Economic Notes by Monte dei Paschi di Siena, vol. 24, no. 1-1995, pp. 69-88

A Characterization of Sequential Equilibrium

GIACOMO BONANNO *

Given an extensive game, we associate with every node t and every player i a subset $K_i(t)$ of the set of terminal nodes, interpreted as player i's information when the play of the game reaches node t. A belief of player i is defined as a map from the set of all nodes into the set of terminal nodes satisfying two properties: what a player believes must be consistent with what she knows, and a player's beliefs must be the same at any two nodes that belong to one of her information sets (since her information is the same at those two nodes). We define four properties of beliefs (Contraction Consistency, Tree Consistency, Individual Rationality and Choice Consistency) and show that these properties are implied by the notion of sequential equilibrium and are sufficient to yield subgame perfection and sequential rationality. In order to obtain consistency as defined by Kreps and Wilson one more property is needed: Minimal Revision. The five properties together provide a characterization of sequential equilibrium.

Introduction

One of the most widely used solution concepts in the literature is sequential equilibrium (Kreps and Wilson, 1982). The formal definition of sequential equilibrium is in terms of an assessment (σ, μ), where σ is a strategy profile and μ is a list of probability distributions, one for each information set. An assessment is a sequential equilibrium if it is sequentially rational and consistent. The substance of sequential rationality is that "the strategy of each player starting from each information set must be optimal starting from there according to some assessment over the nodes in the information set and the strategies of everyone else" (Kreps and Wilson, 1982, p. 871). The notion of consistency places restrictions on out-of-equilibrium beliefs, by requiring μ to be the limit of a sequence of "Bayesian beliefs" obtained from a sequence of completely mixed strategies that converges to the strategy profile under consideration.

* Department of Economics, University of California, Davis, CA 95616-8578, U.S.A., e-mail: gfbonanno@ucdavis.edu. I am grateful to Pierpaolo Battigalli, Larry Samuelson and two anonymous referees for helpful comments.

Recently, a number of authors have tried to shed light on the concept of consistency by relating it to more intuitive notions, such as "structural consistency" (Kreps and Ramey, 1987), "generally reasonable extended assessment" (Fudenberg and Tirole, 1991), "stochastic (quasi) independence" (Kohlberg and Reny, 1991; Battigalli, 1991). In this paper we offer a new perspective on the notion of sequential equilibrium. Intuitively, the notion of sequential equilibrium seems to incorporate a number of quite different concepts. The notion of sequential rationality captures the idea of backward induction but it also embodies the requirement that player's beliefs be conservative (future play is assumed to conform to the originally postulated strategies). The notion of consistency, on the other hand, combines a number of distinct requirements: (i) players' beliefs must reflect the structure of the game-tree, (ii) players must not change their beliefs unless they have to and – when they do – they must switch to "nearby" beliefs, (iii) the beliefs of different players must agree with each other. In this paper we shall try to give an explicit formulation to these concepts and provide a characterization of sequential equilibrium in terms of them.

We base our analysis on the concepts of information and beliefs introduced in Bonanno (1992a, b). Fix an extensive game and let Z be the set of *terminal* nodes. With every pair (i, t), where i is a player and t is a (decision or terminal) node, we associate a subset of Z, denoted by $K_i(t)$, with the following interpretation. Suppose that $K_i(t) = \{z_1, z_3, z_7\}$. Then, when the play of the game reaches node t, player i learns that the play of the game so far has been such that only terminal nodes z_1, z_3 or z_7 can be reached. A belief of player i is defined as a function that associates with every node t an element of the set $K_i(t)$, denoted by $\beta_i(t)$. The interpretation is that if, say, $K_i(t) = \{z_1, z_3, z_7\}$ and $\beta_i(t) = z_3$ then player i knows (is informed) that the outcome of the game can only be either z_1 or z_3 or z_7 and believes that it will be z_3. From a profile of beliefs one can extract a strategy profile in a natural way. In Section 1 we define four properties of beliefs and show that, together, they are sufficient for subgame perfection and sequential rationality. These four properties, however, do not imply consistency of beliefs as defined by Kreps and Wilson. In Section 2 we define one more property of beliefs and show that, together with the previous four, it provides a characterization of sequential equilibrium.

As in Bonanno (1992b), the analysis is carried out entirely in terms of pure strategies and point beliefs. Furthermore, attention is restricted to extensive games without chance moves. The reason for doing so is that one can obtain a very simple and transparent formulation of the concepts involved. Since *the purpose of this paper is not to provide a characterization of sequential equilibrium that is operationally or computationally preferable to the original formulation, but rather to highlight the conceptual* *structure of the notion of sequential equilibrium*, we feel that a simple formulation is preferable to a fully general but considerably more complex one.

1. *Subgame Perfection and Sequential Rationality*

We being by recalling the notions of information and belief used in Bonanno (1992b). Fix a finite extensive game[1]. Let X be the set of *decision* nodes, Z the set of *terminal* nodes, and $T = X \cup Z$. For every $t \in T$, let $\theta(t) \subseteq Z$ be the set of terminal nodes that can be reached from (are the successors of) node t. For every $z \in Z$, we set $\theta(z) = \{z\}$.

We denote by x_0 the root of the tree and for every node $t \neq x_0$ we shall denote the immediate predecessor of t by p_t. Finally, for every node t and for every player i, $\mathcal{H}_i(t)$ is defined as follows: $h \in \mathcal{H}_i(t)$ if and only if h is an information set of player i and there is a node $y \in h$ that is a successor of t.

The information received by player i when the play of the game reaches node t is denoted by $K_i(t)$. The function $K: I \times T \rightarrow \wp(Z)$ (where I is the finite set of players and $\wp(Z)$ denotes the set of subsets of Z) is defined as follows[2]:

(1) For every $i \in I$, $K_i(x_0) = Z$.
(2) For every $z \in Z$ and for every $i \in I$, $K_i(z) = \{z\}$.
(3) If x is a decision node that belongs to information set h of player i, then $K_i(x) = \bigcup_{y \in h} \theta(y)$ [that is, $K_i(x)$ is the set of terminal nodes that can be reached from nodes in h].
(4) If x is a decision node of a player *different* from player i and either $\mathcal{H}_i(x) = \varnothing$ or, for every $h \in \mathcal{H}_i(x)$, and for every $y \in h$, $\theta(y) \subseteq \theta(x)$ (that is, every node in h is a successor of x), then $K_i(x) = \theta(x)$.

[1] We adopt the definition of extensive game given by Selten (1975), (see also Bonanno, 1992b, appendix A).

[2] For a more extensive discussion see Bonanno (1992a). One way of thinking about the proposed definition is as follows. At the root of the tree all players have the same information, namely Z. As the play of the game unfolds and new nodes are reached, an umpire gives (separately) to each player new information according to the following rules. If z is a terminal node, then every player is informed that the game ended at z. If node x belongs to information set h of player, i, then player i is told that her information set h has been reached, but is not told which node in h was reached. If node x does not belong to player i and all the information sets of player i (if any) that are crossed by paths starting at x consist entirely of nodes that are successors of x, then player i is informed that node x has been reached (the justification for this rule is that, later on, at any of her information sets, player i will be able to deduce that the play of the game must have gone through node x; hence player i might as well be told at the time when x is reached). When the above condition is not satisfied, player i's information at x either does not change (that is, player i is not told anything new) or at most reflects the choice made by player i at the immediate predecessor of x, if that node belonged to player i.

(5) Suppose that x is a decision node of a player *different* from player i and the condition given under (4) is not satisfied (that is, there exists a $g \in \mathcal{H}_i(x)$ and a node $y \in g$ such that y is not a successor of x) and p_x (the immediate predecessor of x) belongs to information set $h = \{t_1, t_2, ..., t_m\}$ of player i ($m \geq 1$). Let $c = \{(t_1, y_1), (t_2, y_2), ..., (t_m, y_m)\}$ be the choice of player i to which arc (p_x, x) belongs (that is, for some $j = 1, ..., m$, $t_j = p_x$ and $y_j = x$). Then $K_i(x) = \bigcup_{k=1}^{m} \theta(y_k)$ [that is, $K_i(x)$ is the set of terminal nodes than can be reached from h by following the arcs that constitute choice c].

(6) Finally, if x is a decision node of a player *different* from player i and it satisfies neither condition (4) nor condition (5), then $K_i(x) = K_i(p_x)$[3].

From now on we shall restrict attention to extensive games with perfect recall that have no chance moves.

Definition. A *(pure) belief* of player i as a function

$$\beta_i : T \rightarrow Z$$

satisfying the following properties:

(i) $\beta_i(t) \in K_i(t) \quad \forall t \in T$,

(ii) if x and y belong to the same information set of player i, then $\beta_i(x) = \beta_i(y)$.

Condition (i) says that what a player believes must be consistent with what he knows, and condition (ii) says that a player cannot have different beliefs at two nodes that belong to one of his information sets (since his information is the same at both nodes). *Thus it makes sense to write* $\beta_i(h)$ *for player i's belief at his information set h.*

The interpretation is as follows. Consider again the game of Figure 1. At node x_1 we have $K_1(x_1) = \{z_1, z_2, z_3\}$. This means that player 1 knows (is informed), that only terminal nodes z_1, z_2 or z_3 can be reached. If $\beta_1(x_1) = z_1$ then player 1 believes that the play of the game will actually end at node z_1 (this obviously implies that he believes that player 2 will take actions d and f).

[3] For example, in the game of Figure 1 below, we have:

By (1): $K_i(x_0) = Z = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$ for i = 1, 2

By (2): $K_i(z_j) = \{z_j\}$ for i = 1, 2 and for all j = 1, ..., 7

By (3): $K_2(x_1) = K_2(x_2) = \theta(x_1) \cup \theta(x_2) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$
$K_2(x_3) = K_2(x_4) = \theta(x_3) \cup \theta(x_4) = \{z_1, z_2, z_4, z_5\}$

By (4): $K_1(x_1) = \theta(x_1) = \{z_1, z_2, z_3\}$, $K_1(x_2) = \theta(x_2) = \{z_4, z_5, z_6\}$
$K_1(x_3) = \theta(x_3) = \{z_1, z_2\}$, $K_1(x_4) = \theta(x_4) = \{z_4, z_5\}$

Rules (5) and (6), on the other hand, are superfluous in this case.

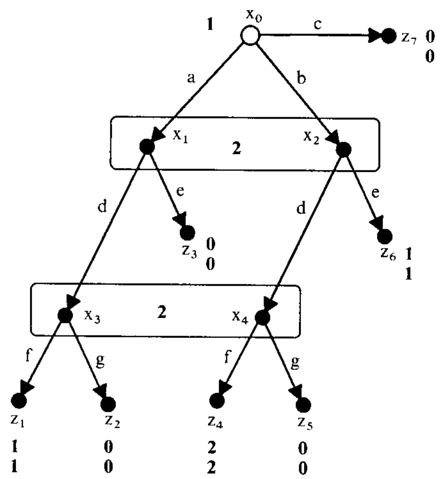

Figure 1

Definition. A *profile of beliefs* is an n-tuple $\beta = (\beta_1, ..., \beta_n)$, where, for every $i = 1, ..., n$, β_1 is a belief of player i.

We shall make use of the following notation: given a decision node x, $S(x)$ denotes the set of *immediate* successors of x.

Definition. We say that a profile of beliefs β is *well-behaved* if it satisfies the following properties (which will be discussed below)[4]:

(i) [Contraction Consistency] For every player i, if y is a successor of x[5] and $\beta_i(x) \in K_i(y)$, then $\beta_i(y) = \beta_i(x)$.

(ii) [Tree Consistency] Fix an arbitrary information set h and let i be the corresponding player. Let $x \in h$ be the predecessor of $\beta_i(h)$. Then $\beta_i(y) \in \theta(y)$, $\forall\, y \in S(x)$.

(iii) [Individual Rationality] Fix an arbitrary information set h and let i be the corresponding player. Let $x \in h$ be the predecessor of $\beta_i(h)$. Then $U_i(\beta_i(h)) \geq U_i(\beta_i(y))$, $\forall\, y \in S(x)$, where $U_i : Z \to \Re$ is player i's payoff function ($\Re$ denotes the set of real numbers).

(iv) [Choice Consistency] Let node x belong to information set h of player i, and let c be the choice at h that precedes $\beta_i(h)$. Then, for every player j, if $\beta_j(x)$ comes after choice d at h, it must be d = c.

[4] In Bonanno (1992b) the word 'rational' is used instead of 'well-behaved'.

[5] It is shown in Bonanno (1992a) that – in an extensive game with perfect recall – if node y is a successor of node x, then, for every player i, $K_i(y) \subseteq K_i(x)$.

Contraction Consistency says that a player will not change his beliefs unless he has to, that is, unless his previous belief is inconsistent with the new information.

Property (ii) (Tree Consistency) requires that a player's beliefs about his opponents' previous moves be independent of his own choices. To see this, consider the game of Figure 1. There we have that $K_2(h) = \{z_1, z_2, z_3, z_4, z_5, z_6\}$ where $h = \{x_1, x_2\}$ is the first information set of player 2, and $K_2(g) = \{z_1, z_2, z_4, z_5\}$ where $g = \{x_3, x_4\}$ is the second information set of player 2. Suppose that $\beta_2(h) = z_6$ and $\beta_2(g) = z_1$. For player 2 to believe in z_6 at h means that she believes that node x_2 was reached and, therefore, that player 1 chose b. Furthermore, it implies that she herself plans to take action e at h. Given this belief, if player 2 takes action d instead, so that the play of the game proceeds to information set g, then node x_4 must be reached, and from x_4 terminal node z_1 cannot be reached. Player 2 can believe in z_1 at g only if she modifies her previous belief concerning player 1's choice at the root. But the only basis for changing her belief concerning player 1 would be that she changed *her own* choice at h from the planned e to d. Formally, Tree Consistency is violated since the predecessor of $\beta_2(h)$ in h is x_2, x_4 is an immediate successor of x_2 and $\beta_2(x_4) = z_1 \notin \theta(x_4) = \{z_4, z_5\}$.

To understand Property (iii) (Individual Rationality), let $z^* = \beta_i(h)$ and let x^* be the unique node in h which is on the path from the root to z^*. Then since player i believes in z^* at his information set h, it means that he believes that node x^* was reached. Property (iii) requires that for every immediate successor y of x^*, $U_i(\beta_i(y)) \le U_i(z^*)$. Suppose instead that there were an

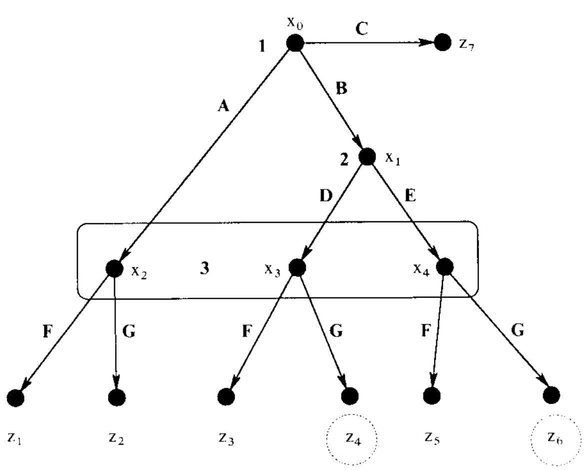

Figure 2

immediate successor y of x^* such that $U_i(\beta_i(y)) > U_i(z^*)$. Then believing in z^* (at h) is irrational for player i because, instead of making the choice required by z^*, he can – according to his beliefs and by making another choice – move the play to node y from where, again according to his beliefs, the game will evolve to outcome $\beta_i(y)$ that he prefers to z^*.

Finally, Property (iv) (Choice Consistency) introduces some degree of agreement across the beliefs of different players. It is a rather weak property, since it is only "forward-looking": it imposes agreement on future choices but not necessarily on past choices. To see this, consider the game of Figure 2.

The following beliefs do not violate Choice Consistency: $\beta_3(x_0) = \beta_3(x_1) = z_7$ [note that, by property 6 of the definition of K, $K_3(x_1) = K_3(x_0)$], $\beta_3(x_2) = \beta_3(x_3) = \beta_3(x_4) = z_6$, $\beta_2(x_0) = z_7$, $\beta_2(x_1) = \beta_2(x_3) = z_4$. Note that, at node x_3, there is disagreement between players 2 and 3 concerning 2's choice (player 3 believes that 2 chose E, while 2 knows that she herself chose D), and at node x_1 Choice Consistency is vacuously satisfied, since $\beta_3(x_1)$ is not a successor of x_1.

We now show that the four properties listed above are sufficient for subgame perfection and sequential rationality. In order to do this we first need to show how to extract an assessment from a profile of beliefs. Recall that an *assessment* is a pair (σ, μ), where σ is a strategy profile and μ is a function (called a "system of beliefs" by Kreps and Wilson) $\mu : T \to [0, 1]$ satisfying the property that, for every information set h, $\sum_{x \in h} \mu(x) = 1$. We shall restrict attention to simple assessments. An assessment (σ, μ) is *simple* if σ is a pure strategy profile and μ satisfies the property that, for every node x, either $\mu(x) = 0$ or $\mu(x) = 1$.

Given a profile of beliefs β we can associate with it a simple assessment $(\sigma, \mu) = (\xi(\beta), \tau(\beta))$ where $\xi(\beta)$ and $\tau(\beta)$ are defined as follows. *Definition of* $\sigma = \xi(\beta)$: if h is an information set of player i and c is the choice at h that precedes $\beta_i(h)$, set $\sigma_i(h) = c$, that is, c is the choice selected (with probability 1) by player i's strategy at information set h. *Definition of* $\mu = \tau(\beta)$: if h is an information set of player i and $x \in h$ is the predecessor of $\beta_i(h)$, then $\mu(x) = 1$ (and $\mu(y) = 0$ for all $y \in h/\{x\}$). (For an example see Remark 1.1 below).

Proposition 1.1. Fix an extensive game G (with perfect recall and no chance moves)[6]. Let β be a well-behaved profile of beliefs and let $(\sigma, \mu) = (\xi(\beta), \tau(\beta))$ be the corresponding simple assessment. Then:

[6] From now on we will omit the reminder that the extensive games we consider have perfect recall and no chance moves.

(i) σ is a subgame-perfect equilibrium of G, and
(ii) (σ, μ) is sequentially rational.
Proof. See Appendix A [7]

Remark 1.1. It may seem that sequential rationality is an "obvious consequence" of the properties of Individual Rationality and Choice Consistency. This is not so. We now give two examples of profiles of beliefs that satisfy Individual Rationality and Choice Consistency (one of the two satisfies also Tree Consistency but not Contraction Consistency, while the other satisfies also Contraction Consistency but not Tree Consistency), whose corresponding simple assessments are not sequentially rational (in fact, they are not even Nash equilibria!). Consider again the game of Figure 1.

Consider first the following profile of beliefs: $\beta_1(x_0) = \beta_1(x_2) = \beta_2(x_0) = \beta_2(x_1) = \beta_2(x_2) = z_6$, $\beta_1(x_1) = z_3$, $\beta_1(x_3) = \beta_2(x_3) = \beta_2(x_4) = z_1$, $\beta_1(x_4) = z_4$. Then β satisfies Contraction Consistency, Individual Rationality (note that $\beta_2(x_4) = z_1$) and Choice Consistency but not Tree Consistency, because $\beta_2(x_4) \notin \theta(x_4)$. If $\sigma = \xi(\beta)$, then $\sigma = (b, (e, f))$, which is not a Nash Equilibrium.

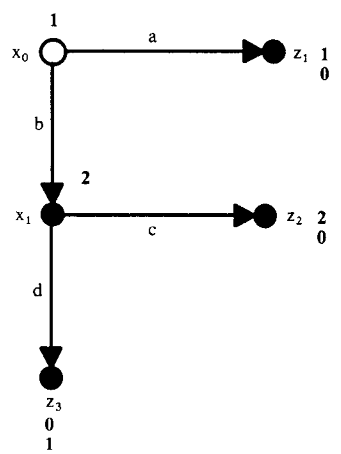

Figure 3

Now consider the game of Figure 3 and the following profile of beliefs: $\beta_1(x_0) = \beta_2(x_0) = z_2$ and $\beta_1(x_1) = \beta_2(x_1) = z_3$. Then β satisfies Choice

[7] Part (i) of this proposition was proved in Bonanno (1992b). Thus in Appendix A we only give a proof of part (ii).

Consistency, Individual Rationality and Tree Consistency[8], but not Contraction Consistency (it is violated for both players). And if $\sigma = \xi(\beta)$ then $\sigma = (b, d)$, which is not a Nash Equilibrium.

Remark 1.2. In Bonanno (1992b) an example is given of a well-behaved profile of beliefs β whose corresponding simple assessment $(\sigma, \mu) = (\xi(\beta), \tau(\beta))$ is not a sequential equilibrium. Thus the properties that define the notion of well-behaved profile of beliefs are sufficient for sequential rationality but not for consistency (as defined by Kreps and Wilson). In the next section we show that, by adding one more property of beliefs, we obtain a necessary and sufficient set of conditions for sequential equilibrium.

Remark 1.3. The properties that define the notion of well-behaved profile of beliefs are sufficient but not necessary for sequential rationality. In order to prove this, we first need to show how to extract a profile of beliefs from a simple assessment (σ, μ).

Definition of $\beta = \chi(\sigma, \mu)$. Given a simple assessment (σ, μ) we can extract from it a profile of beliefs $\beta = \chi(\sigma, \mu)$ as follows (we shall use the following notation: if σ is a pure-strategy profile and t is a node, $\zeta(t \mid \sigma) \in Z$ is the terminal node reached from t by following σ):

(i) For every player i and every node x, if $K_i(x) = \theta(x)$ [recall that, in particular, this is true if x is the root or x is a terminal node], set $\beta_i(x) = \zeta(x \mid \sigma)$;

(ii) If x is a decision node that belongs to information set h of player i, let $\hat{x} \in h$ be the node such that $\mu(\hat{x}) = 1$. Set $\beta_i(x) = \zeta(\hat{x} \mid \sigma)$.

(iii) If $x \neq x_0$ is a decision node that does *not* belong to player i and $\beta_i(p_x) \in K_i(x)$ [recall that p_x denotes the immediate predecessor of x] set $\beta_i(x) = \beta_i(p_x)$.

(iv) If $x \neq x_0$ is a decision node that does *not* belong to player i and $K_i(x) \neq \theta(x)$ and $\beta_i(p_x) \notin K_i(x)$, then it must be $K_i(x) \neq K_i(p_x)$. It follows from the definition of the function $K_i(\cdot)$ that p_x belongs to an information set of player i, call it h. Let $\hat{t} \in h$ be the node such that $\mu(\hat{t}) = 1$ and $\hat{y}$ be the immediate successor of $\hat{t}$ following the choice to which arc (p_x, x) belongs. Set $\beta_i(x) = \zeta(\hat{y} \mid \sigma)$.

Remark 1.3 continued. Consider now the game of Figure 1 modified as follows: the payoff vectors assigned to nodes z_1 and z_4 are interchanged.

[8] In a game with perfect information Tree Consistency is satisfied trivially, since (cf. Bonanno, 1992a) for every player i and every node x, $K_i(x) = \theta(x)$.

Consider the following sequentially rational simple assessment: $\sigma = (b, e, f)$ and $\mu(x_2) = \mu(x_3) = 1$. If $\beta = \chi(\sigma, \mu)$, then β is as follows: $\beta_1(x_0) = \beta_1(x_2) = \beta_2(x_0) = \beta_2(x_1) = \beta_2(x_2) = z_6$, $\beta_1(x_1) = z_3$, $\beta_1(x_3) = \beta_2(x_3) = \beta_2(x_4) = z_1$, $\beta_1(x_4) = z_4$. Then β_2 fails to satisfy the property of Tree Consistency, since $\beta_2(x_4) = z_1 \notin \theta(x_4)$. Thus the properties that define the notion of well-behaved profile of beliefs are not necessary for sequential rationality.

Remark 1.4. It is easy to verify that $(\xi, \tau) = \chi^{-1}$, that is, for every simple assessment (σ, μ), $(\xi(\chi(\sigma, \mu)), \tau(\chi(\sigma, \mu))) = (\sigma, \mu)$.

2. *Sequential Equilibrium*

Fix an extensive game. We shall denote by $\mathcal{A}$ the set of arcs of the game tree [if x and y are nodes and y is an immediate successor of x, then the ordered pair (x, y) is an arc] and by C the set of choices (C is a partition of $\mathcal{A}$).

Definition. A *network*[9] *assignment* is a function

$$\nu : \mathcal{A} \rightarrow \mathbb{N}$$

(where $\mathbb{N}$ denotes the set of non-negative integers) such that:

(i) at every decision node x exactly one arc incident from x is assigned value 0, and

(ii) if arcs a_1 and a_2 belong to the same choice, then $\nu(a_1) = \nu(a_2)$.

Because of property (ii), if c is a choice, every arc in c has the same value and therefore we can write $\nu(c)$ for this common value.

Given a network assignment ν, we define $\lambda_\nu : T \rightarrow \mathbb{N}$ as follows. First of all, we set $\lambda_\nu(x_0) = 0$. If $t \in T/\{x_0\}$ and $\langle x_1, x_1x_2, x_2, x_2x_3, x_3, \ldots, x_{m-1}x_m, x_m\rangle$ is the path from the root to t (thus $x_1 = x_0$, and $x_m = t$), then $\lambda_\nu(t) = \sum_{k=1}^{m-1} \nu((x_k, x_{k+1}))$.

Thus the function λ_ν assigns to every node t the sum of the numbers associated with the arcs that form the path from the root of the tree to t.

Definition. We say that a profile of beliefs β satisfies the property of *Minimal Revision* if there exists a network assignment ν such that, for every player i and for every node t: $\lambda_\nu(\beta_i(t)) \leq \lambda_\nu(z)$, $\forall\, z \in K_i(t)$, with strict inequality if $z \neq \beta_i(t)$.

[9] In graph theory a network is defined as a graph (digraph) in which to every edge (arc) is associated a real number.

Remark 2.1. According to the above definition, β satisfies the property of Minimal Revision if for every player i and for every node t, $\beta_i(t)$ is the unique minimum of the function λ_v on $K_i(t)$. Intuitively, for every choice c, one can think of the number v(c) as the *degree of implausibility* of choice c. The most plausible choices are assigned value zero and the less plausible a choice the higher the number it is assigned. The degree of implausibility of a node is equal to the sum of the degrees of implausibility of the choices that lead to it (from the root). When choosing her belief, a player is required to choose the most plausible node, according this common hierachy. In this sense, belief revision should be minimal [10].

The following lemma shows that Minimal Revision is a strengthening of the properties of Contraction Consistency, Tree Consistency and Choice Consistency.

Lemma 2.1. Let β be a profile of beliefs that satisfies the property of Minimal Revision. Then β satisfies the properties of Contraction Consistency, Tree Consistency and Choice Consistency.
Proof. See Appendix B.

Thus if a profile of beliefs satisfies the properties of Minimal Revision and Individual Rationality then it is a well-behaved profile of beliefs (that the converse is not true follows from Proposition 2.1 and Remark 1.2). The following two propositions provide a characterization of sequential equilibrium in terms of the two properties of Minimal Revision and Individual Rationality.

[10] The above definition can also be interpreted as follows: for every player i, as the play of the game proceeds from a node x to one of its immediate successors y, player i's belief at y is as close as possible to what it was at x, where "closeness" is given by the metric $d_v : Z \times Z \rightarrow \mathbb{N}$ defined as follows:

$$d_v(z, z') = \begin{cases} 0 & \text{if } z = z' \\ \lambda_v(z) + \lambda_v(z') & \text{if } z \neq z' \end{cases}$$

It is easy to check that if β satisfies the property of Minimal Revision then the following is true: for every player i and for every node $t \neq x_0$,

$$d_v(\beta_i(t), \beta_i(p_t)) \leq d_v(z, \beta_i(p_t)) \qquad \forall z \in K_i(t) \text{ with strict inequality if } z \neq \beta_i(t).$$

(recall that p_t denotes the immediate predecessor of t).
It is easy to check that d_v is a metric, that is, it satisfies the following properties:

(1) $\forall z \in Z, d_v(z, z) = 0,$
(2) $\forall z, z' \in Z, z \neq z' \Rightarrow d_v(z, z') > 0,$
(3) $\forall z, z' \in Z, d_v(z, z') = d_v(z', z).$
(4) $\forall z, z', z'' \in Z, d_v(z, z'') \leq d_v(z, z') + d_v(z', z'').$

This is so because, by definition of network assignment, there is a unique $z^* \in Z$ such that $\lambda_v(z^*) = 0$, while for every $z \in Z/\{z^*\}$, $\lambda_v(z) > 0$.

Proposition 2.1. Let β be a profile of beliefs that satisfies the properties of Minimal Revision and Individual Rationality, and let $(\sigma, \mu) = (\xi(\beta), \tau(\beta))$ be the corresponding simple assessment. Then (σ, μ) is a sequential equilibrium.

Proof. See Appendix B.

Proposition 2.2. Let (σ, μ) be a simple assessment and let $\beta = \chi(\sigma, \mu)$. If (σ, μ) is a sequential equilibrium then β satisfies the properties of Minimal Revision and Individual Rationality.

Proof. See Appendix B.

We conclude with an example that shows that the properties of Minimal Revision and Individual Rationality are independent. Consider the game of Figure 3 and the following network assignment: $\nu(a) = \nu(c) = 0$, $\nu(b) = \nu(d) = 1$, so that $\lambda_\nu(z_1) = 0$, $\lambda_\nu(z_2) = 1$ and $\lambda_\nu(z_3) = 2$. Then the following profile of beliefs satisfies the property of Minimal Revision: $\beta_1(x_0) = \beta_2(x_0) = z_1$, $\beta_1(x_1) = \beta_2(x_1) = z_2$. On the other hand, Individual Rationality is violated for both players (for player 1 at x_0 and for player 2 at x_1). Indeed the corresponding strategy profile is (a, c) which is not even a Nash equilibrium.

Concluding Remarks

Given an extensive game, we associated with every node t and every player i a subset $K_i(t)$ of the set of terminal nodes, interpreted as player i's information when the play of the game reaches node t. A belief of player i was then defined as a map from the set of all nodes into the set of terminal nodes satisfying two main properties: what a player believes must be consistent with what she knows, and a player's beliefs must be the same at any two nodes that belong to one of her information sets (since her information is the same at those two nodes). We then defined four properties of beliefs (Contraction Consistency, Tree Consistency, Individual Rationality and Choice Consistency) and showed that these properties are sufficient lo yield subgame perfection and sequential rationality. In order to obtain consistency as defined by Kreps and Wilson one needs a further property, Minimal Revision, which is a strengthening of Contraction Consistency, Tree Consistency and Choice Consistency. The two properties of Minimal Revision and Individual Rationality provide a characterization of sequential equilibrium.

APPENDIX A

In this appendix we prove proposition 1.1. We shall need the following lemma, which is proved in Bonanno (1992b, appendix B). Recall that, if σ is a pure-strategy profile and t a node, then $\zeta(t \mid \sigma) \in Z$ denotes the terminal node reached from t by following σ.

Lemma A. Fix an extensive game (with perfect recall and no chance moves). Let β be a profile of beliefs that satisfies Contraction Consistency and Choice Consistency. Then, for every player i, the following is true: for every node x, if $\beta_i(x) \in \theta(x)$ then $\beta_i(x) = \zeta(x \mid \sigma)$ where $\sigma = \xi(\beta)$.

Proof of proposition 1.1. Fix an extensive game. Let β be a well-behaved profile of beliefs and $(\sigma, \mu) = (\xi(\beta), \tau(\beta))$ the corresponding simple assessment. That σ is a subgame-perfect equilibrium is proved in Bonanno (1992b). Thus we only need to show that (σ, μ) is sequentially rational.

A simple assessment (σ, μ) is *sequentially rational* if it satisfies the following property. Fix an arbitrary information set h and let i be the corresponding player. Let $\hat{x} \in h$ be the node such that $\mu(\hat{x}) = 1$. Write σ as (σ_i, σ_{-i}). Let σ_i' be an arbitrary strategy of player i, and let $\sigma' = (\sigma_i', \sigma_{-i})$. Then

$$U_i(\zeta(\hat{x} \mid \sigma)) \geq U_i(\zeta(\hat{x} \mid \sigma')) \tag{A1}$$

that is, by switching to a different strategy from information set h – given the belief that node $\hat{x}$ was reached with probability 1 and that future play by the other players will be according to σ_{-i} – player i cannot increase his payoff.

Suppose that (σ, μ) is not sequentially rational. Then there exist a player i, an information set h of player i, a strategy σ_i' of player i, such that

$$U_i(\zeta(\hat{x} \mid \sigma)) < U_i(\zeta(\hat{x} \mid \sigma')) \tag{A2}$$

where $\hat{x} \in h$ with $\mu(\hat{x}) = 1$ and $\sigma' = (\sigma_i', \sigma_{-i})$. It follows that $\zeta(\hat{x} \mid \sigma) \neq \zeta(\hat{x} \mid \sigma')$. By definition of $\mu = \tau(\beta)$, $\hat{x} \in h$ must be the predecessor of $\beta_i(h)$. By lemma A, $\beta_i(h) = \zeta(\hat{x} \mid \sigma)$. Let y be the node at which the path from $\hat{x}$ to $\zeta(\hat{x} \mid \sigma)$ and the path from $\hat{x}$ to $\zeta(\hat{x} \mid \sigma')$ diverge [that is, both

$\zeta(\hat{x} \mid \sigma)$ and $\zeta(\hat{x} \mid \sigma')$ belong to $\theta(y)$ and, for every immediate successor w of y, it is not true that both $\zeta(\hat{x} \mid \sigma)$ and $\zeta(\hat{x} \mid \sigma')$ belong to $\theta(w)$]. Then y belongs to an information set of player i, call it g ($g = h$, and hence $y = \hat{x}$, if and only if σ_i and σ'_i select different choices at h) and $\zeta(y \mid \sigma) = \zeta(\hat{x} \mid \sigma)$ and $\zeta(y \mid \sigma' = \zeta(\hat{x} \mid \sigma')$. Thus, from (A2) it follows that

(A3) $$U_i(\zeta(y \mid \sigma)) < U_i(\zeta(y \mid \sigma'))$$

By Contraction Consistency

(A4) $$\beta_i(y) = \zeta(y \mid \sigma)$$

[since $\zeta(y \mid \sigma) \in \theta(y)$ and, by lemma 3 in Bonanno 1992a, $\theta(y) \subseteq K_i(y)$]. Let y_1 be the immediate successor of y on the path from y to $\zeta(y \mid \sigma')$. Clearly,

(A5) $$\zeta(y_1 \mid \sigma') = \zeta(y \mid \sigma')$$

By Tree Consistency, $\beta_i(y_1) \in \theta(y_1)$. By lemma A,

(A6) $$\beta_i(y_1) = \zeta(y_1 \mid \sigma).$$

By Individual Rationality and (A4),

(A7) $$U_i(\zeta(y \mid \sigma)) \geq U_i(\zeta(y_1 \mid \sigma))$$

If $\zeta(y_1 \mid \sigma) = \zeta(y_1 \mid \sigma')$, [which will be the case if no information sets of player i are crossed by the path from y_1 to $\zeta(y_1 \mid \sigma')$ or if σ_i and σ'_i agree on any such information sets] then (A7) and (A5) contradict (A3). If $\zeta(y_1 \mid \sigma) \neq \zeta(y_1 \mid \sigma')$ then we can repeat the same argument (find the node of player i at which the path from y_1 to $\zeta(y_1 \mid \sigma)$ and the path from y_1 to $\zeta(y_1 \mid \sigma')$ diverge, etc.). Eventually, since the number of nodes (and hence the number of information sets of player i) is finite, we will reach a node w such that $\zeta(w \mid \sigma) = \zeta(w \mid \sigma')$. Putting together all the inequalities of the form (A7) and (A5) we will then contradict (A3).

APPENDIX B

In this appendix we prove lemma 2.1 and propositions 2.1 and 2.2 We start with a few lemmas.

Lemma B. Fix an extensive game. Let ν be a network assignment and λ_ν the corresponding function defined on T. Then, for every node $t \in T$,

(B1) $$\min \{\lambda_\nu(z)\}_{z \in \theta(t)} = \lambda_\nu(t)$$

Proof. Fix an arbitrary node t and an arbitrary $z \in \theta(t)$. By definition of $\lambda_\nu(\cdot)$, $\lambda_\nu(z) \geq \lambda_\nu(t)$. If t is a terminal node, then $\theta(t) = \{t\}$ and there is nothing to prove. If t is a decision node, by definition of network assignment, there is an arc incident from t that has value 0. Follow that arc and whenever a decision node is reached continue along an arc with value zero. Eventually a terminal node z^* will be reached. Clearly, $z^* \in \theta(t)$ and $\lambda_\nu(z^*) = \lambda_\nu(t)$.

Notation. From now on, given a network assignment ν, for every decision node t, we shall denote by $z^*_\nu(t)$ the unique terminal node reached from t by following arcs that have value zero.

Lemma C. Let β be a profile of beliefs that satisfies Minimal Revision (relative to network assignment ν and corresponding function λ_ν). Then for every player i and every node t,

(B2) if $\beta_i(t) \in \theta(t)$, then $\beta_i(t) = z^*_\nu(t)$.

Furthermore, if $\sigma = \xi(\beta)$ is the corresponding strategy profile and c is a choice to which the relevant component of σ assigns probability 1, then $\nu(c) = 0$.

Proof. Fix an arbitrary player i and an arbitrary node t. Suppose that $\beta_i(t) \in \theta(t)$. Since, by Minimal Revision, $\beta_i(t)$ is the unique minimum of $\lambda_\nu(\cdot)$ on $K_i(t)$ and, by lemma 3 in Bonanno (1992a), $\theta(t) \subseteq K_i(t)$, it follows that $\beta_i(t)$ is the unique minimum of $\lambda_\nu(\cdot)$ on $\theta(t)$. It follows from lemma B that $\beta_i(t) = z^*_\nu(t)$.

Now fix an arbitrary information set, call it h, and let i be the corresponding player. Let $x \in h$ be the predecessor of $\beta_i(h)$. Then $\beta_i(h) \in \theta(x)$ and, by (B2), $\beta_i(h) = z^*_v(x)$. Let c be the choice at h that precedes $z^*_v(x)$. By definition of $z^*_v(\cdot)$ it must be $v(c) = 0$. By definition of $\sigma = \xi(\beta)$ and the fact that $\beta_i(h) = z^*_v(x)$, c is the choice selected with probability 1 by σ_i.

Lemma D. Let β be a profile of beliefs that satisfies the property of Minimal Revision. Let h be an arbitrary information set and i be the corresponding player. Let $x \in h$ be the predecessor of $\beta_i(h)$. Then,

(B3) $$\lambda_v(x) < \lambda_v(x'), \quad \forall\, x' \in h/\{x\}$$

Proof. By lemma C,

(B4) $$\beta_i(h) = z^*_v(x) \text{ and therefore } \lambda_v(\beta_i(h)) = \lambda_v(x)$$

By Minimal Revision

(B5) $$\lambda_v(\beta_i(h)) < \lambda_v(z) \text{ for every } z \in K_i(h)/\{\beta_i(h)\}$$

If $x' \in h/\{x\}$, then, $z^*_v(x') \neq \beta_i(h)$ and therefore, by (B5),

(B6) $$\lambda_v(\beta_i(h)) < \lambda_v(z^*_v(x')).$$

Since $\lambda_v(x') = \lambda_v(z^*_v(x'))$, (B3) follows from (B4) and (B6).

Proof of Lemma 2.1. Let β be a profile of beliefs that satisfies Minimal Revision with respect to the network assignment v (and corresponding function λ_v). We want so show that β satisfies the properties of Contraction Consistency, Tree Consistency and Choice Consistency.

Contraction Consistency. We need to show that if node y is a successor of node x and $\beta_i(x) \in K_i(y)$ then $\beta_i(y) = \beta_i(x)$. Since the game has perfect recall, by proposition 1′ in Bonanno (1992a), $K_i(y) \subseteq K_i(x)$. By Minimal Revision, $\beta_i(x)$ is the unique minimum of λ_v on $K_i(x)$. Since $\beta_i(x) \in K_i(y)$, it follows that $\beta_i(x)$ is the unique minimum of λ_v on $K_i(y)$. By Minimal Revision, $\beta_i(y) = \beta_i(x)$.

Tree Consistency. Fix an arbitrary information set h and let i be the corresponding player. Let $x \in h$ be the predecessor of $\beta_i(h)$ and let y be an immediate successor of x. We want to show that $\beta_i(y) \in \theta(y)$. If $K_i(y) = \theta(y)$ there is nothing to prove, since $\beta_i(y) \in K_i(y)$ by definition of belief. Suppose therefore that $K_i(y)$ is a proper superset of $\theta(y)$. If $\beta_i(x) \in \theta(y)$, by Contraction Consistency $\beta_i(y) = \beta_i(x)$ [since, by lemma 3 in Bonanno (1992a), $\theta(y) \subseteq K_i(y)$] and there is nothing to prove. Suppose therefore that $\beta_i(y) \notin \theta(y)$. Let $c = \{(t_1, w_1), \ldots, (t_m, w_m)\}$ be the choice at h that leads from x to y. Then $m \geq 2$ and for some $j = 1, \ldots, m$, $t_j = x$ and $w_j = y$. Two cases are possible: (i) y is not a decision node of player i, and (ii) y is a decision node of player i. In case (i), by (5) of the definition of $K_i(\cdot)$ – since $K_i(y) \neq \theta(y)$ – it must be $K_i(y) = \theta(w_1) \cup ... \cup \theta(w_m)$. Thus if $\beta_i(y) \notin \theta(y)$, then there exists a node $x' \in h$ with $x' \neq x$ such that $\beta_i(y) \in \theta(y')$, where y′ is the immediate successor of x′ following choice c. By lemma C,

(B7) $$\beta_i(y) = z^*_v(y')$$

By minimal revision, since both $z^*_v(y')$ and $z^*_v(y)$ belong to $K_i(y)$ and $z^*_v(y') \neq z^*_v(y)$, using (B7) we obtain

(B8) $$\lambda_v(z^*_v(y')) < \lambda_v(z^*_v(y))$$

But, by definition of $z^*_v(\cdot)$,

(B9) $$\lambda_v(z^*_v(y')) = \lambda_v(y') \text{ and } \lambda_v(z^*_v(y)) = \lambda_v(y).$$

Furthermore,

(B10) $$\lambda_v(y') = \lambda_v(x') + v(c) \text{ and } \lambda_v(y) = \lambda_v(x) + v(c).$$

Thus, using (B8)-(B10) we obtain

(B11) $$\lambda_v(x') < \lambda_v(x)$$

which contradicts lemma D.

Consider now case (2) where y is a decision node of player i. Let g be the information set of player i to which node y belongs. Then, since $\beta_i(y) \notin \theta(y)$, there must exist a node $s \in g$ such that $s \neq y$ and $\beta_i(y) \in \theta(s)$. By perfect recall s comes after choice c at h. Let $x' \in h$ be the predecessor of s (hence $x' \neq x$) and let y′ be the immediate successor of x′ following choice c. Then either $y' = s$ or y′ is a predecessor of s, so that $\theta(s) \subseteq \theta(y')$. If $y' = s$ the argument of (B7)-(B11) applies directly. If y′ is a predecessor of s then $\beta_i(y) \in \theta(y')$. Then we can apply the argument of (B7)-(B11) to y′ and reach a contradiction with lemma D, as before.

Choice Consistency. Let h be an arbitrary information set and let i be the corresponding player. Let c be the choice at h that precedes $\beta_i(h)$. Fix an arbitrary $y \in h$ and an arbitrary player j. Suppose that $\beta_j(y)$ comes after choice d at h. We want to show that d = c. By lemma C, $v(c) = 0$. Let $y' \in h$ be predecessor of $\beta_j(y)$. Then $\beta_j(y) \in \theta(y')$ and, by lemma C, $\beta_j(y) = z^*_v(y')$. Hence $v(d) = 0$. By definition of network assignment, exactly one choice at every information set is assigned value 0. Hence c = d.

Proof of Proposition 2.1. Let β be a profile of beliefs that satisfies Minimal Revision (with respect to network assignment ν) and Individual Rationality and let $(\sigma, \mu) = (\xi(\beta), \tau(\beta))$ be the corresponding simple assessment. By lemma 2.1 and proposition 1.1, (σ, μ) is sequentially rational. Thus is only remains to prove that (σ, μ) is consistent in the sense of Kreps and Wilson. By definition of $\tau(\cdot)$ and by lemma D, for every information set h and for every node $x \in h$, $\mu(x) = 1$ if and only if x is the unique minimum of λ_v on h. By definition of $\xi(\cdot)$ and by lemma C, $v(c) = 0$ if and only if c is a choice to which the relevant component of σ assigns positive probability (in our case, probability 1). Let $B = N \cup C_\sigma$, where N is the set of nodes x such that $\mu(x) = 1$ and C_σ is the set of choices to which σ assigns probability 1. Then, using the terminology of Kreps and Wilson (1982, p. 887), B is a basis and v is a B labeling. By lemma A1 in Kreps and Wilson (1982, p. 887), (σ, μ) is consistent.

Proof of Proposition 2.2. Let the simple assessment (σ, μ) be a sequential equilibrium and let $\beta = \chi(\sigma, \mu)$. We first show that β satisfies the property of Minimal Revision. By lemma A1 in Kreps and Wilson (1982, p. 887) there is a function $N : C \to \mathbb{N}$ (where C is the set of choices and $\mathbb{N}$ is the set of non-negative integers) such that:

(i) if h is an arbitrary information set and $\tilde{x} \in h$ is the node such that $\mu(\tilde{x}) = 1$, then

(B12) $$\Lambda_N(\tilde{x}) < \Lambda_N(x) \quad \forall\, x \in h/\{\tilde{x}\}$$

[where $\Lambda_N : T \to \mathbb{N}$ is the function that associates with every node t the sum of the values of the choices that precede t], and

(ii) $N(c) = 0$ if and only if c is a choice to which the relevant component of σ assigns probability 1.

Define $v : \mathcal{A} \to \mathbb{N}$ as follows: if arc a belongs to choice c, then $v(a) = N(c)$. Then v is a network assignment. We want to show that, for every player i and for every node t:

(B13) $\lambda_v(\beta_i(t)) \leq \lambda_v(z) \quad \forall\, z \in K_i(t)$, with strict inequality if $z \neq \beta_i(t)$.

Fix an arbitrary note t and an arbitrary player i. We shall consider all possible cases.

Case 1. Suppose first that t belongs to information set h of player i. Let $\tilde{x} \in h$ be the node such that $\mu(\tilde{x}) = 1$. By (ii) of the definition of $\beta = \chi(\sigma, \mu)$, $\beta_i(h) = \zeta(\tilde{x} \mid \sigma)$ and by (ii) above $\zeta(\tilde{x} \mid \sigma) = z_v^*(\tilde{x})$. Thus $\lambda_v(\beta_i(h)) = \lambda_v(z_v^*(\tilde{x})) = \lambda_v(\tilde{x})$. By (ii) above, if $z \in K_i(h)/\{z_v^*(\tilde{x})\}$ then $\lambda_v(z) > \lambda_v(z_v^*(\tilde{x}))$. Thus, (B13) is satisfied.

Case 2. Suppose that t is *not* a decision node of player i and also that $K_i(t) = \theta(t)$. By (i) of the definition of $\beta = \chi(\sigma, \mu)$, $\beta_i(t) = \zeta(t \mid \sigma)$. By (ii) above, $\zeta(t \mid \sigma) = z_v^*(t)$ and if $z \in \theta(t)/\{z_v^*(t)\}$ then $\lambda_v(z) > \lambda_v(t)$. Since $\lambda_v(z_v^*(t)) = \lambda_v(t)$, (B13) is satisfied.

*Case 3.*Suppose that $t \neq x_0$ is *not* a decision node of player i [note that, by definition of $K_i(\cdot)$, the case $t = x_0$ must fall under case 1 or case 2] and $K_i(t)$

is a proper superset of $\theta(t)$. Suppose that $\beta_i(p_t) \notin K_i(t)$. It follows from the definition of the function $K_i(\cdot)$ that p_t belongs to an information set of player i, call it h. Let $\hat{t} \in h$ be the node such that $\mu(\hat{t}) = 1$ and $\hat{y}$ be the immediate successor of $\hat{t}$ following the choice to which arc (p_t, t) belongs (call it choice c). By (iv) of the definition of $\beta = \chi(\sigma, \mu)$, $\beta_i(t) = \zeta(\hat{y} \mid \sigma)$. Thus $\lambda_v(\beta_i(t)) = \lambda_v(\hat{y}) = \lambda_v(\hat{t}) + v(c)$. Fix an arbitrary $z \in K_i(t)/\{\beta_i(t)\}$. Then, by definition of $K_i(\cdot)$, z comes after choice c at h. Let $w \in h$ bew the predecessor of z (thus $w \neq \hat{t}$) and w′ be the immediate successor of w following choice c. Then $\lambda_v(z) \geq \lambda_v(w') = \lambda_v(w) + v(c)$. By (B12), $\lambda_v(w) > \lambda_v(\hat{t})$. Thus (B13) is satisfied.

Case 4. Suppose that t is *not* a decision node of player i, $K_i(t)$ is a proper superset of $\theta(t)$ and $\beta_i(p_t) \in K_i(t)$. Then by definition of $\beta = \chi(\sigma, \mu)$, $\beta_i(t) = \beta_i(p_t)$. If p_t falls under one of the previous cases, the proof is complete [recall that by proposition 1′ in Bonanno, 1992a, $K_i(p_t) \supseteq K_i(t)$], otherwise consider the immediate predecessor of p_t, and so on. Eventually we will reach a node that falls under one of the previous cases.

It only remains to show that β satisfies the property of Individual Rationality. Fix an arbitrary information set, call it h, and let i be the corresponding player. Let $\tilde{x} \in h$ be the node such that $\mu(\tilde{x}) = 1$. By definition of $\beta = \chi(\sigma, \mu)$, $\beta_i(h) = \zeta(\tilde{x} \mid \sigma)$. Let y be an arbitrary immediate successor of $\tilde{x}$. We want to show that

$$U_i(\beta_i(h)) \geq U_i(\beta_i(y)). \tag{B14}$$

By Minimal Revision and lemma 2.1, β satisfies Contraction Consistency. Thus if $\zeta(\tilde{x} \mid \sigma) \in \theta(y)$, then $\beta_i(y) = \zeta(\tilde{x} \mid \sigma)$ [since by lemma 3 in Bonanno, 1992a, $\theta(y) \subseteq K_i(y)$] and therefore (B14) is satisfied as an equality. Suppose therefore that $\zeta(\tilde{x} \mid \sigma) \notin \theta(y)$. Then node y comes after a choice at h which is different from the choice selected (with probability 1) by σ at h. By Minimal Revision and lemma 2.1, β satisfies Tree Consistency and Choice Consistency. Thus $\beta_i(y) \in \theta(y)$ and, by Lemma A (cf. Appendix A) $\beta_i(y) = \zeta(\tilde{x} \mid \sigma)$. Let σ_i' the strategy of player i obtained by modifying σ_i only at information set h and so that the choice selected with probability 1 by σ_i' at h is the choice that precedes node y. Let $\sigma = (\sigma_i', \sigma_{-i})$. Then, by construction,

$$\zeta(\tilde{x} \mid \sigma') = \zeta(y \mid \sigma') = \zeta(y \mid \sigma). \tag{B15}$$

By sequential rationality,

$$U_i(\zeta(\tilde{x} \mid \sigma)) \geq U_i(\zeta(\tilde{x} \mid \sigma')). \tag{B16}$$

Thus, using (B15) and (B16) and the facts that $\beta_i(y) = \zeta(y \mid \sigma)$ and $\beta_i(h) = \zeta(\tilde{x} \mid \sigma)$, we obtain (B14).

REFERENCES

P. BATTIGALLI (1991), "Strategic Independence, Generally Reasonable Extended Assessments and Consistent Assessment", mimeo, Bocconi University, Milano (Italy) [forthcoming in the *Journal of Economic Theory*].

G. BONANNO (1992a), "Players' Information in Extensive Games", *Mathematical Social Sciences*, 24, pp. 35-48.

G. BONANNO (1992b), "Rational Beliefs in Extrensive Games", *Theory and Decision*, 33, pp. 153-176.

D. FUNDENBERG - J. TIROLE (1991), "Perfect Bayesian Equilibrium and Sequential Equilibrium", *Journal of Economic Theory*, 53, pp. 236-260.

E. KOHLBERG - P. RENY (1991), "On the Rationale for Perfect Equilibrium", mimeo, University of Western Ontario, London (Canada).

D. KREPS - G. RAMEY (1987), "Structural Consistency, Consistency, and Sequential Rationality", *Econometrica*, 55 (6), pp. 1331-1348.

D. KREPS - R. WILSON (1982), "Sequential Equilibria", *Econometrica*, 50 (4), pp. 863-894.

R. SELTEN (1975), "Re-examination of the Perfectness Concept for Equilibrium Points in Extensive Games", *International Journal of Game Theory*, 4, pp. 25-55.