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Players' information in extensive games

Giacomo Bonanno*

Department of Economics, University of California, Davis, CA 95616, USA

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This paper suggests a way of formalizing the amount of information that can be conveyed to each player along every possible play of an extensive game. The information given to each player i when the play of the game reaches node x is expressed as a subset of the set of terminal nodes. Two definitions are put forward, one expressing the minimum amount of information and the other the maximum amount of information that can be conveyed without violating the constraint represented by the information sets. Our definitions provide intuitive characterizations of such notions as perfect recall, perfect information and simultaneity.

Key words: Information; extensive game; information set.

1. Introduction

What information is, or can be, conveyed to the players during the play of an extensive game? A partial answer to this question is implicit in the notion of an information set: if two decision nodes x and y belong to the same information set of player i, then player i does not know whether she is making a choice at x or at y. However, in the notion of an information set one cannot find an answer to questions such as: (i) Who informs a player when it is her turn to move? (ii) Does, or can, the play of an extensive game follow a well-defined temporal structure? (iii) Is a player given (or can she be given) any information when the play of the game reaches a decision node that belongs to another player? (iv) What is the content of the information given to the players during every possible play of the game?

In this paper we do not deal with the problem of who gives the relevant information to each player.[1] Instead we suggest one way of formalizing the

Correspondence to: G. Bonanno, Department of Economics, University of California, Davis, CA 95616, USA.

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[1] One possibility is that there is an outside agent (a referee or umpire) who provides different players with different amounts of information as the play of the game unfolds.

information that is conveyed to the players along every possible play of an extensive game. With every node t and every player i we associate a subset of the set of terminal nodes representing what player i knows when node t is reached. The interpretation is that if—when node t is reached—player i's information is given by, say, the set $\{z_1, z_2, z_5, z_8\}$, then player i is informed that the play of the game so far has been such that only terminal nodes z_1, z_2, z_5 or z_8 can be reached.

If t is a decision node that belongs to information set h of player i, we define player i's information at t as the set of all the terminal nodes that can be reached from nodes in h. It seems that, as long as a player's information at her decision nodes is a faithful representation of her information sets, there is a lot of freedom concerning the specification of her information at decision nodes of *other* players. We put forward two different ways of dealing with this degree of freedom, i.e. we suggest two alternative definitions of information, denoted by $N_i(t)$ and $K_i(t)$, respectively. The first represents the situation where each player is given the *minimum* amount of information, while the second represents the situation where each player is given the *maximum* amount of information compatible with the structure of the game.

One of the advantages of our definitions is that they provide an intuitive characterization of such notions as perfect recall, perfect information and simultaneity. This will be shown in Sections 3 and 4. Another advantage of our approach is that it suggests a new way of thinking about the solutions of an extensive game. For example, in Bonanno (1991) the notion of a rational profile of beliefs is introduced and it is shown that it gives rise to a refinement of the notion of subgame-perfect equilibrium.[2]

2. Preliminary definitions

This section introduces the notation and defines some functions that will be used throughout the paper.

Fix a finite extensive form.[3] Let X be the set of *decision* nodes, Z the set of *terminal* nodes and $T = X \cup Z$. (In general, we shall denote a decision node by x or y, a terminal node by z and a generic node—decision or terminal—by t.) Let x_0 be the root of the tree and, for every node $t \neq x_0$, let π_t be the immediate predecessor of t. For every $i \in I$ (where I is the finite set of players), P_i is the set of decision

[2] A belief of player i is defined as a function β_i that associates with every node t an element of the set $K_i(t)$. The interpretation is as follows. Suppose $K_i(t) = \{z_1, z_3, z_8\}$ and $\beta_i(x) = z_3$. Then when node t is reached, player i knows that the play of the game so far has been such that only terminal nodes z_1, z_3 or z_8 can be reached and she actually believes that z_3 will be the final outcome. A profile of beliefs is a list of beliefs, one for each player.

[3] An extensive form is an extensive game without the payoff functions. The notion of extensive game was introduced by Kuhn (1953). We shall adopt Selten's (1975) formulation of it. The details of this definition, as well as a generalization of it, can also be found in Dubey and Kaneko (1984).

nodes of player i, H_i is the set of information sets of player i and, for every $h \in H_i$, $C_i(h)$ is the set of choices of player i at h.

For every $t \in T$, let $\theta(t) \subseteq Z$ be the set of terminal nodes that can be reached from t (if $t \in Z$, then $\theta(t) = \{t\}$). For example, in the extensive form of Fig. 1, $\theta(x_2) = \{z_4, z_5, z_6, z_7\}$.

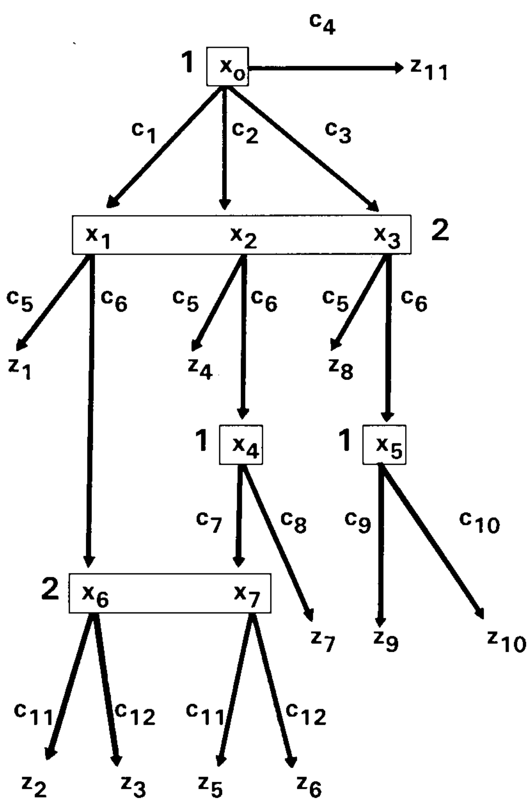

Fig. 1.

We shall omit the proof of the following lemma, since it is an immediate consequence of uniqueness of plays in extensive forms:

Lemma 1. *Fix an arbitrary extensive form. Then*:

(a) *if node t is a successor of node x, $\theta(t) \subseteq \theta(x)$*;

(b) *if t and w are such that $t \neq w$ and neither t is a predecessor of w nor w is a predecessor of t, then $\theta(t) \cap \theta(w) = \emptyset$.*

For every information set h, we denote by $\theta^*(h)$ the set of terminal nodes that can be reached from nodes in h: $\theta^*(h) = \bigcup_{x \in h} \theta(x)$. For example, in the extensive form of Fig. 1, $\{x_6, x_7\}$ is an information set of player 2 and $\theta^*(\{x_6, x_7\}) = \{z_2, z_3, z_5, z_6\}$.

Recall that a choice c at information set $h = \{x_1, \ldots, x_m\}$ is a set of edges $c = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$, where node y_k is an immediate successor of node x_k $(k = 1, \ldots, m)$. Define

$$\mu(c) = \theta(y_1) \cup \theta(y_2) \cup \cdots \cup \theta(y_m),$$

that is, $\mu(c)$ is the set of terminal nodes that can be reached from nodes in h by following the edges that constitute choice c. For example, in the extensive form of Fig. 1, $\mu(c_6) = \{z_2, z_3, z_5, z_6, z_7, z_9, z_{10}\}$. Note that if $h \in H_i$, then $\theta^*(h) = \bigcup_{c \in C_i(h)} \mu(c)$.

3. The first definition of players' information

The definition given in this section has the following intuitive content. At the root of the tree all players have the same information, represented by the set of all terminal nodes. A player is given new information only when one of her information sets is reached (in which case the information she receives is represented by the set of terminal nodes reachable from that information set) or when a terminal node is reached (in which case the information she receives is represented by the singleton set containing that terminal node). In other words, players are given new information only when necessary: to advise them that they have to move or to advise them that the game has ended. The definition also takes into account the fact that a player's knowledge may change even if the player is not given any new information: after she has made a choice at one of her information sets, she will know that the outcome of the game is restricted to the set of terminal nodes that can be reached by that choice.

Define the function $N: I \times T \to 2^Z$ (where 2^Z denotes the set of subsets of Z) by the following conditions (we shall write $N_i(t)$ instead of $N(i, t)$):

(1) For every player $i \in I$, set $N_i(x_0) = Z$ (recall that x_0 is the root of the tree).

(2) For every terminal node $z \in Z$ and for every player $i \in I$, set $N_i(z) = \{z\}$.

(3) If node $x \neq x_0$ belongs to information set h of player i, set $N_i(x) = \theta^*(h)$, that is, $N_i(x)$ is the set of terminal nodes that can be reached from nodes in h.[4]

(4) If $x \neq x_0$ is a decision node that does *not* belong to player i while π_x belongs to information set h of player i and c is the choice at h that leads from π_x to x, set $N_i(x) = \mu(c)$ (recall that π_x denotes the immediate predecessor of x and $\mu(c)$ is the set of terminal nodes that can be reached from h by following the edges that constitute choice c).

[4] It follows that if x and y are two nodes that belong to information set h of player i, then $N_i(x) = N_i(y)$.

(5) Finally, if $x \neq x_0$ is a decision node that does *not* belong to player i and also π_x does not belong to player i, set $N_i(x) = N_i(\pi_x)$.[5]

The above conditions define a unique non-empty subset of Z for every $i \in I$ and $t \in T$.[6] As an illustration consider the extensive form of Fig. 1.

By (1): $N_1(x_0) = N_2(x_0) = Z = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}\}$.

By (4): $N_1(x_1) = \mu(c_1) = \{z_1, z_2, z_3\}$; $N_1(x_2) = \mu(c_2) = \{z_4, z_5, z_6, z_7\}$;

$N_1(x_3) = \mu(c_3) = \{z_8, z_9, z_{10}\}$.

By (3): $N_2(x_1) = N_2(x_2) = N_2(x_3) = \theta^*(\{x_1, x_2, x_3\})$

$= \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}\}$.

By (3): $N_1(x_4) = \theta(x_4) = \{z_5, z_6, z_7\}$; $N_1(x_5) = \theta(x_5) = \{z_9, z_{10}\}$.

By (4): $N_2(x_4) = N_2(x_5) = \mu(c_6) = \{z_2, z_3, z_5, z_6, z_7, z_9, z_{10}\}$.

By (5): $N_1(x_6) = N_1(x_1) = \{z_1, z_2, z_3\}$; by (4): $N_1(x_7) = \mu(c_7) = \{z_5, z_6\}$.

By (3): $N_2(x_6) = N_2(x_7) = \theta^*(\{x_6, x_7\}) = \{z_2, z_3, z_5, z_6\}$.

By (2): $N_1(z_j) = N_2(z_j) = \{z_j\}$ for all $j = 1, 2, \ldots, 11$.

This example can be used to stress an important point of interpretation of the definition of players' information given above. If x is the immediate predecessor of y and $N_i(x) \neq N_i(y)$, then the suggested interpretation is that, as the play of the game moves from node x to node y, player i is given new information, *except* for the case covered by (4), namely the case where x is a decision node of player i and y is a decision node of another player. In this case player i's information changes, *not* because player i is given new information, but simply because the player knows what action he took. The alternative interpretation (a player's information changes *only if* he is given new information) is subject to the following criticism. Consider the extensive form of Fig. 1 and the two paths from x_0 to x_6 and from x_0 to x_7. Along the first path, player 2's information changes twice (when node x_1 is reached and when node x_6 is reached), while along the second path player 2's information changes three times (at node x_2, at node x_4 and at node x_7). If we interpret a player's change of information as determined by the fact that the player is necessari-

[5] It follows that if decision node y is a successor of node x and none of the nodes on the path from x to y belongs to player i, then $N_i(x) = N_i(y)$.

[6] This can be proved by induction, as follows. First of all we show that if y is an immediate successor of x and $N_i(x)$ is a non-empty subset of Z, then so is $N_i(y)$. If $y \in Z$, $N_i(y) = \{y\} \neq \emptyset$, by (2). If y belongs to information set h of player i, $N_i(y) = \theta^*(h) \neq \emptyset$, by (3). If x is a decision node of player i while y is a decision node of another player, $N_i(y) = \mu(c) \neq \emptyset$, where c is the choice of player i that leads from x to y. Finally, in every other case $N_i(y) = N_i(x)$, by (5). To complete the proof it is sufficient to note that $N_i(x_0) = Z$ and that there is a unique path from x_0 to any other node.

ly *given* new information, then, by looking at the tree, player 2—having been informed that his second information set has been reached—can deduce that if he did not receive the information embodied in $N_2(x_4)$, then he must be at node x_6, while, if he did, then he must at node x_7. On the contrary, according to our interpretation, player 2 will *not* be able to distinguish between nodes x_6 and x_7, since his information at node x_4 is different to what it was before (at node x_2), not because the player has been told anything, but simply because he knows that he took action c_6. *After taking action c_6 player 2 knows that the outcome of the game is restricted to the set $\mu(c_6)$ but does not know whether the next piece of information that he will receive is $\{z_7\}$, $\{z_9\}$, $\{z_{10}\}$ or $\{z_2, z_3, z_5, z_6\}$.*[7]

Lemma 2. *For every node t and every player i, $\theta(t) \subseteq N_i(t)$.*

Proof. We prove this by induction. We first prove that if t is an immediate successor of x and $\theta(x) \subseteq N_i(x)$, then $\theta(t) \subseteq N_i(t)$. If t is a terminal node, then $\theta(t) = N_i(t) = \{t\}$. If node t belongs to information set h of player i, then $N_i(t) = \theta^*(h) \supseteq \theta(t)$. If t satisfies condition (4), then $N_i(t) = \mu(c) \supseteq \theta(t)$ (where c is player i's choice that precedes t). Finally, in every other case, $N_i(t) = N_i(x)$. By Lemma 1, $\theta(t) \subseteq \theta(x)$ and by our supposition $\theta(x) \subseteq N_i(x)$. To complete the proof we only need to recall that $N_i(x_0) = \theta(x_0) = Z$. □

Note that when the play of the game reaches node t, the set of outcomes that are still possible is $\theta(t)$. Thus Lemma 2 says that the information that player i has when node t is reached is correct, although it may be imprecise (it may be a proper superset of $\theta(t)$).

The following proposition gives a partial characterization of the notion of perfect recall.[8]

Proposition 1. *An extensive form with perfect recall satisfies the following property: if node t is a successor of node x, then, for every player i, $N_i(t) \subseteq N_i(x)$. In other words, at every node each player has at least as much information as she had before that node was reached.*

Proof. Fix an extensive form with perfect recall. By transitivity of inclusion it is sufficient to prove that if y is an immediate successor of x, then $N_i(y) \subseteq N_i(x)$, for every player i. We shall consider all the possible cases.

[7] Alternatively one could make the definition of $N_i(t)$ coarser by dropping (4) and extending (5) to every decision node that does not belong to player i. It is easy to check that all the results given in this section are true also for this coarser definition.

[8] An extensive form is said to have *perfect recall* if it satisfies the following property: for every player i and for every two information sets h and g of player i, if one node $x \in g$ comes after a choice c at h, then every node $y \in g$ comes after this choice.

Case 1: x belongs to information set h of player i and y belongs to information set g of player i. Let c be the choice at h that leads from x to y. By perfect recall, for every $v \in g$, $\theta(v) \subseteq \mu(c)$. Since $N_i(x) = \bigcup_{d \in C_i(h)} \mu(d)$ (where $C_i(h)$ is the set of choices at h), and $N_i(y) = \theta^*(g) = \bigcup_{v \in g} \theta(v)$, it follows that $N_i(y) \subseteq \mu(c) \subseteq N_i(x)$.

Case 2: x is a decision node of player i, while y is not. Then, either $N_i(y) = \theta(y) = \{y\}$ (if y is a terminal node) or $N_i(y) = \mu(c)$ (if y is a decision node), where c is player i's choice that leads from x to y. Since $\theta(y) \subseteq \mu(c) \subseteq N_i(x)$, it follows that $N_i(y) \subseteq N_i(x)$.

Case 3: Neither x nor y are decision nodes of player i. Then if y is a decision node, $N_i(y) = N_i(x)$ by (5), while if y is a terminal node, $N_i(y) = \theta(y) = \{y\}$. By Lemma 1, $\theta(y) \subseteq \theta(x)$, and by Lemma 2, $\theta(x) \subseteq N_i(x)$.

Case 4: x is not a decision node of player i, while y is. Consider the path from the root to x. If none of the nodes on this path belongs to player i, then $N_i(x) = N_i(x_0) = Z \supseteq N_i(y)$. Otherwise, let t be the last node on this path that belongs to player i and let c be the choice at t that precedes x. Let v be the immediate successor of t on this path. Then, $N_i(v) = \mu(c)$ and $N_i(x) = N_i(v)$. By perfect recall, $N_i(y) \subseteq \mu(c)$. □

The extensive form shown in Fig. 2 is one with *imperfect* recall and yet it satisfies the property of Proposition 1. Hence the converse of Proposition 1 is not true.[9]

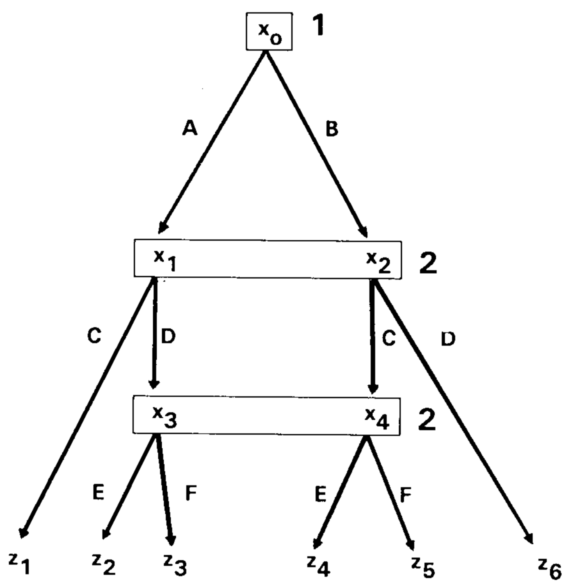

Fig. 2.

[9] Here we have: $N_1(x_0) = N_2(x_0) = N_2(x_1) = N_2(x_2) = Z$, $N_1(x_1) = N_1(x_3) = \{z_1, z_2, z_3\}$, $N_1(x_2) = N_1(x_4) = \{z_4, z_5, z_6\}$, $N_2(x_3) = N_2(x_4) = \{z_2, z_3, z_4, z_5\}$.

Define an extensive form to be *simultaneous* if every play crosses all the information sets.[10]

Proposition 2. *An extensive form (with or without perfect recall) is simultaneous if and only if it satisfies the following property: if node x belongs to player i, then* $N_i(x)=Z$. *That is, when a player has to move she knows as much as she did at the root of the tree (her information has not improved since the beginning of the game).*

Proof.

Necessity. Fix a simultaneous extensive form. Choose an arbitrary information set h. Since every play crosses h, $\theta^*(h)=Z$. Thus, if h belongs to player i, for every $x \in h$, $N_i(x)=Z$.

Sufficiency. Consider an extensive form satisfying the property that, for every player i and for every node x that belongs to player i, $N_i(x)=Z$. Suppose that the extensive form is not simultaneous. Then there exists a player i, an information set h of player i and a terminal node z such that the play to z does not cross h. Then for every $y \in h$, $z \notin \theta(y)$. Hence, $z \notin \bigcup_{y \in h} \theta(y)=\theta^*(h)$. Since, for every $y \in h$, $N_i(y)=\theta^*(h)$, it follows that $N_i(y)$ is a proper subset of Z, a contradiction. □

A characterization of the notion of perfect information is given in the following proposition. First of all, note that, according to the definition of information given in this section, it is not true that in extensive forms of perfect information at every node all the players have the same information.[11]

Proposition 3. *An extensive form with perfect recall has* perfect information *if and only if, at every decision node, the player whose turn it is to move knows at least as much as every other player, that is, if and only if it satisfies the following property: if node x belongs to player i, then* $N_i(x) \subseteq N_j(x)$ *for all* $j \in I$.

Proof.

Necessity. In a game of perfect information, if h is an information set, then $h=\{x\}$ for some node x. Hence $\theta^*(h)=\theta(x)$. It follows that if x belongs to player i, $N_i(x)=\theta(x)$. By Lemma 2, $\theta(x) \subseteq N_j(x)$ for every player j.

Sufficiency. Consider a game with perfect recall that satisfies the property of Proposition 3 and assume there is a player i and an information set h of player i such that $h=\{x_1,\dots,x_m\}$, with $m \geq 2$. Let w_k be the immediate successor of x_0 on

[10] It can be shown that in a simultaneous game *with perfect recall* where every player has at least two choices at every information set, it must be the case that every player has exactly one information set.

[11] Consider, for example, a perfect information game with three players, three decision nodes (one for each player) and two choices at every decision node. Then at the second decision node (immediate successor of the root) the player who has just moved and the player whose turn it is to move have more information than the remaining player.

the path from x_0 to x_k ($k = 1, \ldots, m$). Let c_k be the choice at the root to which edge (x_0, w_k) belongs (in fact, it must be $c_k = \{(x_0, w_k)\}$). By the definition of extensive form, $k \neq k'$ implies $c_k \neq c_{k'}$ and, by (b) of Lemma 1, $\theta(w_k) \cap \theta(w_{k'}) = \emptyset$. If x_0 belongs to player i, then x_1 comes after choice c_1 and x_2 comes after choice c_2, with $c_1 \neq c_2$, contradicting the hypothesis of perfect recall. Thus x_0 belongs to some player $j \neq i$. Now, for each $k = 1, \ldots, m$, $N_j(w_k) = \theta(w_k)$. In fact, if node w_k does not belong to player j, this is true by definition of N_j, since $\mu(c_k) = \theta(w_k)$. If, on the other hand, w_k belongs to information set f of player j, then it must be $f = \{w_k\}$, because if there were a $v \neq w_k$ that also belonged to f, then by perfect recall v would have to come after choice c_k and this would imply that v would be a successor of w_k—contradicting the definition of extensive form (no play can intersect the same information set more than once). By Proposition 1, $N_j(x_1) \subseteq N_j(w_1) = \theta(w_1)$. However, $N_i(x_1)$ is a superset of $\theta(x_1) \cup \theta(x_2)$ (both non-empty sets) and, by Lemma 1, $\theta(x_2) \subseteq \theta(w_2)$ and $\theta(w_1) \cap \theta(w_2) = \emptyset$. Hence $\theta(x_2) \cap \theta(w_1) = \emptyset$. It follows that it cannot be $N_i(x_1) \subseteq N_j(x_1)$, contradicting the hypothesis that at node x_1 player i knows at least as much as every other player. □

Note that if perfect recall is not assumed, then Proposition 3 is false: any one-person game with imperfect recall satisfies the property of Proposition 3.

4. The second definition of information

While the definition of information given in the previous section represents the *minimum* amount of information that must be given to each player during any play of the game, the definition given in this section represents the *maximum* amount of information that can be conveyed to the players.[12] Its intuitive content is as follows. As before, (i) at the root of the tree all players have the same information; (ii) if z is a terminal node, every player is informed that the game ended at that node; (iii) if node x belongs to information set h of player i, then player i is told that h has been reached. The new feature is the following: if node x does not belong to player i and all the information sets of player i (if any) that are crossed by paths starting at x consist entirely of nodes that are successors of x, then player i is informed that node x has been reached (the justification for this rule is that later on, at any of her information sets, player i will be able to deduce that the play of the game must have gone through node x; hence player i might as well be told at the time when x is reached). When the above condition is not satisfied, player i's information at x either does not change, or at most reflects the choice made by player i at the immediate predecessor of x, if that node belonged to player i.

[12] The question of whether the definition given in this section indeed represents the maximum amount of information that can be given to the players is discussed in detail in the appendix.

First a new piece of notation. For every node $t \in T$ and for every player $i \in I$, let $H_i(t)$ be the subset of H_i (recall that H_i is the set of information sets of player i) defined by the following condition: $h \in H_i(t)$ if and only if there is a node $y \in h$ that is a successor of t. Define the function $K: I \times T \to 2^Z$ by the following conditions.[13]

(1′) For every $i \in I$ set $K_i(x_0) = Z$.

(2′) For every $z \in Z$ and for every player $i \in I$, set $K_i(z) = \{z\}$.

(3′) If $h \in H_i$, then for every $x \in h$ set $K_i(x) = \theta^*(h)$.

(4′) If $x \notin P_i$ (recall that P_i is the set of decision nodes of player i) and either $H_i(x) = \emptyset$ or, for every $h \in H_i(x)$, $\theta^*(h) \subseteq \theta(x)$ (that is, every node in h is a successor of x), then set $K_i(x) = \theta(x)$.

(5′) If $x \notin P_i$ and the condition given under (4′) is not satisfied (that is, there exists an $h \in H_i(x)$ and a node $y \in h$ such that y is not a successor of x) and π_x (the immediate predecessor of x) is a decision node of player i and c is the choice of player i that leads from π_x to x, then set $K_i(x) = \mu(c)$.

(6′) Finally, in every other case set $K_i(x) = K_i(\pi_x)$.

Lemma 3. *For every $i \in I$ and $t \in T$, $\theta(t) \subseteq K_i(t) \subseteq N_i(t)$.*

Proof. Since (1′)–(3′) are identical to (1)–(3), (5′) implies (4), and (6′) implies (5), it follows from (4′) that $K_i(t) \neq N_i(t)$ implies $K_i(t) = \theta(t)$. By Lemma 1, $\theta(t) \subseteq N_i(t)$. □

For example, in the extensive form of Fig. 1 we have that $K_1(t) = \theta(t)$ for every node t and $K_2(t) = N_2(t)$ for every node $t \neq x_5$, while, by (4′), $K_2(x_5) = \theta(x_5) = \{z_9, z_{10}\}$.

The characterization of perfect recall and simultaneity obtained for N are true also for K.

Proposition 1′. *An extensive form with perfect recall satisfies the following property: if y is a successor of x, then, for every player i, $K_i(y) \subseteq K_i(x)$.*

Proof. In order to adapt the proof of Proposition 1 to the function K we only need to show that if x is not a decision node of player i and y is an immediate successor of x that belongs to information set h of player i, then $K_i(y) \subseteq K_i(x)$. Three cases are possible: (i) $K_i(x) = \theta(x)$; (ii) π_x belongs to information set g of player i, c is the choice to which edge (π_x, x) belongs and $K_i(x) = \mu(c)$; (iii) π_x does not belong to player i and $K_i(x) = K_i(\pi_x)$. Case (i) requires $\theta^*(h) \subseteq \theta(x)$. Since $K_i(y) = \theta^*(h)$, it follows that $K_i(y) \subseteq K_i(x)$. In case (ii), since y comes after choice c, by perfect recall every node in h comes after choice c. Hence $K_i(y) \subseteq \mu(c) = K_i(x)$. Finally, consider case (iii), where π_x does not belong to player i and $K_i(x) = K_i(\pi_x)$. Then we

[13] With an argument similar to the one used in footnote 6, it can be shown that (1′)–(6′) define a unique non-empty subset of Z for every $i \in I$ and $t \in T$.

can apply to π_x the same reasoning as above: if either (i) or (ii) above applies to π_x, then the proof is complete. If case (iii) applies to π_x, then consider the immediate predecessor of π_x and proceed the same way. Eventually either case (i) or case (ii) applies, because $K_i(x_0)=\theta(x_0)$. □

As before, the converse of Proposition 1′ is not true (cf. the extensive form of Fig. 2).

Proposition 2′. *An extensive form is* simultaneous *if and only if it satisfies the following property: if x is a decision node of player i, then $K_i(x)=Z$.*

Proof. Since for every $x\in P_i$, $K_i(x)=N_i(x)$, Proposition 2′ follows from Proposition 2. □

What becomes different is the characterization of the notion of perfect information. With the definition given in this section perfect information games are those where at every node all the players have the same information.

Proposition 3′. *An extensive form with perfect recall has* perfect information *if and only if for every $t\in T$ and for every $i,j\in I$, $K_i(t)=K_j(t)$.*

Proof.

Necessity. In a game of perfect information, if h is an information set, then $h=\{x\}$ for some node x. Hence $\theta^*(h)=\theta(x)$. Furthermore, for every player i, if $g\in H_i(x)$, then $g=\{y\}$, where y is a successor of x. Hence, by Lemma 1, $\theta^*(g)=\theta(y)\subseteq\theta(x)$. It follows that $K_i(x)=\theta(x)$ for all i.

Sufficiency. Consider a game with perfect recall that satisfies the property of Proposition 3′ and assume there is a player i and an information set h of player i such that $h=\{x_1,\ldots,x_m\}$, with $m\geq 2$. Let v be the unique node that satisfies the following properties: (1) v is a predecessor of x_k, for all $k=1,\ldots,m$; (2) no successor of v satisfies (1). Such a node v exists because the root satisfies property (1) and the number of nodes is finite; it is unique because of uniqueness of plays in extensive forms. Let w_k be the immediate successor of v on the path from v to x_k $(k=1,\ldots,m)$. By definition of v, there must exist k and k' such that $w_k\neq w_{k'}$. Let c be the choice (at the information set that contains v) to which edge (v,w_k) belongs and c' be the choice (at the same information set) to which edge $(v,w_{k'})$ belongs. Then, by the uniqueness of the plays in extensive forms and by the definition of choice, $\mu(c)\cap\mu(c')=\emptyset$. If v belongs to player i, then x_k comes after choice c and $x_{k'}$ comes after choice c', contradicting the hypothesis of perfect recall. Thus v belongs to some player $j\neq i$. Then, since $K_j(w_k)\subseteq\mu(c)$ and $K_j(w_{k'})\subseteq\mu(c')$ and $\mu(c)\cap\mu(c')=\emptyset$, $K_j(w_k)\cap K_j(w_{k'})=\emptyset$. By Proposition 1′, $K_j(x_k)\subseteq K_j(w_k)$ and $K_j(x_{k'})\subseteq K_j(w_{k'})$. Thus $K_j(x_k)\cap K_j(x_{k'})=\emptyset$. Since $K_i(x_k)=K_i(x_{k'})$, it cannot be that $K_i(x_k)=K_j(x_k)$ and $K_i(x_{k'})=K_j(x_{k'})$, contradicting the hypothesis of Proposition 3′. □

5. Conclusion

We suggested a way of formalizing the amount of information that can be conveyed to each player along every possible play of an extensive form. The information given to player i when the play of the game reaches node t is expressed as a subset of the set of terminal nodes and has a natural interpretation. Two definitions were put forward, one (Section 3) expressing the minimum amount of information and the other (Section 4) the maximum amount of information that can be conveyed to the players without violating the constraint represented by the information sets. We showed that our definitions provide intuitive characterizations of such notions as perfect recall, perfect information and simultaneity. Other advantages of our approach are explored in Bonanno (1991).

Appendix

In this appendix we discuss whether the function K defined in Section 4 can indeed be interpreted as the *maximum* amount of information that can be conveyed to the players. A referee suggested that players can be given more information than the function K allows:

> Suppose the play of the game reaches node x. Let w be a predecessor of x and let c be a choice incident out of w, but not on the path from w to x; then player i at x can be told that c was not chosen unless there is an information set $h \in H_i$ and two nodes $u, v \in h$ such that v comes after choice c and u is either a successor of x or coincides with x. In other words, i is informed of all past events he can learn without having to forget them at some information set after x or containing x.

Let $\hat{K}: I \times T \to 2^Z$ be the function that represents the above suggestion (it will be defined shortly). The difference between K and $\hat{K}$ can be seen in the extensive form of Fig. 1, where $K_2(x_4) = \mu(c_6) = \{z_2, z_3, z_5, z_6, z_7, z_9, z_{10}\}$, while $\hat{K}_2(x_4) = \{z_2, z_3, z_5, z_6, z_7\}$, that is, according to $\hat{K}$, player 2 at node x_4 can be informed that player 1 did not take action c_3. A possible objection to the function $\hat{K}$ is related to the problem of interpretation discussed in Section 3. With the function K, the change in player 2's information as the play of the game proceeds from node x_2 to node x_4 can be interpreted as a mere reflection of the fact that player 2 knows that she took action c_6 (and she does not know if the next piece of information that she will receive is $\{z_2, z_3, z_5, z_6\}$ or $\{z_9, z_{10}\}$), so that player 2 is not actually given any new information when node x_4 is reached. With $\hat{K}$, on the other hand, as the play of the game reaches node x_4, player 2 learns that outcomes z_9 and z_{10} are no longer possible: information that she *cannot* deduce from the knowledge of having taken action c_6. Thus player 2 *does* receive new information as node x_4 is reached and this fact enables her to discriminate between nodes x_6 and x_7 (depending on whether her information changed directly from $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_9, z_{10}\}$ to $\{z_2, z_3, z_5, z_6\}$ or first from $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_9, z_{10}\}$ to $\{z_2, z_3, z_5, z_6, z_7\}$ and then from the latter to $\{z_2, z_3, z_5, z_6\}$). The difference between K and $\hat{K}$ can be 'solved' by adding a dummy information set of player 2 containing two nodes, one between x_1 and x_6 and the other between x_2 and x_4, where player 2 has only one choice (note that adding only a node between x_1 and x_6 and a dummy player at that node with only one choice would not be enough). After this 'inessential' transformation of the extensive form, K and $\hat{K}$ coincide. Thus $\hat{K}$ can be seen as a refinement of K which is invariant to *some* 'inessential' transformations.

The referee suggested the following definition:

(a) In cases (1′)–(3′), $\hat{K}_i(t) = K_i(t)$.

(b) If $x \notin P_i$ and $\pi_x \notin P_i$, then

$$\hat{K}_i(x) + \theta(x) \cup \left(\bigcup_{h \in H_i(x)} \theta^*(h) \right).$$

(c) If $x \notin P_i$ and $\pi_x \in P_i$ and $(\pi_x, x) \in c$, then

$$\hat{K}_i(x) = \left(\theta(x) \cup \left(\bigcup_{h \in H_i(x)} \theta^*(h) \right) \right) \cap \mu(c).$$

It can be shown that $\theta(t) \subseteq \hat{K}_i(t)$, for every $i \in I$ and $t \in T$ and that in extensive forms with perfect recall $\hat{K}_i(t) \subseteq K_i(t)$ (if perfect recall is not assumed, then the latter inclusion is not true in general). Furthermore, Propositions 1′, 2′ and 3′ are true also for $\hat{K}$. (Proofs of these claims can be obtained from the author.)

The referee also noted that if we assume at the outset that the game has perfect recall, then $\hat{K}$ can elegantly be defined by a unique condition. First define

$$H_i^*(t) = \begin{cases} H_i(t), & \text{if } t \notin P_i, \\ H_i(t) \cup h, & \text{if } t \in h \in H_i. \end{cases}$$

Then $\hat{K}$ can be defined as follows:

$$\hat{K}_i(t) = \theta(t) \cup \left(\bigcup_{h \in H_i^*(t)} \theta^*(h) \right).$$

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