector* μ *of initial resources* (that is, all the commodity vectors into which the initial resources can be transformed). In the example of Figure 1.2, where the initial resources are 4 units of commodity p_1 and 3 units of commodity p_2, it is possible, for instance, to double the quantity of commodity p_2 without reducing the quantity of commodity p_1: the commodity vector (4,6) can be obtained from the vector of initial resources (4,3) [cf. Figure 3.1].

The reachability set of a marking μ can be obtained from the reachability digraph of the Petri net by considering its maximal subgraph with μ as source. $R(\mu)$ will then coincide with the vertex set of this subgraph.

Furthermore, the subgraph would also give complete information concerning all the possible sequences of transition firings that lead from μ to any marking μ' reachable from μ. Unfortunately, this subgraph may be very large and in most cases is infinite. A more practical tool for analyzing questions of reachability will be discussed in Section 6.

The reachability digraph does not provide any information concerning *time*, which may be an important issue in modeling production. However, there is a straightforward way of modifying the reachability digraph so as to represent the amount of time involved in moving from a vector of initial resources μ to a new vector $\mu' \in R(\mu)$. Let τ_j be the number of units of time required to operate the production process represented by transition t_j. Given a set $S = \{t_{j_1}, ..., t_{j_r}\}$ of transitions, let τ_s be the amount of time that elapses if all the production processes in S are run *simultaneously*. Clearly $\tau_s = \max\{\tau_{j_1}, ..., \tau_{j_r}\}$. Then we can extend the label associated with an arc in the reachability digraph by adding the corresponding amount of time τ_s. For example, consider again the marked Petri net of Figure 1.2 and suppose that it takes 1 unit of time to operate process t_1 and 3 units of time to operate process t_2 (that is, $\tau_1 = 1$ and $\tau_2 = 3$). Hence it takes 3 units of time to operate the two processes simultaneously. It is then easy to see from Figure 3.1 that the minimum amount of time it takes to transform (4,3) into (4,6) is 9 periods (achieved by following the path $\langle(4,3), \{t_1, t_2\}, (5,3), \{t_1, t_2\}, (6,3)\ t_2, (4,6)\rangle$, that is, by running the two processes simultaneously twice and then t_2 only)[3].

4. *The Algebraic Representation of Petri Nets*

Given a Petri net $\langle P, T, A, \nu\rangle$ with n places and m transitions, the associated *input matrix* is the $n \times m$ matrix $A = (a_{ij})_{n \times m}$ where $a_{ij} = \nu(p_i, t_j)$; the associated *output matrix* is the $n \times m$ matrix $B = (b_{ij})_{n \times m}$ where $b_{ij} = \nu(t_j, p_i)$. For example, for the Petri net shown in Figure 1.1, the associated input matrix is

transitions

places

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

[3] In general, however, the minimum amount of time required to go from μ to μ' may be less than the sum of the time labels associated with the arcs that form the (relevant) path from μ to μ'. This is because when some processes are run simultaneously and one of them takes less time than the others, then it may be possible to restart this process (if required by the path) while the others are still running. Thus the sum of the time labels only gives an upper bound to the minimum time required.

while the associated output matrix is

$$\text{places}\quad B = \overset{\text{transitions}}{\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}}$$

Using the input and output matrices A and B we can recast the previous definitions in vector and matrix terms. Let $e_j \in \mathbb{N}^m$ be the unit vector whose j^{th} component is 1 and every other component is zero. Transition t_j is represented by the unit vector e_j. Now, transition t_j is enabled at marking μ if and only if

$$\mu \geq A\, e_j$$

and the result of firing transition t_j at marking μ is the new marking μ' given by

$$\mu' = \mu - A\, e_j + B\, e_j = \mu + (B - A)\, e_j.$$

On the other hand, starting at marking μ and firing the sequence of transitions $\sigma = t_{j_1} t_{j_2} \dots t_{j_k}$ leads to the new marking μ'' given by

$$\mu'' = \mu + (B - A)\, e_{j_1} + (B - A)\, e_{j_2} + \dots + (B - A)\, e_{j_k} = \mu + (B - A) f(\sigma)$$

where

$$f(\sigma) = e_{j_1} + e_{j_2} + \dots + e_{j_k}.$$

The non-negative integer vector $f(\sigma)$ is called the *firing vector* of the sequence $\sigma = t_{j_1} t_{j_2} \dots t_{j_k}$. The j^{th} component of $f(\sigma)$ is the number of times that transition t_j fires in the sequence $t_{j_1} t_{j_2} \dots t_{j_k}$.

Now, if marking μ' is reachable from marking μ, there exists a sequence σ (possibly empty) of transition firings that leads from μ to μ'. This implies that $f(\sigma)$ is a solution, in non-negative integers, for x in the following matrix equation[4]:

$$\mu' = \mu + (B - A)\, x \qquad (1)$$

Thus, if μ' is reachable from μ, then Equation (1) has a solution in non-negative integers; if Equation (1) has no such solution, then μ' is not

[4] Equation (1) can also be written as $\mu' - \mu = (B - A)x$, where the left-hand side represents the addition to the initial resources μ (net output). Thus it is a generalization of the well-known Leontief equation: $y = (I - A)x$. Note that, unlike in the Leontief case, the matrices A and B are not necessarily square and each process can produce several outputs (that is, joint production is allowed).

reachable from μ. Consider, for example, the Petri net shown in Figure 1.1, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. Let $\mu = \begin{pmatrix} 0 \\ 9 \end{pmatrix}$ and $\mu' = \begin{pmatrix} 0 \\ 8 \end{pmatrix}$. Then Equation (1) has a unique solution $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, which is not in non-negative integers. Hence $\begin{pmatrix} 0 \\ 8 \end{pmatrix}$ is not reachable from $\begin{pmatrix} 0 \\ 9 \end{pmatrix}$ (in fact, we know from Figure 3.1 that the latter is a dead-end). On the other hand, let $\mu = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mu' = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$. Then Equation (1) has the unique solution $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which corresponds to either the firing sequence $t_1\, t_2\, t_2$, to $t_2\, t_1\, t_2$ or $\{t_1, t_2\}\, t_2$, but to no other sequence (cf. Figure 3.1).

Note that the existence of a solution in non-negative integers of Equation (1) is a necessary *but not sufficient* condition for μ' to be reachable from μ. For example, consider again the Petri net shown in Figure 1.1, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$. Let $\mu = \begin{pmatrix} 0 \\ 9 \end{pmatrix}$, $\mu' = \begin{pmatrix} 1 \\ 12 \end{pmatrix}$. Then Equation (1) has the solution $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. However, $\begin{pmatrix} 1 \\ 12 \end{pmatrix}$ is not reachable from $\begin{pmatrix} 0 \\ 9 \end{pmatrix}$, since the latter is a dead-end (cf. Figure 3.1)[5].

We saw above that with a Petri net one can associate a pair of $n \times m$ (input-output) matrices (A, B) whose entries are non-negative integers. Conversely, given a pair of $n \times m$ matrices (A, B) whose entries are non-negative integers one can associate with it a Petri net as follows. Associate one place with each row of A and one transition with each column of A. Draw an arc from place p_i to transition t_j if and only if a_{ij} (the entry in the i^{th} row and j^{th} column of A) is positive and assign multiplicity a_{ij} to that arc. Finally, draw an arc from transition t_j to place p_i if and only if b_{ij} (the entry in the i^{th} row and j^{th} column of B) is positive and assign multiplicity b_{ij} to that arc. *Thus the notion of Petri nets is equivalent to the notion of a pair of $n \times m$ matrices (A, B) whose entries are non-negative integers.* The graph-theoretic definition has the obvious advantages that come from a visual representation of the underlying structure. For example, consider the problem of deciding

[5] It may seem that this problem can be solved as follows. First determine if Equation (1) has a solution in non-negative integers. If it does not, then μ' is not reachable from μ. Otherwise, let x be a solution in non-negative integers. Let k be the sum of the components of x (that is, $k = x_1 + x_2 + \ldots + x_m$). Consider the, at most k!, possible sequences of transition firings compatible with x (for example, if $x = (1,2)$, then all the possible firing sequences compatible with x are $t_1\, t_2\, t_2$, $t_2\, t_1\, t_2$ and $t_2\, t_2\, t_1$). If (at least) one of them is legal, μ' is reachable from μ. The problem with this procedure is that Equation (1) might have more than one solution in non-negative integers, even an infinite number of solutions (for an example see Peterson, 1981, p. 111; in that example $n < m$).

whether, in the marked Petri net of Figure 4.1 (where the multiplicity of each arc is 1), it is possible to obtain an arbitrarily large number of tokens in some or all of the places, and if so, how.

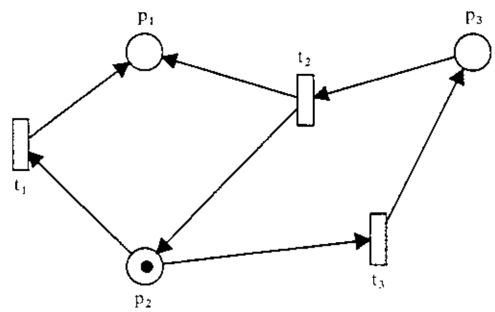


Figure 4.1

From the matrix representation the answer is not immediately obvious. The input matrix A and the output matrix B of the Petri net of Figure 4.1 are as follows:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From the graph-theoretic representation, however, it is apparent that the answer is "Yes" for place p_1 (all is needed is that the sequence $t_3\, t_2$ be fired a sufficiently large number of times) and "No" for the other two places.

The "producibility" problem – whether or not it is possible to obtain an arbitrarily large number of tokens in a given place, or set of places (that is, if it is possible to increase, through production, the quantity of a commodity or set of commodities) – is an important one and will be dealt with in Section 7.

5. *Petri Nets, Externalities and Returns to Scale*

We suggested an interpretation of Petri nets in terms of the production possibilities of a firm, or group of firms, or the entire economy: each place represents a commodity and each transition represents a production process. With this interpretation in mind, one might be tempted to conclude that a Petri net is "nothing more than a von Neumann input-output system with the added restriction that the entries of the input and output matrices be integers". The relationship between Petri nets and input-output systems is examined more thoroughly in Appendix C. Here we shall highlight the fact that two assumptions that have been inherently associated with linear production models – namely the absence of externalities between processes and the presence of constant returns to scale – are *not* implied by the notion of Petri nets [6].

We shall first of all show that externalities *can* be represented in a Petri net. Consider the following simple example. There are two firms near a lake. One firm is an oil refinery that uses one unit of oil to produce one unit of gasoline. Production of gasoline leaves chemical waste that the firm discharges in the lake. This chemical waste is a pollutant that reduces the population of fish in the lake. The other firm (a fisherman) uses a boat to fish. If the lake is not polluted, the fisherman can fish an average of 100 fish in one trip. If the lake is polluted, the catch will only be 20 fish per trip. This is a clear example of external diseconomies between the two production processes. Suppose that, initially, the lake is not polluted *and oil refining has not started yet.* For simplicity, we shall also suppose that the supply of oil is unilimited. This situation can be represented by the marked Petri net of Figure 5.1 (where the place corresponding to oil is marked with the symbol ∞ to denote the unlimited supply of oil; furthermore, for simplicity, the arc multiplicity has been omitted whenever it is equal to 1). The only restriction that we need to impose is that if μ_0 is an arbitrary *initial* marking, then

$$\mu_0[p_2] + \mu_0[p_3] = 1$$

expressing the fact that the conditions "the lake is polluted" and "the lake is not polluted" are mutually exclusive and each condition can only be either true (marking of 1) or false (marking of 0) [7]. A number of things should be noted about the Petri net of Figure 5.1:

(i) A *place does not necessarily represent a commodity* (if by commodity we mean a physical entity for which there is a market or price). Thus we have not only places representing the commodities: boats (p_1), fish (p_4), oil (p_5) and gasoline (p_6) but also places representing the condition, or state of nature, "the lake is polluted" (p_3) and the condition "the lake is unpolluted" (p_2).

(ii) Fishing does not create pollution and, therefore, if the lake is unpolluted

[6] It is also worth noting that, in principle, there is no need to assume that every row of B has at least one positive entry (this assumption – which is often made in the context of the von Neumann growth model – means that every commodity is produced by at least one process): some of the places of the Petri net could represent non-producible commodities (e.g. various kinds of labor). Also it is meaningful to have one or more columns of B consisting entirely of zeros: the corresponding transitions would then represent disposal processes.

[7] Note that if the initial marking μ_0 satisfies the above condition, then every marking reachable from μ_0 also satisfies that condition.

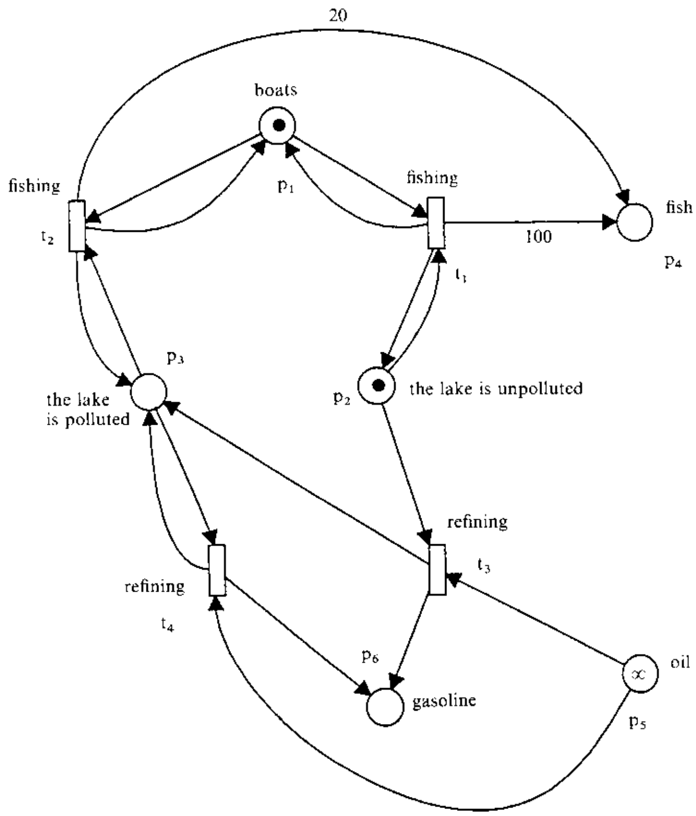


Figure 5.1

and fishing takes place at the high rate of 100 fish per trip, the lake remains unpolluted (p_2 is both an input to and an output of t_1).

(iii) The oil-refining process has been represented as two separate processes, one (transition t_3) using the unpolluted lake, and the other (transition t_4) using the polluted lake, as input. The former can be activated at most once (activating t_3 removes the token in p_2 for ever). After that, the lake becomes polluted and fishing at the *high* rate of 100 fish per trip (transition t_1) is no longer possible.

(iv) Fishing at the *low* rate of 20 fish per trip (transition t_2) and refining can coexist for ever (reflecting the simplifying assumption that the amount of pollution is constant, for example because of a constant inflow of clean water into the lake and a constant outflow of polluted water, e.g. through a river).

(v) The boats have been modeled as an infinitely lived capital good (p_1 is an input to, as well as an output of, both t_1 and t_2).

This example also shows that the graph-theoretic representation of Petri nets can be a very effective modeling tool. While it is very easy to grasp the situation depicted in Figure 5.1, it would be extremely hard to gain the same understanding by mere inspection of the corresponding input and output matrices, which are as follows (A is the input matrix and B the output matrix):

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 100 & 20 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Before we address the issue of returns to scale, we shall discuss the question of whether the main restriction embedded in the notion of Petri nets – namely that the entries of the input and output matrices are (non-negative) integers – is indeed a restriction. We argue that, from the point of view of applications, it is not. First of all, many commodities (e.g. pianos, washing machines, etc.) are produced in indivisible units and for them the integer constraint is actually a requirement rather than a restriction. Other commodities (e.g. milk, cream cheese, etc.) are produced in divisible units. However, for practical reasons, for each such commodity there is a minimum unit of measurement below which no further division takes place (e.g. for cream cheese grams or ounces). Taking the smallest possible (from a practical point of view) units of measurement for each such commodity, the integer constraint will obviously be satisfied. If one accepts the above argument in favor of integer constraints, then one must also accept that *each production process must have a minimum scale of operation*, bounded below by the production of the smallest (practically measurable) unit of each output.

We now show that Petri nets can also model situations where there are increasing returns to scale[8]. Of course, this is trivially true if one takes the

[8] What about *decreasing* returns to scale? From a logical point of view, the notion of decreasing returns to scale does not make sense. The notion of decreasing returns to a *factor* is certainly meaningful (it is illustrated in the classical problem of the continual addition of labor to a fixed amount of land, say, one acre). Decreasing returns to *scale*, on the other hand, means that

point of view that integer constraints (that is, indivisibilities) are the essence of the notion of increasing returns to scale. According to this point of view – which is not the one taken here – Petri nets can model *only* increasing returns to scale! We shall adopt a definition of returns to scale which separates the integer constraint problem from the issue of whether a production process can be scaled up or down. For a detailed discussion the reader is referred to Appendix A. Intuitively, constant returns to scale means that by doubling all the inputs one obtains exactly double the amount of output, while increasing returns to scale means that by doubling all the inputs one can more than double the output. If a production process is characterized by constant returns to scale, then – for the type of questions examined in this paper (reachability, coverability, producibility, etc.) – it is sufficient to list the minimum scale intensity of the process: scaling the process up by a factor of n is the same as firing the corresponding transition n times. On the other hand, if a production process is characterized by increasing returns to scale, then it can be represented by a number of different transitions, each representing a minimum scale of operation. Consider the following simple example: if a group of fishermen use one boat, they can get, on average, 100 fish per trip, while if they use two boats, their catch is, on average, 250 fish.

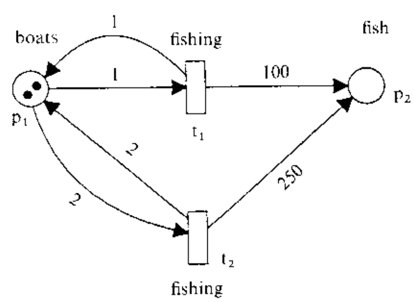


Figure 5.2

This situation can be represented by the Petri net of Figure 5.2. If the initial marking is, for example, $\mu[p_1] = 2$ and $\mu[p_2] = 0$, then one can either fire transition t_1 twice or transition t_2 once. In the first case the new marking

doubling *all* the inputs (in the previous example, land as well as labor) leads to less than double the amount of output. However, if all the inputs to a production process are listed, then *repetition or duplication of the process must yield double amount of each output!* It is generally agreed that "decreasing returns to scale require the presence of an extra input, not listed in the arguments in the production function, that cannot be duplicated" (Silvestre, 1987, p. 80). In a Petri net the availability of inputs is represented by the notion of marking, which is independent of the notion of transition.

will be (2, 200), in the second case it will be (2, 250) [note that the Petri net of Figure 5.2 reflects the assumption that boats are infinitely lived capital goods].

6. *Production Possibilities and the Karp-Miller Tree*

In Section 3 we defined, for a given Petri net with initial marking (vector of resources) μ, the reachability set $R(\mu)$ as the set of those markings (commodity vectors) that can be obtained from μ by some sequence of transition firings (operations of production processes). We saw that one way of obtaining the set $R(\mu)$ is by constructing the reachability digraph. There may be cases, however, where one is interested not in constructing the whole set $R(\mu)$, but in establishing whether or not a particular commodity vector μ' belongs to this set, that is, is reachable from the initial vector of resources μ. This is the so-called *reachability problem* for Petri nets:

Reachability problem: given an initial marking μ and a marking μ', is μ' reachable from μ? Mayr (1984) showed that it is decidable whether or not a marking μ' can be reached from μ. However, the corresponding algorithm is of exponential complexity (in storage space and time). On the other hand, the *coverability* problem can be decided with a much simpler algorithm (yielding the so-called Karp-Miller coverability tree).

Coverability problem. Given an initial marking μ and a marking μ', does there exist a marking μ'' such that:

(i) μ'' is reachable from μ, and

(ii) $\mu'' \geq \mu'$?

Karp and Miller (1969) constructed an algorithm that yields the so-called *coverability tree.* The aim is to replace the (usually infinite) reachability digraph with a *finite tree.* Associated with each node of the tree is an *extended marking.* Recall that a marking is a point in $\mathbb{N}^n$ (where n is the number of places). An extended marking is a point in the set $(\mathbb{N} \cup \{\infty\})^n$. The simbol ∞ stands for "infinity" and represents a number of tokens that can be made arbitrarily large. For any integer k, we define:

$$\infty + k = \infty$$

$$\infty - k = \infty$$

$$k < \infty$$

$$\infty \leq \infty.$$

Given an initial marking μ_0, we associate μ_0 with the root of the tree. We then proceed as in the reachability digraph, except that: (i) we create a new node for every marking (this is a necessary condition for the result to be a tree), (ii) we introduce rules aimed at making every path from the root finite. Thus starting from the root any path leads to a terminal node. Obvious terminal nodes are *dead ends* (that is, markings at which no transition is enabled), or nodes whose associated markings are duplicates of markings previously obtained. The symbol ∞ is used to obtain the remaining terminal nodes. Consider a sequence of transition firings σ which starts at a marking μ and ends at a marking μ' with $\mu' \geq \mu$, $\mu' \neq \mu$ (thus, at least one component of μ' is greater than the corresponding component of μ). Clearly, all the transitions that were enabled at μ are also enabled at μ'. Thus the sequence σ can be fired again starting from μ' and will lead to a new marking $\mu'' = \mu' + (\mu' - \mu)$ [since the sequence σ adds the vector of tokens $(\mu' - \mu)$]. If we fire σ n times, we add the vector of tokens $(\mu' - \mu)$ n times. Thus, for those places which gained tokens from the sequence σ, we can create an arbitrarily large number of tokens simply by repeating the sequence σ as often as desired. For example, it can be seen from Figure 3.1 that firing the sequence $\sigma = t_1 t_2$ from a marking (x, y) leads to marking (x + 1, y): firing σ from (4,3) leads to (5,3), firing σ from (2,6) leads to (3,6), etc. When we obtain a marking $\mu' \geq \mu$, $\mu' \neq \mu$, we can replace μ' with an extended marking where there is the symbol ∞ in place of those components of μ' that are greater than the corresponding components of μ.

The Karp-Miller coverability tree is constructed by the following algorithm (which will be illustrated in Figures 6.1-6.5). Every node v of the tree is assigned two labels: an extended marking $\mu[v]$ and the label $\lambda[v] \in$ {temporary, interior, duplicate, dead-end, infinite}. The algorithm terminates when there are no nodes v such that $\lambda[v]$ = "temporary".

Step 1. Let μ_0 be the initial marking. Label the root v_0 as follows: $\mu[v_0] = \mu_0$, $\lambda[v_0]$ = "temporary".

Step 2. While nodes v such that $\lambda[v]$ = "temporary" exist, do the following:

Step 2.1. Select a node v such that $\lambda[v]$ = "temporary".

Step 2.2. If $\mu[v]$ is identical to $\mu[v']$ for some node $v' \neq v$ with $\lambda[v'] \neq$ "duplicate", set $\lambda[v]$ = "duplicate".

Step 2.3. If no transition is enabled at $\mu[v]$, set $\lambda[v]$ = "dead-end".

Step 2.4. If each coordinate of $\mu[v]$ is the symbol ∞, set $\lambda[v]$ = "infinite".

Step 2.5. While there exist enabled transitions at $\mu[v]$, do the following for each enabled transition at $\mu[v]$.

Step 2.5.1. Set $\lambda[v]$ = "interior". Draw a new vertex w and an arc from v to w. Label the arc with transition t. Obtain the marking μ' that results from firing t at $\mu[v]$.

Step 2.5.2. If on the path from the root to v there exists a node $z \neq v$ such that $\mu' \geq \mu[z]$ and $\mu' \neq \mu[z]$, then replace each component of μ' which is greater than the corresponding component of $\mu[z]$ with the symbol ∞.

Step 2.5.3. Set $\mu[w] = \mu'$ and $\lambda[w]$ = "temporary".

It can be shown that the Karp-Miller algorithm terminates (all nodes are labeled as either interior or duplicate or dead-end or infinite) and therefore yields a finite tree. Thus the coverability problem is decidable.

Example 6.1. Consider the marked Petri net of Figure 1.2. With the aid of Figure 3.1 it is easy to see that the Karp-Miller algorithm yields the coverability tree of Figure 6.1. In fact, at (4,3) both t_1 and t_2 are enabled. Firing t_1 leads to (7,0) while firing t_2 leads to (2,6). At (7,0) the only transition that is enabled is t_2. Firing t_2 at (7,0) yields (5,3) which is greater than (4,3) (the initial marking): the first component is greater while the second is equal. Thus we replace 5 with ∞ and attach label $(\infty, 3)$ to node v_3. Going now to node v_2, at

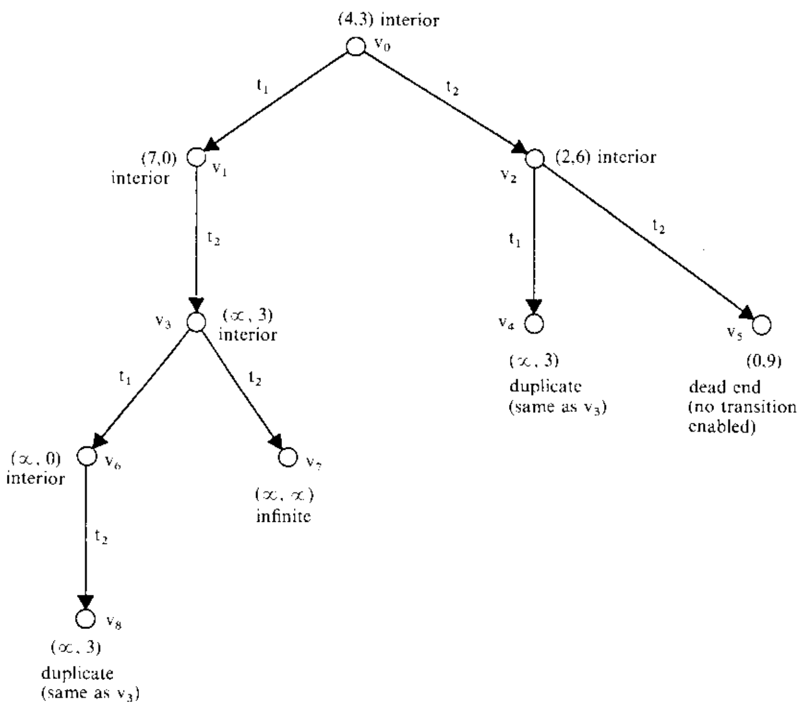


Figure 6.1

(2,6) both transitions are enabled. Firing t_2 leads to (0,9) which is a dead end (no transition is enabled). Firing t_1 leads to $(5,3) \geq (4,3)$, $(5,3) \neq (4,3)$. Thus we replace 5 with ∞ and attach $(\infty, 3)$ to node v_4. We have obtained a duplicate of node v_3. Now go back to node v_3: both transitions are enabled. Firing t_1 leads to ($\infty + 3 = \infty$, 0). Firing t_2 leads to ($\infty - 3 = \infty$, 6) which is greater than (∞, 3), the second component being greater. Thus we replace 6 with ∞ and obtain the label (∞, ∞) for node v_7. Now we go back to node v_6, firing transition t_2 leads to ($\infty - 2 = \infty$, 3) which is a duplicate of node v_3.

Example 6.2. Consider the marked Petri net of Figure 6.2. Using the Karp-Miller algorithm we obtain the tree shown in Figure 6.3.

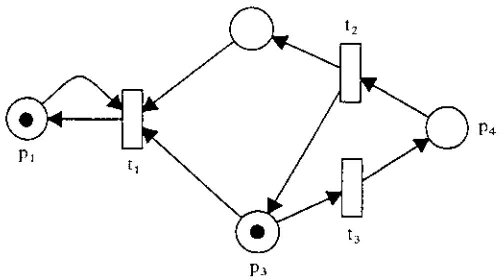


Figure 6.2

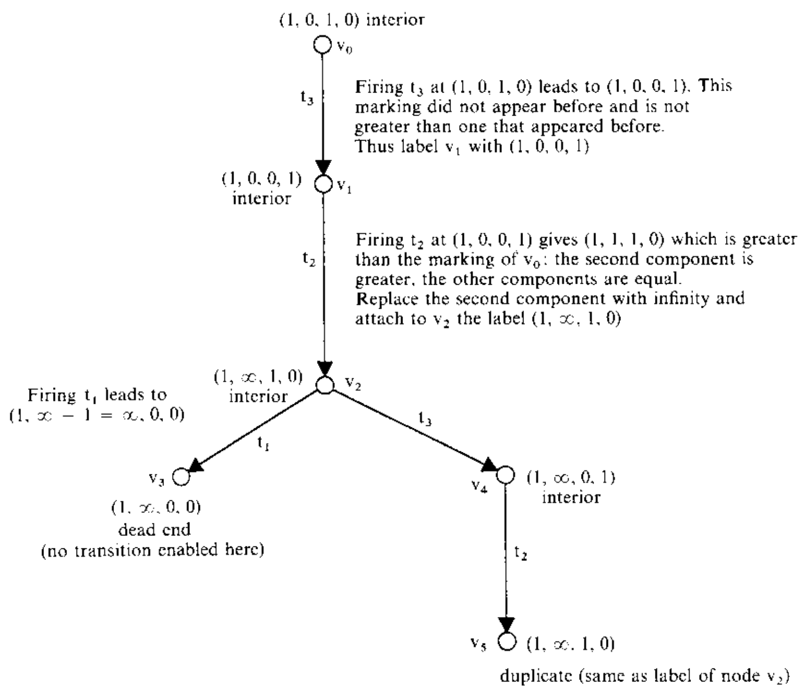


Figure 6.3

Although the Karp-Miller tree answers the coverability question, the use of the symbol ∞ involves an important loss of information. For example:
(i) two essentially different Petri nets might have the same coverability tree (for an example see Peterson, 1981, p. 104);
(ii) even if the coverability tree has no nodes labeled "dead-end" (and even if there is a node labeled "infinite"), the net may deadlock. Consider for example the marked Petri net of Figure 6.4, whose reachability tree is shown in Figure 6.5.

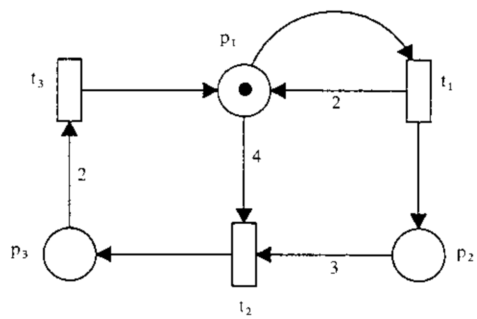


Figure 6.4

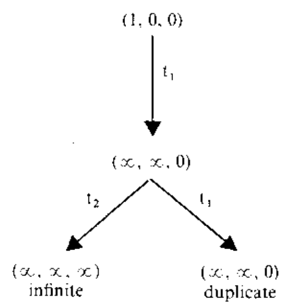


Figure 6.5

The following firing sequence leads to a deadlock:

$$(1,0,0) \xrightarrow{t_1} (2,1,0) \xrightarrow{t_1} (3,2,0) \xrightarrow{t_1} (4,3,0) \xrightarrow{t_2} (0,0,1) \text{ deadlock.}$$

7. *Augmentable Commodities*

Given an initial marking (interpreted, economically, as a vector of initial resources), it is easy to check if it is possible to produce an arbitrarily large number of units of commodity i by simple inspection of the corresponding Karp-Miller coverability tree: if there is a node in the tree whose corresponding extended marking has the symbol ∞ as its i^{th} component, then the answer is affirmative, otherwise it is negative. However, one could ask the same question *without reference to a specific vector of initial resources.* This motivates the following definition.

Definition. Commodity i (represented by place p_i) is *augmentable* if there exists an initial marking μ_0 such that, for every positive integer N, there exists a marking μ reachable from μ_0 whose i^{th} component is greater than or equal to N.

In other words, commodity i is augmentable if there is at least one initial marking μ_0 with the property that, in the associated Karp-Miller coverability tree, there is a node whose corresponding extended marking has the symbol ∞ as its i^{th} component. With this interpretation in mind it is easy to see that the following lemma is true.

Lemma 7.1. Consider a Petri net with corresponding input matrix A and output matrix B. Commodity i is augmentable if and only if there exists an $x \in \mathbb{N}^m$ such that

$$(B - A)x \geq e_i$$

(where $e_i \in \mathbb{N}^n$ is the vector whose i^{th} component is 1 and every other component is 0).

The following proposition shows that one can check whether or not a commodity is augmentable by solving a simple linear program without having to impose the constraint that the solution be in integers (that is, not an integer program)[9].

Proposition 7.1. Consider a Petri net with corresponding $n \times m$ input matrix A and output matrix B. Fix an arbitrary $j \in \{1, 2, ..., m\}$ and let $e_j \in \mathbb{N}^m$ be the unit vector whose j^{th} coordinate is 1 and every other coordinate is 0. Then, for every $i = 1, ..., n$, commodity i is augmentable if and only if the following *linear* program (note: *not* integer program) has a solution:

$$\text{minimize } x \cdot e_j$$

$$\text{subject to:} \quad x \in \mathbb{R}^m, \quad (B - A)\, x \geqq e_i \text{ and } x \geq 0,$$

where $e_i \in \mathbb{N}^n$ is the i^{th} unit vector (note that $x \cdot e_j$ is the j^{th} coordinate of x).

Proof. See Appendix B.

The following lemma and proposition are the dual of Lemma 7.1 and Proposition 7.2. A proof can be found in Appendix B.

[9] There does not seem to be a clear economic interpretation of Propositions 7.1 and 7.2.

Lemma 7.2. Commodity i is *not* augmentable if and only if there exists a $y \in \mathbb{N}^n$ such that

$$(B - A)^T y \leq 0 \text{ and } (y)_i \geq 1$$

where 'T' denotes transpose and $(y)_i$ denotes the i^{th} coordinate of y.

Proposition 7.2. Consider a Petri net with corresponding $n \times m$ input matrix A and output matrix B. Fix an arbitrary $k \in \{1, 2, ..., n\}$ and let $e_k \in \mathbb{N}^n$ be the unit vector whose k^{th} coordinate is 1. Then, for every $i = 1, ..., n$, commodity i is *not* augmentable if and only if the following linear program (note: *not* integer program) has a solution:

$$\text{minimize } y \cdot e_k$$

$$\text{subject to:} \quad y \in \mathbb{R}^n, \quad \begin{pmatrix} (A - B)^T \\ e_i \end{pmatrix} y \geqq \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \quad \text{and } y \geq 0,$$

where $e_i \in \mathbb{N}^n$ is the i^{th} unit vector and $\mathbf{0} \in \mathbb{N}^m$ is the vector all of whose coordinates are 0 (note that $y \cdot e_k$ is the k^{th} coordinate of y).

Concluding Remarks

The purpose of this paper was to bring to the attention of economists Petri nets, a tool developed in computer science. Although, from a purely formal point of view, Petri nets are not a new tool (since a Petri net is equivalent to a generalized input-output system with integer coefficients), they do seem to provide a new perspective on models of production, whether it is at the level of a firm, of a group of firms or of the whole economy. First of all, the graph-theoretic representation of Petri nets makes it possible to see things that would be hard to detect from a purely algebraic formulation of the same problem. Secondly, the formal definition of a Petri net allows one to introduce a wedge between the notions of input and output (to a production process) and the notion of commodity. Among the inputs to (and outputs of) a production process one can include states of nature, logical conditions, etc. This enabled us to show that one of the assumptions which is usually considered to be inherent to linear models of production, namely the absence of external economies and diseconomies among processes, is not required in a Petri net model of production. We also showed that Petri nets do *not* require another assumption normally associated with activity analysis, namely that of constant returns to scale. Finally, Petri nets allow a simple analysis of a problem, which so far has received little attention in general input-output analysis, namely what commodity vectors can be obtained from a given vector of initial resources.

APPENDIX A

In this appendix we discuss the notion of returns to scale. Let n be the number of commodities and $Y \subseteq \mathbb{R}^n$ be a production set. Debreu (1959, pp. 40-41) gives the following definitions:

Constant returns to scale (each production vector can be scaled up or down):

$y \in Y, \lambda \in \mathbb{R}^+ \Rightarrow \lambda y \in Y$ (where $\mathbb{R}^+$ is the set of non-negative real numbers).

Non-decreasing returns to scale (each production vector can be scaled up):

$$y \in Y, \lambda \in \mathbb{R}, \lambda > 1 \Rightarrow \lambda y \in Y.$$

Increasing returns to scale: there are non-decreasing returns to scale and there is a possible production for which the scale of operations cannot be arbitrarily decreased.

According to this definition, whenever there are integer constraints, constant returns to scale are ruled out *by definition*. One is therefore forced to say that Petri nets can only model increasing returns to scale. In what follows we shall put forward alternative definitions which are tailored to the case where there are integer constraints. In order to isolate the integer constraint problem from the notion of returns to scale we shall take the commodity space to be not $\mathbb{R}^n$ but $\mathbb{N}^n$. A production set is a subset $Y \subseteq \mathbb{N}^n \times \mathbb{N}^n$. If $(x, y) \in Y$ then x represents inputs and y outputs.

Definition. Following Debreu, we say that production set $Y \subseteq \mathbb{N}^n \times \mathbb{N}^n$ has *non-decreasing returns to scale* if

$$(x, y) \in Y, \quad \lambda \in \mathbb{N}, \lambda \geq 1 \Rightarrow \lambda(x, y) \in Y.$$

Definition. $P \subseteq \mathbb{N}^n \times \mathbb{N}^n$ is a *linear production process* with minimum scale $(x_0, y_0) \neq 0$ if:

(i) $$P = \{(x, y) : (x, y) = \lambda(x_0, y_0) \text{ for some } \lambda \in \mathbb{N}/\{0\}\}$$

(ii) $$\text{for every } \lambda \in \mathbb{N} \text{ with } \lambda > 1, \frac{1}{\lambda}(x_0, y_0) \notin P.$$

For example, Figure A.1a shows a linear production process with minimum scale (4,1) while Figure A.1b shows a linear production process with minimum scale (6,2) [10].

[10] Figure A.1a can be interpreted as follows. There are two goods, 1 and 2. Good 1 is an input only and good 2 is an output only. The process represented by Figure A.1a is the set $P = \{(4, 0, 0, 1), (8, 0, 0, 2), (12, 0, 0, 6) ...\}$. Thus P is a 4-dimensional set and Figure A.1a gives a 2-dimensional projection. Similarly for Figure A.1b.

In a Petri net each transition represents the minimum scale of operation of a linear production process.

The production set associated with a Petri net is the set generated by a finite number of linear production processes with minimum scale of operation represented by the corresponding transition. It is clear that the production set of a Petri net has non-decreasing returns to scale. In order to disentangle indivisibilities from scale economies we suggest the following definition of constant and increasing returns to scale.

Definition. Given a production set $Y \subseteq \mathbb{N}^n \times \mathbb{N}^n$, the following set is the set of *efficient production vectors*:

$$Y^E = \{(x, y) \in Y : \forall\, (x', y') \in \mathbb{N}^n \times \mathbb{N}^n \text{ with } (x', y') \neq (x, y)\ x' \leq x \text{ and } y' \geq y,\ (x', y') \notin Y\}.$$

Definition. A production set Y has *constant returns to scale* if

$$(x, y) \in Y^E,\ \lambda \in \mathbb{N}/\{0\} \Rightarrow \lambda(x, y) \in Y^E.$$

Definition. A production set Y has *increasing returns to scale* if (i) it has non-decreasing returns to scale and (ii) $\exists\,(x, y) \in Y^E, \exists\,\lambda \in \mathbb{N}/\{0\}$ such that $\lambda(x, y) \notin Y^E$.

Thus, for example, in both Figure A.1a and Figure A.1b we have a production set (consisting of a single process) displaying constant returns to scale. If Y is the production set generated by these two processes, then Y has increasing returns to scale. In fact letting $(x, y) = (4,1)$ and $\lambda = 3$, we have that $(x, y) \in Y^E$ but $\lambda(x, y) = (12,3) \notin Y^E$, because $(x', y') = (12,4) \in Y$[11].

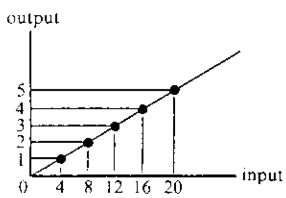


Figure A.1a

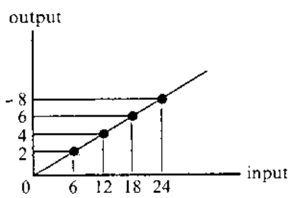


Figure A.1b

[11] To be consistent with our notation we should have written the production vectors as (4, 0, 0, 1), (12, 0, 0, 3) and (12, 0, 0, 4): cf. the previous footnote.

APPENDIX B

In this appendix we prove Propositions 7.1 and 7.2 and Lemma 7.2. We first prove some lemmas.

Lemma B.1. Let A and B be $n \times m$ matrices whose entries are integers and let $e_i \in \mathbb{N}^n$ be the unit vector whose i[th] coordinate is 1 and every other coordinate is 0. Let $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x \geq 0\}$ (where $\mathbb{R}$ denotes the set of real numbers). Then the following conditions are equivalent:

(i) $\exists\, x \in \mathbb{N}^m$ such that $(B - A)\, x \geq e_i$,

(ii) $\exists\, x \in \mathbb{R}^m_+$ such that $(B - A)\, x \geq e_i$.

Proof. That (1) $\Rightarrow$ (2) is obvious, since $\mathbb{N}^m \subseteq \mathbb{R}^m_+$. We now show that (2) $\Rightarrow$ (1). Suppose that $T = \{x \in \mathbb{R}^m_+ \mid (B - A)\, x \geq e_i\}$ is non-empty. Since all the entries of A, B and e_i are integers, there must be a point $x_0 \in T \cap \mathbb{Q}^m_+$, where $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{Q}^m_+ = \{x \in \mathbb{Q}^m \mid x \geq 0\}$ [see Chvatal (1983)]. Then the j[th] coordinate of x_0 is equal to $\frac{p_j}{q_j}$ for some $p_j, q_j \in \mathbb{N}$, with $q_j \neq 0$. Let $\alpha = \prod_{j=1}^{m} q_j$. Then $\alpha \in \mathbb{N}$, $\alpha > 0$, $\alpha x_0 \in \mathbb{N}^m$ and $(B - A)\,(\alpha x_0) = \alpha (B - A)\, x_0 \geq \alpha\, e_i \geq e_i$.

Proof of proposition 7.1. By Lemma 7.1, commodity i is augmentable if and only if the set $S = \{x \in \mathbb{N}^m_+ \mid (B - A)\, x \geq e_i\}$ is non-empty. By Lemma B.1, S is non-empty if and only if $T = \{x \in \mathbb{R}^m_+ \mid (B - A)\, x \geq e_i\}$ is non-empty. Thus it only remains to show that T is non-empty if and only if, for an arbitrary $j \in \{1, 2, ..., m\}$, the following linear program has a solution:

$$\text{minimize } x \cdot e_j$$

$$\text{subject to:} \quad x \in \mathbb{R}^m, \quad (B - A)\, x \geqq e_i \quad \text{and } x \geq 0,$$

where e_j is the j[th] unit vector in $\mathbb{N}^m$ and e_i is the i[th] unit vector in $\mathbb{N}^n$. It is clear that if the above program has a solution, then T is non-empty. We only need to show the converse. Suppose that T is non-empty. Then [cf. Nef (1967), Theorems 2 and 3, pp. 144 and 146] $T = K_1 + K_2$, where K_1 is a convex polyhedron, whose vertices can be taken to be the basic solutions of the system of inequalities $(B - A)\, x \geq e_i$ and $x \geq 0$, and K_2 is a convex pyramid. Let $a_1, ..., a_r$ be the vertex vectors of K_1 and let $b_1, ..., b_s$ be the generators of K_2. Thus if $x_0 \in T$ then there exist non-negative real numbers $\lambda_1, ..., \lambda_r, ..., \mu_s$ such that $\sum_{k=1}^{r} \lambda_k = 1$ and

$$x_0 = \sum_{k=1}^{r} \lambda_k a_k + \sum_{k=1}^{s} \mu_k b_k$$

Thus

$$x_0 \cdot e_j = \sum_{k=1}^{r} \lambda_k (a_k)_j + \sum_{k=1}^{s} \mu_k (b_k)_j$$

Since, for every k = 1, ..., s, $b_k \geq 0$, it follows that $(b_k)_j \geq 0$ and therefore [see Nef (1967), p. 150] the linear program has at least one solution (furthermore, there is at least one solution which is a basic solution).

Proof of lemma 7.2. By lemma 7.1, commodity i is not augmentable if and only if the set $S = \{x \in \mathbb{N}^m \mid (B - A)x \geq e_i\}$ is empty. By Lemma B.1, S is empty if and only if $T = \{x \in \mathbb{R}^m \mid (B - A)x \geq e_i\}$ is empty. By the Minkowski-Farkas Lemma [see, for example, Hu (1969), pp. 8-9], T is empty if and only if $U = \{y \in \mathbb{R}^n_+ \mid (B - A)^T y \leq 0$ and $y \cdot e_i > 0\}$ is non-empty. By an argument similar to the one used in the proof of Lemma B.1 (that is, by appealing to the fact that the entries of A, B and e_i are non-negative integers), one can show that U is non-empty if and only if $V = \{y \in \mathbb{N}^n \mid (B - A)^T y \leq 0$ and $y \cdot e_i = (y)_i \geq 1\}$ is non-empty (note that $y \in \mathbb{N}^n$ and $y \cdot e_i > 0$ implies $y \cdot e_i \geq 1$), which in turn is equivalent to non-emptiness of $W = \{y \in \mathbb{R}^n_+ \mid (B - A)^T y \leq 0$ and $y \cdot e_i \geq 1\}$.

Proof of proposition 7.2. In the proof of Lemma 7.2 it was shown that commodity i is not augmentable if and only if the set $W = \{y \in \mathbb{R}^n_+ \mid (B - A)^T y \leq 0$ and $y \cdot e_i \geq 1\}$ is non-empty. The inequalities $(B - A)^T y \leq 0$ and $y \cdot e_i \geq 1$ can also be written [since $(B - A)^T y \leq 0$ is equivalent to $-(B - A)^T y \geq 0$, which in turn is equivalent to $(A - B)^T y \geq 0$] as:

$$\text{(B1)} \qquad \begin{pmatrix} (A - B)^T \\ e_i \end{pmatrix} y \geqq \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$$

where $\mathbf{0}$ is the origin in $\mathbb{R}^m$. By an argument similar to the one used in the proof of Proposition 7.1, the subset of $\mathbb{R}^n_+$ that satisfies (B1) is non-empty if and only if the following linear program has a solution:

$$\text{minimize } y \cdot e_k$$

$$\text{subject to:} \quad y \in \mathbb{R}^n, \quad \begin{pmatrix} (A - B)^T \\ e_i \end{pmatrix} y \geqq \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} \quad \text{and } y \geq 0.$$

APPENDIX C

Input-output Systems and Activity Analysis

In this appendix we discuss the relationship between Petri nets and input-output systems, which were introduced by von Neumann in 1937 [12]. An *input-output system* is a pair of real, non-negative, $n \times m$ matrices (A, B), where A is the input matrix and B is the output matrix. Each row of A (and B) represents a commodity and each column of A (and B) represents a basic production process. The j^{th} column of A, a_j, gives, for each commodity, the quantity (possibly zero) *used* by basic process j, while the j^{th} column of B, b_j, gives, for each commodity, the quantity (possibly zero) *produced* by basic process j. In input-output (or activity) analysis it is assumed that for every $j = 1, \ldots, m$ and for every real number $\lambda > 0$, the production process that transforms λa_j into λb_j is technologically feasible (constant returns to scale) and that if j and k are basic processes, then the process that transforms $a_j + a_k$ into $b_j + b_k$ is also technologically feasible (additivity). Thus every vector $x \in \mathbb{R}^n$, $x \geq 0$, called an *intensity vector*, represents a feasible production process that transforms Ax into Bx.

Now we turn to a discussion of the relationship between the notion of augmentable commodity introduced in Section 7 and the two notions of von Neumann growth rate and of productivity of a Leontief matrix.

Von Neumann's (1945) objective was to determine the maximum rate of *proportional* growth of an arbitrary input-output system [we shall follow the version of von Neumann's model given by Gale (1956)]. To this purpose, given an intensity vector x and a commodity i, define the expansion rate $\alpha_i(x)$ of i in x as follows (recall that if y is a vector, $(y)_i$ denotes the i^{th} component of y):

$$\alpha_i(x) = \begin{cases} \dfrac{(Bx)_i}{(Ax)_i} & \text{if } (Ax)_i > 0 \\ \infty & \text{if } (Bx)_i > 0 \text{ and } (Ax)_i = 0 \\ \text{undefined} & \text{if } (Bx)_i = (Ax)_i = 0 \end{cases}$$

[12] The so-called "activity analysis" [see, for example, Koopmans (1951)] is covered by the notion of (von Neumann) input-output system (if the number of processes is finite). In the special case where $n = m$ and B is the identity matrix – each production process is an industry and each industry produces a single, homogeneous, product – then A is called a *Leontief matrix* (Leontief, 1941).

The technological expansion rate of intensity vector x, $\alpha(x)$, is defined by

$$\alpha(x) = \min_{i=1,\ldots,n} \alpha_i(x).$$

Finally, the technological expansion rate of the system (A, B), α, is defined by

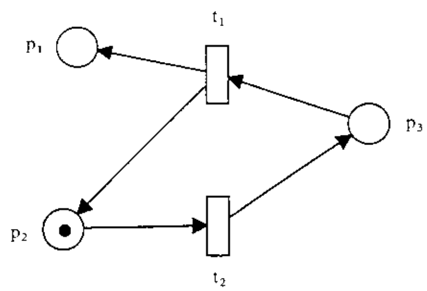
$$\alpha = \max_{x \in \mathbb{R}^m, x \geq 0} \alpha(x).$$

An intensity vector $\hat{x}$ such that $\alpha(\hat{x}) = \alpha$ is called *optimal*. Gale (1956) showed that α is well-defined and $0 < \alpha < \infty$, *if and only if* the following condition holds: every column of A has a positive entry (i.e. every basic process requires at least one input) and every row of B has at least one positive entry (i.e. every commodity is produced by at least one basic process). If the technological expansion rate α is greater than 1 *and* there is a corresponding optimal intensity vector $\hat{x}$ such that *all* of its components are positive, then the system can grow at the rate of $100(\alpha - 1)\%$ per period, in the sense that every commodity will grow *at least* at that rate, although some commodities may grow at a faster rate.

What is the relationship between the von Neumann expansion rate α and the notion of augmentable commodity discussed in Section 7? First of all, $\alpha > 1$ does not imply that every commodity is augmentable. For example, let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$. Then $\alpha = 3$ and yet only commodity 1 is augmentable. Secondly, even if $\alpha = 1$ it may be possible to produce arbitrarily large quantities of some (up to $n - 1$) commodities. For example, let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\alpha = 1$ and every optimal intensity vector is a scalar multiple of (1,1). While commodities 2 and 3 are not augmentable, commodity 1 is. An easy way of seeing this is by means of the Karp miller coverability tree shown in Figure C.1, where it is assumed that the initial vector of resources (the initial marking) is (0,1,0). Thus from the fact that $\alpha > 1$ one cannot deduce that every commodity is augmentable and from the fact that $\alpha = 1$ one cannot deduce that no commodity is augmentable.

The notion of augmentable commodity is also related (but not identical) to the notion of a productive Leontief matrix. Gale (1960, p. 296) defines a Leontief matrix to be productive if there exists a non-negative vector $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} > A\bar{x}$. This definition can be extended to a general input-output system (A, B) by calling it productive whenever there exists a non-negative vector $\bar{x} \in \mathbb{R}^n$ such that $B\bar{x} > A\bar{x}$. Two observations can be made concerning this definition. First of all, apart from the simple case of a Leontief matrix [13], there is no simple way of checking whether or not an input-output system is productive. Secondly, it may be possible to have an economy where all the commodities, except one, are *augmentable* (that is, their quantity can be increased through production) and yet, according to the above definition, the economy is *not* productive (an example of this was given above).

[13] It can be shown [see Gale (1960), chapter 9] that a Leontief matrix A is productive if and only if the maximum eigenvalue of A is less than 1.



The input-output system is: $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ with initial resources (0,1,0).

The corresponding coverability tree is:

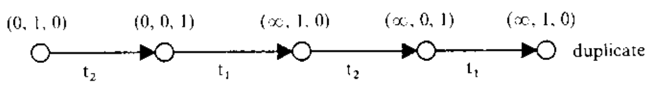


Figure C.1

REFERENCES

G. BONANNO (1993), "The Coverability Problem in Input-Output Systems. *Economics Letters*", 43, pp. 15-21.

V. CHVATAL (1983), *Linear Programming*, W. H. Freeman, New York.

G. DEBREU (1959), *Theory of Value*, Yale University Press, New Haven.

D. GALE (1956), "The Closed Linear Model of Production", in H. W. Kuhn and A. W. Tucker (Eds.), *Linear Inequalities and Related Systems*, pp. 285-303, Princeton University Press, Princeton.

D. GALE (1960), *The Theory of Linear Economic Models*, McGraw Hill, New York.

T. C. HU (1969), *Integer Programming and Network Flows*, Addison-Wesley, Reading (Mass.).

R. M. KARP - R. E. MILLER (1969), "Parallel Program Schemata", *Journal of Computer and System Science*, 3, pp. 247-295.

T. KOOPMANS (Ed.) (1951), *Activity Analysis of Production and Allocation*, John Wiley & Sons, New York.

W. LEONTIEF (1941), *The Structure of the American Economy 1919-1929*, Harvard University Press, Harvard.

E. W. MAYR (1984), "An Algorithm for the General Petri Net Reachability Problem", *SIAM Journal of Computing*, 13, 3, pp. 441-460.

T. MURATA (1989), "Petri Nets: Properties, Analysis and Applications", *Proceedings of the IEEE*, 77 (4), pp. 541-580.

W. NEF (1967), *Linear Algebra*, Dover Publications, New York.

J. L. PETERSON (1981), *Petri Net Theory and the Modeling of Systems*, Prentice-Hall, Englewood Cliffs, N. J.

C. A. PETRI (1962), "Kommunication mit Automaten", Bonn Institut für Instrumentelle Mathematik, Schriften des IfM Nr. 3.

W. REISIG (1985), *Petri Nets: an Introduction*, Springer-Verlag, Berlin.

W. REISIG (1992), *A Primer in Petri Net Design*, Springer-Verlag, Berlin.

C. REUTENAUER (1990), *The Mathematics of Petri Nets*, Prentice Hall, Hemel Hempstead (U. K.).

J. SILVESTRE (1987), "Economics and Diseconomies of Scale", in J. Earwell, M. Milgate and P. Newman (Eds.), *The New Palgrave*, Macmillan, New York, vol. 1.

J. VON NEUMANN (1945), "A Model of General Economic Equilibrium", *Review of Economic Studies*, 13 (this is the English translation of a paper published in German in 1937).