A modal logic translation of the AGM axioms for belief revision

Giacomo Bonanno

Department of Economics, University of California, Davis, USA
gfbonanno@ucdavis.edu

Abstract

Building on the analysis of Bonanno (2025) we introduce a simple modal logic containing three modal operators: a unimodal belief operator B, a bimodal conditional operator $>$ and the unimodal global operator $\Box$. For each AGM axiom for belief revision, we provide a corresponding modal axiom. The correspondence is as follows: each AGM axiom is characterized by a property of the Kripke-Lewis frames considered in Bonanno (2025) and, in turn, that property characterizes the proposed modal axiom.

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1 Introduction

In Bonanno (2025) a new semantics for both belief update and belief revision was provided in terms of frames consisting of a set of states, a Kripke belief relation and a Lewis selection function. In this paper we focus on AGM belief revision and make use of that semantics to establish a correspondence between each AGM axiom and a modal axiom in a logic that contains three modal operators: a unimodal belief operator B, a bimodal conditional operator $>$ and the unimodal global operator $\Box$. Adding a valuation to a frame yields a model. Given a model and a state s, we identify the initial belief set K with the set of formulas that are believed at state s (that is, $K = \{\phi : s \models B\phi\}$) and we identify the revised belief set $K * \phi$ (prompted by the input represented by formula ϕ) as the set of formulas that are the consequent of conditionals that (1) are believed at state s and (2) have ϕ as antecedent (that is, $K * \phi = \{\psi : s \models B(\phi > \psi)\}$). The next section briefly reviews the AGM axioms for belief revision and the approach put forward in Bonanno (2025), while Section 3 introduces the modal logic and provides axioms that correspond to the semantic properties that characterize the AGM axioms. Thus there is a precise sense in which each proposed modal axiom corresponds to the respective AGM axiom.

2 AGM axioms and their semantic characterization

We consider a propositional logic based on a countable set `At` of atomic formulas. We denote by Φ_0 the set of Boolean formulas constructed from `At` as follows: $\texttt{At} \subset \Phi_0$ and if $\phi, \psi \in \Phi_0$ then $\neg\phi$ and $\phi \vee \psi$ belong to Φ_0. Define $\phi \rightarrow \psi$, $\phi \wedge \psi$, and $\phi \leftrightarrow \psi$ in terms of $\neg$ and $\vee$ in the usual way.

Given a subset K of Φ_0, its deductive closure $Cn(K) \subseteq \Phi_0$ is defined as follows: $\psi \in Cn(K)$ if and only if there exist $\phi_1, ..., \phi_n \in K$ (with $n \geq 0$) such that $(\phi_1 \wedge ... \wedge \phi_n) \rightarrow \psi$ is a tautology. A set $K \subseteq \Phi_0$ is *consistent* if $Cn(K) \neq \Phi_0$; it is *deductively closed* if $K = Cn(K)$. Given a set $K \subseteq \Phi_0$ and a formula $\phi \in \Phi_0$, the *expansion* of K by ϕ, denoted by $K+\phi$, is defined as follows: $K+\phi = Cn\left(K \cup \{\phi\}\right)$.

Let $K \subseteq \Phi_0$ be a *consistent and deductively closed* set representing the agent's initial beliefs and let $\Psi \subseteq \Phi_0$ be a set of formulas representing possible inputs for belief change. A *belief change function* based on Ψ and K is a function $\circ : \Psi \rightarrow 2^{\Phi_0}$ (2^{Φ_0} denotes the set of subsets of Φ_0) that associates with every formula $\phi \in \Psi$ a set $K \circ \phi \subseteq \Phi_0$, interpreted as the change in K prompted by the input ϕ. We follow the common practice of writing $K \circ \phi$ instead of $\circ(\phi)$ which has the advantage of making it clear that the belief change function refers to a given, *fixed*, K. If $\Psi \neq \Phi_0$ then $\circ$ is called a *partial* belief change function, while if $\Psi = \Phi_0$ then $\circ$ is called a *full-domain* belief change function.

Definition 2.1. Let $\circ : \Psi \rightarrow 2^{\Phi_0}$ be a partial belief change function (thus $\Psi \neq \Phi_0$) and $\circ' : \Phi_0 \rightarrow 2^{\Phi_0}$ a full-domain belief change function. We say that $\circ'$ is an *extension* of $\circ$ if $\circ'$ coincides with $\circ$ on the domain of $\circ$, that is, if, for every $\phi \in \Psi$, $K \circ' \phi = K \circ \phi$.

2.1 AGM axioms

We consider the notion of belief revision proposed Alchourrón, Gärdenfors and Makinson in Alchourrón et al. (1985).

Definition 2.2. A *belief revision function*, based on the consistent and deductively closed set $K \subset \Phi_0$ (representing the initial beliefs), is a full domain belief change function $* : \Phi_0 \to 2^{\Phi_0}$ that satisfies the following axioms: $\forall \phi, \psi \in \Phi_0$,

$(K*1)$ $K * \phi = Cn(K * \phi)$.

$(K*2)$ $\phi \in K * \phi$.

$(K*3)$ $K * \phi \subseteq K + \phi$.

$(K*4)$ if $\neg\phi \notin K$, then $K \subseteq K * \phi$.

$(K*5)$ $(K*5a)$ If $\neg\phi$ is a tautology, then $K * \phi = \Phi_0$.
$(K*5b)$ If $\neg\phi$ is not a tautology, then $K * \phi \neq \Phi_0$.

$(K*6)$ if $\phi \leftrightarrow \psi$ is a tautology then $K * \phi = K * \psi$.

$(K*7)$ $K * (\phi \wedge \psi) \subseteq (K * \phi) + \psi$.

$(K*8)$ if $\neg\psi \notin K * \phi$, then $(K * \phi) + \psi \subseteq K * (\phi \wedge \psi)$.

$K * \phi$ is interpreted as the revised belief set in response to receiving the input represented by formula ϕ. For a discussion of the above axioms, known as the AGM axioms, see, for example Fermé and Hansson (2018), Gärdenfors (1988).

2.2 Kripke-Lewis semantics

Definition 2.3. A *Kripke-Lewis frame* is a triple $\langle S, \mathcal{B}, f \rangle$ where

1. S is a set of *states*; subsets of S are called *events*.

2. $\mathcal{B} \subseteq S \times S$ is a binary *belief relation* on S which is serial: $\forall s \in S, \exists s' \in S$, such that $s\mathcal{B}s'$ ($s\mathcal{B}s'$ is an alternative notation for $(s, s') \in \mathcal{B}$). We denote by $\mathcal{B}(s)$ the set of states that are reachable from s by $\mathcal{B}$: $\mathcal{B}(s) = \{s' \in S : s\mathcal{B}s'\}$. $\mathcal{B}(s)$ is interpreted as the set of states that initially the agent considers doxastically possible at state s.

3. $f : S \times (2^S \setminus \{\varnothing\}) \to 2^S$ is a *selection function* that associates with every state-event pair (s, E) (with $E \neq \varnothing$) a set of states $f(s, E) \subseteq S$.[1]

   $f(s, E)$ is interpreted as the set of states that are closest (or most similar) to s, conditional on event E.

Adding a valuation to a frame yields a model. Thus a *model* is a tuple $\langle S, \mathcal{B}, f, V\rangle$ where $\langle S, \mathcal{B}, f\rangle$ is a frame and $V : \texttt{At} \to 2^S$ is a valuation that assigns to every atomic formula $p \in \texttt{At}$ the set of states where p is true.

Definition 2.4. Given a model $M = \langle S, \mathcal{B}, f, V\rangle$ define truth of a Boolean formula $\phi \in \Phi_0$ at a state $s \in S$ in model M, denoted by $s \models_M \phi$, as follows:

1. if $p \in \texttt{At}$ then $s \models_M p$ if and only if $s \in V(p)$,
2. $s \models_M \neg\phi$ if and only if $s \not\models_M \phi$,
3. $s \models_M (\phi \vee \psi)$ if and only if $s \models_M \phi$ or $s \models_M \psi$ (or both).

We denote by $||\phi||_M$ the truth set of formula ϕ in model M:

$$||\phi||_M = \{s \in S : s \models_M \phi\}.$$

Given a model $M = \langle S, \mathcal{B}, f, V\rangle$ and a state $s \in S$, let $K_{s,M} = \{\phi \in \Phi_0 : \mathcal{B}(s) \subseteq ||\phi||_M\}$; thus a Boolean formula ϕ belongs to $K_{s,M}$ if and only if at state s the agent believes ϕ (in the sense that ϕ is true at every state that the agent considers doxastically possible at state s). We identify $K_{s,M}$ with the agent's *initial beliefs at state s*. It is shown in Bonanno (2025) that the set $K_{s,M} \subseteq \Phi_0$ so defined is deductively closed and consistent (recall the assumption that the belief relation $\mathcal{B}$ is serial).

Next, given a model $M = \langle S, \mathcal{B}, f, V\rangle$ and a state $s \in S$, let $\Psi_M = \{\phi \in \Phi_0 : ||\phi||_M \neq \varnothing\}$[2] and define the following *partial* belief change function $\circ : \Psi_M \to 2^{\Phi_0}$ based on $K_{s,M}$:

$$\psi \in K_{s,M} \circ \phi \;\text{ if and only if, }\; \forall s' \in \mathcal{B}(s),\; f\left(s', ||\phi||_M\right) \subseteq ||\psi||_M. \tag{RI}$$

[1] Note that $f(s, E)$ is defined only if $E \neq \varnothing$. The reason for this will become clear later. In Bonanno (2025) the selection function was assumed to satisfy the following standard properties:

(3.1) $f(s, E) \neq \varnothing$ (Consistency).

(3.2) $f(s, E) \subseteq E$ (Success).

(3.3) if $s \in E$ then $s \in f(s, E)$ (Weak Centering).

Here we do not require any properties to start with, because we want to highlight the role of each frame property in the characterization of the AGM axioms.

[2] Since, in any given model there are formulas ϕ such that $||\phi||_M = \varnothing$ (at the very least all the contradictions), Ψ_M is a proper subset of Φ_0.

Given the customary interpretation of selection functions in terms of conditionals, (RI) can be interpreted as stating that $\psi \in K_{s,M} \circ \phi$ if and only if at state s the agent believes that "if ϕ is (were) the case then ψ is (would be) the case". This interpretation will be made explicit in the modal logic considered in Section 3. In what follows, when stating an axiom for a belief change function, we implicitly assume that it applies to every formula *in its domain*. For example, the axiom $\phi \in K \circ \phi$ asserts that, for all ϕ in the domain of $\circ$, $\phi \in K \circ \phi$.

Definition 2.5. An axiom for belief change functions is *valid on a frame* F if, for every model based on that frame and for every state s in that model, the partial belief change function defined by (RI) satisfies the axiom. An axiom is *valid on a set of frames* $\mathcal{F}$ if it is valid on every frame $F \in \mathcal{F}$.

2.3 Frame correspondence

A stronger notion than validity is that of frame correspondence. The following definition mimics the notion of frame correspondence in modal logic.

Definition 2.6. We say that an axiom A of belief change functions *is characterized by*, or *corresponds to*, or *characterizes*, a property P of frames if the following is true:

(1) axiom A is valid on the class of frames that satisfy property P, and

(2) if a frame does not satisfy property P then axiom A is not valid on that frame, that is, there is a model based on that frame and a state in that model where the partial belief change function defined by (RI) violates axiom A.

The table of Figure 2 shows, for every AGM axiom, the characterizing property of frames. In Bonanno (2025) it is shown that the class $\mathcal{F}_*$ of frames that satisfy the properties listed in Figure 2 characterizes the set of AGM belief revision functions in the following sense (for simplicity we omit the subscript M that refers to a given model; e.g. we write $||\phi||$ instead of $||\phi||_M$):

(A) For every model based on a frame in $\mathcal{F}_*$ and for every state s in that model, the belief change function $\circ$ (based on $K_s = \{\phi \in \Phi_0 : \mathcal{B}(s) \subseteq ||\phi||\}$ and $\Psi = \{\phi \in \Phi_0 : ||\phi|| \neq \varnothing\}$) defined by (RI) can be extended to a full-domain belief revision function $*$ that satisfies the AGM axioms $(K*1)$-$(K*8)$.

(B) Let $K \subset \Phi_0$ be a consistent and deductively closed set and let $* : \Phi_0 \to 2^{\Phi_0}$ be a belief revision function based on K that satisfies the AGM axioms

$(K*1)$-$(K*8)$. Then there exists a frame in $\mathcal{F}_*$, a model based on that frame and a state s in that model such that (1) $K = K_s = \{\phi \in \Phi_0 : \mathcal{B}(s) \subseteq ||\phi||\}$ and (2) the partial belief change function $\circ$ (based on K_s and and $\Psi = \{\phi \in \Phi_0 : ||\phi|| \neq \varnothing\}$) defined by (RI) is such that $K_s \circ \phi = K_s * \phi$ for every consistent formula ϕ.

The characterization for $(K * 4)$, $(K * 7)$ and $(K * 8)$ is proved in Propositions 9, 3 and 10, respectively, in Bonanno (2025). For the remaining axioms we note the following.

- Validity of $(K*1)$ on all frames is a consequence of Part (B) of Lemma 1 in Bonanno (2025).

- Sufficiency for $(K * 2)$ is proved in Item 2 of Proposition 1 in Bonanno (2025). For necessity, consider a frame that violates Property $(P * 2)$, that is, there exist $s \in S$, $\varnothing \neq E \subseteq S$ and $s' \in \mathcal{B}(s)$ such that $f(s', E) \nsubseteq E$. Construct a model where, for some atomic sentence p, $||p|| = E$. Then, since $f(s', ||p||) \nsubseteq ||p||$ and $s' \in \mathcal{B}(s)$, $p \notin K_s * p$, yielding a violation of $(K*2)$.

- The proof for $(K * 3)$ is as follows.
  (A) Sufficiency. Consider a model based on a frame that satisfies Property $(P * 3)$. Let $\phi \in \Phi_0$ be such that $||\phi|| \neq \varnothing$ and let $s \in S$ and $\psi \in \Phi_0$ be such that $\psi \in K_s * \phi$, that is, for every $s' \in \mathcal{B}(s)$, $f(s', ||\phi||) \subseteq ||\psi||$, so that $\bigcup_{x \in \mathcal{B}(s)} f(x, ||\phi||) \subseteq ||\psi||$. Fix an arbitrary $s' \in \mathcal{B}(s)$. If $s' \notin ||\phi||$ (that is, $s' \models \neg\phi$) then $s' \models (\phi \to \psi)$. If $s' \in ||\phi||$, then by Property $(P * 3)$, $s' \in \bigcup_{x \in \mathcal{B}(s)} f(x, ||\phi||)$ and hence $s' \in ||\psi||$; thus $s' \models (\phi \to \psi)$. It follows that $(\phi \to \psi) \in K_s$ and thus $\psi \in K_s + \phi$ (since, by Lemma 2 in Bonanno (2025), K_s is deductively closed) .
  (B) Necessity. Consider a frame that violates $(P * 3)$, that is, there exist $s \in S$, $s' \in \mathcal{B}(s)$ and $E \subseteq S$ such that $s' \in E$ and $s' \notin \bigcup_{x \in \mathcal{B}(s)} f(x, E)$. Construct a model based on this frame where, for some atomic sentences p and q, $||p|| = E$ and $||q|| = \bigcup_{x \in \mathcal{B}(s)} f(x, E) = \bigcup_{x \in \mathcal{B}(s)} f(x, ||p||)$. Then, for every $x \in \mathcal{B}(s)$, $f(x, ||p||) \subseteq ||q||$ so that $q \in K_s * p$. Since $s' \in ||p||$ and $s' \notin ||q||$, $s' \not\models (p \to q)$ from which it follows (since $s' \in \mathcal{B}(s)$) that $(p \to q) \notin K_s$ and thus $q \notin K_s + p$. Hence $K_s * p \nsubseteq K_s + p$.

AGM axiom	Frame property
$(K*1)$ $K*\phi = Cn(K*\phi)$	No additional property
$(K*2)$ $\phi \in K*\phi$	$(P*2)$ $\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \forall s' \in \mathcal{B}(s)$, $f(s', E) \subseteq E$
$(K*3)$ $K*\phi \subseteq K+\phi$	$(P*3)$ $\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \forall s' \in \mathcal{B}(s)$, if $s' \in E$ then $s' \in \bigcup_{x \in \mathcal{B}(s)} f(x, E)$
$(K*4)$ if $\neg\phi \notin K*\phi$ then $K \subseteq K*\phi$	$(P*4)$ $\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}$, if $\mathcal{B}(s) \cap E \neq \varnothing$ then, $\forall s' \in \mathcal{B}(s), f(s', E) \subseteq \mathcal{B}(s) \cap E$
$(K*5b)$ If $\neg\phi$ is not a tautology then $K*\phi \neq \Phi_0$	$(P*5)$ $\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \exists s' \in \mathcal{B}(s)$ such that $f(s', E) \neq \varnothing$
$(K*6)$ if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$	No additional property
$(K*7)$ $K*(\phi \wedge \psi) \subseteq (K*\phi)+\psi$	$(P*7)$ $\forall s \in S, \forall E, F, G \in 2^S$ with $E \cap F \neq \varnothing$, if, $\forall s' \in \mathcal{B}(s)$, $f(s', E \cap F) \subseteq G$, then, $\forall s' \in \mathcal{B}(s)$, $f(s', E) \cap F \subseteq G$
$(K*8)$ If $\neg\psi \notin K*\phi$, then $(K*\phi)+\psi \subseteq K*(\phi \wedge \psi)$	$(P*8)$ $\forall s \in S, \forall E, F \in 2^S \setminus \{\varnothing\}$ if $\exists \hat{s} \in \mathcal{B}(s)$ such that $f(\hat{s}, E) \cap F \neq \varnothing$, then, $\forall s' \in \mathcal{B}(s)$, $f(s', E \cap F) \subseteq \bigcup_{x \in \mathcal{B}(s)} (f(x, E) \cap F)$.

Figure 1: Semantic characterization of the AGM axioms.

- For axiom $(K * 5b)$ the proof is as follows.
  Sufficiency. Consider a model based on a frame that satisfies Property $(P * 5)$. Let $\phi \in \Phi_0$ be such that $||\phi|| \neq \emptyset$ (thus ϕ is consistent) and fix an arbitrary $s \in S$. By $(P * 5)$ there exists an $s' \in \mathcal{B}(s)$ such that $f(s', ||\phi||) \neq \emptyset$. Let $p \in \mathtt{At}$ be an atomic sentence . Than $f(s', ||\phi||) \nsubseteq ||p \wedge \neg p|| = \emptyset$. Hence, since $s' \in \mathcal{B}(s)$, $(p \wedge \neg p) \notin K_s * \phi$ so that $K_s * \phi \neq \Phi_0$.

  Necessity, consider a frame that violates $(P * 5)$, that is, there exist $s \in S$ and $\emptyset \neq E \subseteq S$ such that, $\forall s' \in \mathcal{B}(s)$, $f(s', E) = \emptyset$. Construct a model based on this frame where, for some atomic sentence p, $||p|| = E \neq \emptyset$. Fix an arbitrary $\phi \in \Phi_0$. Then, $\forall s' \in \mathcal{B}(s)$, $f(s', ||p||) = \emptyset \subseteq ||\phi||$, so that, by (RI), $\phi \in K_s * p$. Thus $K_s * p = \Phi_0$.

- Validity of $(K * 6)$ on all frames is a consequence of the fact that if $\phi \leftrightarrow \psi$ is a tautology, then, in every model, $||\phi|| = ||\psi||$.

3 A modal logic for belief revision

We now introduce a simple language with three modal operators: a unimodal belief operator B, a bimodal conditional operator $>$ and the unimodal global operator $\Box$. The interpretation of $B\phi$ is "the agent believes ϕ", the interpretation of $\phi > \psi$ is "if ϕ is (or were) the case then ψ is (or would be) the case" and the interpretation of $\Box\phi$ is "ϕ is necessarily true".
The set Φ of formulas in the language is defined as follows.

- Let Φ_0 be the set of Boolean formulas defined in Section 2.
- Let $\Phi_>$ be the set of formulas of the form $\phi > \psi$ with $\phi, \psi \in \Phi_0$.[3]
- Let Φ_1 be the set of Boolean combinations of formulas in $\Phi_0 \cup \Phi_>$.
- Let Φ_B be the set of formulas of the form $B\phi$ with $\phi \in \Phi_1$.[4]
- Let $\Phi_\Box$ be the set of formulas of the form $\Box\phi$ with $\phi \in \Phi_0$.[5]
- Finally, let Φ be the set of Boolean combinations of formulas in $\Phi_1 \cup \Phi_B \cup \Phi_\Box$.

[3]Thus, for example, $\phi > (\psi > \chi)$ is *not* a formula in $\Phi_>$.
[4]Thus, for example, $B(B\phi \to \phi)$ and $B\Box\phi$ are *not* formulas in Φ_B.
[5]Thus, for example, $\Box(\phi > \psi)$ is *not* a formula in $\Phi_\Box$.

3.1 Kripke-Lewis semantics

As semantics for this modal logic we take the Kripke-Lewis frames of Definition 2.3. A model based on a frame is obtained, as before, by adding a valuation $V : \texttt{At} \to 2^S$. The following definition expands Definition 2.4 by adding validation rules for formulas of the form $\Box\phi$, $\phi > \psi$ and $B\phi$.

Definition 3.1. Truth of a formula ϕ at state s in model M (denoted by $s \models_M \phi$) is defined as follows:

1. if $p \in \texttt{At}$ then $s \models_M p$ if and only if $s \in V(p)$.
2. For $\phi \in \Phi$, $s \models_M \neg\phi$ if and only if $s \not\models_M \phi$.
3. $\phi \in \Phi$, $s \models_M (\phi \vee \psi)$ if and only if $s \models_M \phi$ or $s \models_M \psi$ (or both).
4. For $\phi \in \Phi_0$, $s \models_M \Box\phi$ if and only if, $\forall s' \in S$, $s' \models_M \phi$ (thus $s \models_M \neg\Box\neg\phi$ if and only if, for some $s' \in S$, $s' \models_M \phi$).
5. For $\phi, \psi \in \Phi_0$, $s \models_M (\phi > \psi)$ if and only if, either
   (a) $s \models_M \Box\neg\phi$ (that is, $||\phi||_M = \varnothing$), or
   (b) $s \models_M \neg\Box\neg\phi$ (that is, $||\phi||_M \neq \varnothing$) and, for every $s' \in f(s, ||\phi||_M)$, $s' \models_M \psi$ (that is, $f(s, ||\phi||_M) \subseteq ||\psi||_M$).[6]
6. For $\phi \in \Phi_1$, $s \models_M B\phi$ if and only if, $\forall s' \in \mathcal{B}(s)$, $s' \models_M \phi$ (that is, $\mathcal{B}(s) \subseteq ||\phi||_M$).

The definition of validity is as in the previous section.

Definition 3.2. A formula $\phi \in \Phi$ is *valid on a frame F* if, for every model M based on that frame and for every state s in that model, $s \models_M \phi$. A formula $\phi \in \Phi$ is *valid on a set of frames* $\mathcal{F}$ if it is valid on every frame $F \in \mathcal{F}$.

3.2 Frame correspondence

The definition of frame correspondence is the standard definition in modal logic.

Definition 3.3. A formula $\phi \in \Phi$ *is characterized by*, or *corresponds to*, or *characterizes*, a property P of frames if the following is true:

(1) ϕ is valid on the class of frames that satisfy property P, and

[6]Recall that, by definition of frame, $f(s, E)$ is defined only if $E \neq \varnothing$.

(2) if a frame does not satisfy property P then ϕ is not valid on that frame.

The table in Figure 2 shows, for every property of frames considered in Figure 1, the modal formula that corresponds to it. The proofs of the characterizations results are given in the Appendix.

	Frame property	Modal formula (for $\phi, \psi, \chi \in \Phi_0$)
$(P*2)$	$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \forall s' \in \mathcal{B}(s)$, $f(s', E) \subseteq E$	$B(\phi > \phi)$
$(P*3)$	$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \forall s' \in \mathcal{B}(s)$, if $s' \in E$ then $s' \in \bigcup_{x \in \mathcal{B}(s)} f(x, E)$	$\big(\neg\Box\neg\phi \wedge B(\phi > \psi\big) \rightarrow B(\phi \rightarrow \psi)$
$(P*4)$	$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}$, if $\mathcal{B}(s) \cap E \neq \varnothing$ then, $\forall s' \in \mathcal{B}(s), f(s', E) \subseteq \mathcal{B}(s) \cap E$	$\big(\neg B\neg\phi \wedge B(\phi \rightarrow \psi)\big) \rightarrow B(\phi > \psi)$
$(P*5)$	$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \exists s' \in \mathcal{B}(s)$, such that $f(s', E) \neq \varnothing$	$\big(\neg\Box\neg\phi \wedge B(\phi > \psi)\big) \rightarrow \neg B(\phi > \neg\psi)$
$(P*7)$	$\forall s \in S, \forall E, F, G \in 2^S$ with $E \cap F \neq \varnothing$, if, $\forall s' \in \mathcal{B}(s),\ f(s', E \cap F) \subseteq G$, then, $\forall s' \in \mathcal{B}(s),\ f(s', E) \cap F \subseteq G$	$\big(\neg\Box\neg(\phi \wedge \psi)\ \wedge\ B((\phi \wedge \psi) > \chi)\big)$ $\rightarrow B\big(\phi > (\psi \rightarrow \chi)\big)$
$(P*8)$	$\forall s \in S, \forall E, F \in 2^S \setminus \{\varnothing\}$ if $\exists \hat{s} \in \mathcal{B}(s)$ such that $f(\hat{s}, E) \cap F \neq \varnothing$ then, $\forall s' \in \mathcal{B}(s)$, $f(s', E \cap F) \subseteq \bigcup_{x \in \mathcal{B}(s)} (f(x, E) \cap F)$.	$\neg B(\phi > \neg\psi) \wedge B(\phi > (\psi \rightarrow \chi))$ $\rightarrow B\big((\phi \wedge \psi) > (\psi \wedge \chi)\big)$

Figure 2: Modal characterization of the frame properties of Figure 1.

Putting together Figures 1 and 2, we have a modal-logic characterization of AGM axioms $(K*2)$, $(K*3)$, $(K*4)$, $(K*5b)$, $(K*7)$ and $(K*8)$. For instance, the modal axiom $B(\phi > \phi)$ corresponds to AGM axiom $(K*2)$ $(\phi \in K*\phi)$ in the following sense: $(K*2)$ is characterized by frame Property $(P*2)$ which, in turn, characterizes $B(\phi > \phi)$; in other words, $B(\phi > \phi)$ can be viewed as a translation into our modal logic of AGM axiom $(K*2)$.

To complete the analysis we need to account for AGM axioms $(K*1)$, $(K*5a)$ and $(K*6)$.

- The modal counterpart of $(K*1)$ $(K*\phi = Cn(K*\phi))$ can be taken to be the following axiom: for $\phi, \psi, \chi \in \Phi_0$,

$$\big(B(\phi > \psi) \wedge B(\phi > (\psi \to \chi))\big) \to B(\phi > \chi)$$

  which is valid on all the frames considered in this paper.[7].

- AGM axiom $(K*5a)$ (if ϕ is a contradiction, then $K*\phi = \Phi_0$) can be captured in our logic by the following rule of inference: if $\neg\phi$ is a tautology (a theorem of propositional calculus) then $B(\phi > \psi)$ is a theorem.

- AGM axiom $(K*6)$ (if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$) can be captured in our logic by the following rule of inference: if $\phi \leftrightarrow \psi$ is a tautology then $B(\phi > \chi) \leftrightarrow B(\psi > \chi)$ is a theorem (for $\phi, \psi, \chi \in \Phi_0$), which is clearly validity preserving in every model, since if $\phi \leftrightarrow \psi$ is a tautology then $||\phi|| = ||\psi||$.

The table in Figure 3 synthesizes Figures 1 and 2 by showing the correspondence between each AGM axiom and its modal counterpart.

4 Related literature

A connection between conditional logic and AGM belief revision has been pointed out in several contributions, in particular Giordano et al. (1998; 2001; 2005), Günther and Sisti (2022). Although there are some similarities between our approach and those contributions, there are also some important differences. For a detailed discussion see Bonanno (2025).

[7]Fix an arbitrary model, an arbitrary state s and arbitrary $\phi, \psi, \chi \in \Phi_0$ and suppose that $s \models B(\phi > \psi) \wedge B(\phi > (\psi \to \chi))$. If $||\phi|| = \varnothing$ then, by (a) of Item 5 of Definition 3.1, $\forall s' \in \mathcal{B}(s)$, $s' \models \phi > \chi$ and thus $s \models B(\phi > \chi)$. If $||\phi|| \neq \varnothing$, then, $\forall s' \in \mathcal{B}(s)$, $f(s', ||\phi||) \subseteq ||\psi||$ [because $s \models B(\phi > \psi)$] and $f(s', ||\phi||) \subseteq ||\psi \to \chi|| = (S \setminus ||\psi||) \cup ||\chi||$ [because $s \models B(\phi > (\psi \to \chi))$]. Since $||\psi|| \cap \big((S \setminus ||\psi||) \cup ||\chi||\big) = ||\psi|| \cap ||\chi|| \subseteq ||\chi||$, we have that, $\forall s' \in \mathcal{B}(s)$, $f(s', ||\phi||) \subseteq ||\chi||$, that is, $s \models B(\phi > \chi)$.

AGM axiom	Modal axiom / Rule of Inference (for $\phi, \psi, \chi \in \Phi_0$)
$(K*1)$ $K*\phi = Cn(K*\phi)$	$B(\phi > \psi) \wedge B(\phi > (\psi \rightarrow \chi))$ $\rightarrow B(\phi > \chi)$
$(K*2)$ $\phi \in K*\phi$	$B(\phi > \phi)$
$(K*3)$ $K*\phi \subseteq K+\phi$	$\big(\neg\Box\neg\phi \wedge B(\phi > \psi\big) \rightarrow B(\phi \rightarrow \psi)$
$(K*4)$ if $\neg\phi \notin K*\phi$ then $K \subseteq K*\phi$	$\big(\neg B\neg\phi \wedge B(\phi \rightarrow \psi)\big) \rightarrow B(\phi > \psi)$
$(K*5a)$ If $\neg\phi$ is a tautology, then $K*\phi = \Phi_0$	Rule of inference: if $\neg\phi$ is a tautology then $B(\phi > \psi)$ is a theorem
$(K*5b)$ If $\neg\phi$ is not a tautology then $K*\phi \neq \Phi_0$	$\big(\neg\Box\neg\phi \wedge B(\phi > \psi)\big) \rightarrow \neg B(\phi > \neg\psi)$
$(K*6)$ if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$	Rule of inference: if $\phi \leftrightarrow \psi$ is a tautology then $B(\phi > \chi) \leftrightarrow B(\psi > \chi)$ is a theorem
$(K*7)$ $K*(\phi \wedge \psi) \subseteq (K*\phi)+\psi$	$\big(\neg\Box\neg(\phi \wedge \psi) \ \wedge \ B((\phi \wedge \psi) > \chi)\big)$ $\rightarrow B\big(\phi > (\psi \rightarrow \chi)\big)$
$(K*8)$ If $\neg\psi \notin K*\phi$, then $(K*\phi)+\psi \subseteq K*(\phi \wedge \psi)$	$\neg B(\phi > \neg\psi) \wedge B(\phi > (\psi \rightarrow \chi))$ $\rightarrow \ B\big((\phi \wedge \psi) > (\psi \wedge \chi)\big)$

Figure 3: The correspondence between AGM axioms and their modal counterparts.

5 Appendix

Proposition 1. *The modal axiom*

$$B(\phi > \phi) \qquad (\phi \in \Phi_0) \tag{A2}$$

is characterized by the following property of frames:

$$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \forall s' \in \mathcal{B}(s),\ f(s', E) \subseteq E. \tag{$P*2$}$$

Proof. First we show that Axiom (A2) is valid on every frame that satisfies Property ($P*2$). Fix a model based on a frame that satisfies Property ($P*2$), arbitrary $s \in S$, $\phi \in \Phi_0$ and $s' \in \mathcal{B}(s)$; we need to show that $s' \models (\phi > \phi)$. If $\|\phi\| = \varnothing$, then, by (a) of Item 5 of Definition 3.1, $s' \models (\phi > \phi)$. If $\|\phi\| \neq \varnothing$ then, by Property ($P*2$), $f(s', \|\phi\|) \subseteq \|\phi\|$ and thus, by (b) of Item 5 of Definition 3.1, $s' \models (\phi > \phi)$.

Next we show that Axiom (A2) is not valid on a frame that violates Property ($P*2$). Fix such a frame, that is, a frame where there exist $s \in S$, $\varnothing \neq E \subseteq S$ and $s' \in \mathcal{B}(s)$ such that $f(s', E) \nsubseteq E$. Let $p \in \texttt{At}$ be an atomic formula and construct a model based on this frame where $\|p\| = E \neq \varnothing$. Then, since $f(s', \|p\|) \nsubseteq \|p\|$, $s' \not\models (p > p)$ and thus $s \not\models B(p > p)$ so that axiom (A2) is not valid on the given frame. □

Proposition 2. *The modal axiom*

$$\big(\neg\Box\neg\phi \wedge B(\phi > \psi)\big) \rightarrow B(\phi \rightarrow \psi) \qquad (\phi, \psi \in \Phi_0) \tag{A3}$$

is characterized by the following property of frames:

$$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \forall s' \in \mathcal{B}(s), \textit{ if } s' \in E \textit{ then } s' \in \bigcup_{x \in \mathcal{B}(s)} f(x, E). \tag{$P*3$}$$

Proof. First we show that Axiom (A3) is valid on every frame that satisfies Property ($P*3$). Fix a model based on a frame that satisfies Property ($P*3$), arbitrary $s \in S$ and $\phi, \psi \in \Phi_0$ and suppose that $s \models \neg\Box\neg\phi \wedge B(\phi > \psi)$. We need to show that $s \models B(\phi \rightarrow \psi)$. Since $s \models \neg\Box\neg\phi$, $\|\phi\| \neq \varnothing$. Thus, since $s \models B(\phi > \psi)$, $\forall s' \in \mathcal{B}(s)$, $f(s', \|\phi\|) \subseteq \|\psi\|$. It follows that

$$\bigcup_{x \in \mathcal{B}(s)} f(x, \|\phi\|) \subseteq \|\psi\|. \tag{1}$$

Fix an arbitrary $s' \in \mathcal{B}(s)$. Is $s' \notin ||\phi||$ then $s' \models \phi \to \psi$. If $s' \in ||\phi||$, then by Property ($P*3$), $s' \in \bigcup_{x \in \mathcal{B}(s)} f(x, ||\phi||)$ and thus, by (1), $s' \in ||\psi||$, so that $s' \models \phi \to \psi$. Hence $s \models B(\phi \to \psi)$.

Next we show that Axiom (A3) is not valid on a frame that violates Property ($P*3$). Fix such a frame, that is, a frame where there exist $s \in S$, $\varnothing \neq E \subseteq S$ and $s' \in \mathcal{B}(s)$ such that $s' \in E$ but $s' \notin \bigcup_{x \in \mathcal{B}(s)} f(x, E)$. Let $p, q \in \mathtt{At}$ and construct a model based on this frame where $||p|| = E$ and $||q|| = \bigcup_{x \in \mathcal{B}(s)} f(x, E) = \bigcup_{x \in \mathcal{B}(s)} f(x, ||p||)$. Then $s' \models p$ but $s' \not\models q$, so that $s' \not\models p \to q$, from which it follows that $s \not\models B(p \to q)$, that is, $s \models \neg B(p \to q)$. To obtain a violation of (A3) it only remains to show that $s \models \neg\Box\neg p \wedge B(p > q)$. That $s \models \neg\Box\neg p$ is a consequence of the fact that, by hypothesis, $\varnothing \neq E = ||p||$. Furthermore, since $||p|| \neq \varnothing$ and $||q|| = \bigcup_{x \in \mathcal{B}(s)} f(x, ||p||)$, for every $y \in \mathcal{B}(s)$, $f(y, ||p||) \subseteq ||q||$ and therefore $y \models (p > q)$, so that $s \models B(p > q)$. □

Proposition 3. *The modal axiom*

$$\big(\neg B\neg\phi \wedge B(\phi \to \psi)\big) \to B(\phi > \psi) \qquad (\phi, \psi \in \Phi_0) \tag{A4}$$

is characterized by the following property of frames:

$$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \text{ if } \mathcal{B}(s) \cap E \neq \varnothing \text{ then}, \forall s' \in \mathcal{B}(s), f(s', E) \subseteq \mathcal{B}(s) \cap E. \quad (P*4)$$

Proof. First we show that Axiom (A4) is valid on every frame that satisfies Property ($P*4$). Fix a model based on a frame that satisfies Property ($P*4$), an arbitrary $s \in S$ and arbitrary $\phi, \psi \in \Phi_0$ and suppose that $s \models \neg B\neg\phi \wedge B(\phi \to \psi)$. We need to show that $s \models B(\phi > \psi)$. Since $s \models \neg B\neg\phi$, there exists an $s' \in \mathcal{B}(s)$ such that $s' \models \phi$, that is, $\mathcal{B}(s) \cap ||\phi|| \neq \varnothing$. Thus, by Property ($P*4$),

$$\forall s' \in \mathcal{B}(s),\ f(s', ||\phi||) \subseteq \mathcal{B}(s) \cap ||\phi||. \tag{2}$$

Since $s \models B(\phi \to \psi)$, $\mathcal{B}(s) \subseteq ||\phi \to \psi|| = ||\neg\phi|| \cup ||\psi||$. Hence

$$\mathcal{B}(s) \cap ||\phi|| \subseteq \big(||\neg\phi|| \cup ||\psi||\big) \cap ||\phi|| = ||\phi|| \cap ||\psi|| \subseteq ||\psi||. \tag{3}$$

It follows from (2) and (3) that, $\forall s' \in \mathcal{B}(s)$, $f(s', ||\phi||) \subseteq ||\psi||$, that is, $s' \models (\phi > \psi)$ and thus $s \models B(\phi > \psi)$.

Next we show that Axiom (A4) is not valid on a frame that violates Property ($P*4$). Fix such a frame, that is, a frame where there exist $s \in S$, $\hat{s} \in \mathcal{B}(s)$ and $E \in 2^S \setminus \{\varnothing\}$ such that $\mathcal{B}(s) \cap E \neq \varnothing$ and $f(\hat{s}, E) \nsubseteq \mathcal{B}(s) \cap E$. Let $p, q \in \mathtt{At}$ and construct a model based on this frame where $||p|| = E$ and $||q|| = \mathcal{B}(s) \cap E$. Then $f(\hat{s}, ||p||) \nsubseteq ||q||$, that is, $\hat{s} \not\models p > q$ and thus $s \not\models B(p > q)$, that is,

$$s \models \neg B(p > q). \tag{4}$$

Since $\mathcal{B}(s) \cap ||p|| \neq \varnothing$,

$$s \models \neg B \neg p. \tag{5}$$

Finally, since $\mathcal{B}(s) = \big(\mathcal{B}(s) \cap (S \setminus ||p||)\big) \bigcup \big(\mathcal{B}(s) \cap ||p||\big) \subseteq (S \setminus ||p||) \cup \big(\mathcal{B}(s) \cap ||p||\big) = ||\neg p|| \cup ||q|| = ||p \to q||$,

$$s \models B(p \to q). \tag{6}$$

From (4), (5) and (6) we get a violation of Axiom (A4). $\square$

Proposition 4. *The modal axiom*

$$\big(\neg\Box\neg\phi \wedge B(\phi > \psi)\big) \to \neg B(\phi > \neg\psi) \qquad (\phi, \psi \in \Phi_0) \tag{A5}$$

is characterized by the following property of frames:

$$\forall s \in S, \forall E \in 2^S \setminus \{\varnothing\}, \exists s' \in \mathcal{B}(s), \text{ such that } f(s', E) \neq \varnothing. \tag{$P*5$}$$

Proof. First we show that Axiom (A5) is valid on every frame that satisfies Property ($P*5$). Fix a model based on a frame that satisfies Property ($P*5$), an arbitrary $s \in S$ and arbitrary $\phi, \psi \in \Phi_0$ and suppose that $s \models \neg\Box\neg\phi \wedge B(\phi > \psi)$. Then

$$\begin{array}{lll} (a) & ||\phi|| \neq \varnothing & (\text{since } s \models \neg\Box\neg\phi) \\ (b) & \forall s' \in \mathcal{B}(s),\ s' \models \phi > \psi & (\text{since } s \models B(\phi > \psi)). \end{array} \tag{7}$$

Thus, by (a) of (7) and Property ($P*5$), that there exists an $s' \in \mathcal{B}(s)$, such that $f(s', ||\phi||) \neq \varnothing$. By (b) of (7), $f(s', ||\phi||) \subseteq ||\psi||$, so that (since $f(s', ||\phi||) \neq \varnothing$) $f(s', ||\phi||) \nsubseteq \big(S \setminus ||\psi||\big) = ||\neg\psi||$, that is, $s' \not\models (\phi > \neg\psi)$ and, therefore, $s \not\models B(\phi > \neg\psi)$, that is, $s \models \neg B(\phi > \neg\psi)$.

Next we show that Axiom (A5) is not valid on a frame that violates Property ($P*5$). Fix such a frame, that is, a frame where there exist $s \in S$ and $\varnothing \neq E \subseteq S$ such that, $\forall s' \in \mathcal{B}(s)$, $f(s', E) = \varnothing$. Construct a model based on this frame where, for some $p, q \in \texttt{At}$, $||p|| = E$ and $||q|| = \varnothing$. Then (since $\varnothing \neq E = ||p||$) $s \models \neg\Box\neg p$. Furthermore, $\forall s' \in \mathcal{B}(s)$, $f(s', ||p||) \subseteq ||q||$ and $f(s', ||p||) \subseteq ||\neg q||$. Thus, $\forall s' \in \mathcal{B}(s)$, $s' \models p > q$ and $s' \models p > \neg q$, so that $s \models B(p > q)$ and $s \models B(p > \neg q)$ yielding a violation of (A5). □

Proposition 5. *The modal axiom*

$$\big(\neg\Box\neg(\phi \wedge \psi) \wedge B((\phi \wedge \psi) > \chi)\big) \rightarrow B\big(\phi > (\psi \rightarrow \chi)\big) \qquad (\phi, \psi, \chi \in \Phi_0) \tag{A7}$$

is characterized by the following property of frames:

$$\begin{array}{l} \forall s \in S, \forall E, F, G \in 2^S \text{ with } E \cap F \neq \varnothing, \\ \text{if, } \forall s' \in \mathcal{B}(s),\ f(s', E \cap F) \subseteq G, \text{ then, } \forall s' \in \mathcal{B}(s),\ f(s', E) \cap F \subseteq G. \end{array} \tag{$P*7$}$$

Proof. First we show that Axiom (A7) is valid on every frame that satisfies Property ($P*7$). Fix a model based on a frame that satisfies Property ($P*7$), arbitrary $s \in S$ and $\phi, \psi, \chi \in \Phi_0$ and suppose that $s \models \neg\Box\neg(\phi\wedge\psi) \wedge B((\phi\wedge\psi) > \chi)$. Since $s \models \neg\Box\neg(\phi \wedge \psi)$, $||\phi \wedge \psi|| \neq \varnothing$ (note that $||\phi \wedge \psi|| = ||\phi|| \cap ||\psi||$, so that we also have that $||\phi|| \neq \varnothing$ and $||\psi|| \neq \varnothing$). Thus, since $s \models B((\phi \wedge \psi) > \chi)$,

$$\forall s' \in \mathcal{B}(s),\ s' \models (\phi \wedge \psi) > \chi, \text{ that is, } f(s', ||\phi|| \cap ||\psi||) \subseteq ||\chi||. \tag{8}$$

We need to show that $s \models B\big(\phi > (\psi \rightarrow \chi)\big)$, that is, that, for all $s' \in \mathcal{B}(s)$, $s' \models \phi > (\psi \rightarrow \chi)$, that is, $f(s', ||\phi||) \subseteq ||\psi \rightarrow \chi||$. By (8) and Property ($P*7$) (with $E = ||\phi||$, $F = ||\psi||$ and $G = ||\chi||$) ,

$$\forall s' \in \mathcal{B}(s), f(s', ||\phi||) \cap ||\psi|| \subseteq ||\chi|| \subseteq \big(S \setminus ||\psi||\big) \cup ||\chi|| \tag{9}$$

which is equivalent to the desired property, since $\big(S \setminus ||\psi||\big) \cup ||\chi|| = ||\psi \rightarrow \chi||$.

Next we show that Axiom (A7) is not valid on a frame that violates Property ($P*7$). Fix such a frame, that is, a frame where there exist $s \in S$ and $E, F, G \in 2^S$ with $E \cap F \neq \varnothing$ such that

$$\begin{array}{l} \forall s' \in \mathcal{B}(s),\ f(s', E \cap F) \subseteq G, \text{ but} \\ \exists \hat{s} \in \mathcal{B}(s) \text{ such that } f(\hat{s}, E) \cap F \nsubseteq G. \end{array} \tag{10}$$

Let $p, q, r \in \texttt{At}$ and construct a model based on this frame where $||p|| = E$, $||q|| = F$ and $||r|| = G$. Since $E \cap F \neq \varnothing$,

$$s \models \neg\Box\neg(p \wedge q) \tag{11}$$

Furthermore, by (10), $\forall s' \in \mathcal{B}(s)$, $f(s', ||p|| \cap ||q||) \subseteq ||r||$, that is (since $||p|| \cap ||q|| = ||p \wedge q||$), $s' \models (p \wedge q) > r$. Hence

$$s \models B\big((p \wedge q) > r\big). \tag{12}$$

By (10), $f(\hat{s}, ||p||) \cap ||q|| \nsubseteq ||r||$, which is equivalent to $f(\hat{s}, ||p||) \nsubseteq \big(S \setminus ||q||\big) \cup ||r|| = ||q \to r||$, so that $\hat{s} \not\models p > (q \to r)$, from which it follows (since $\hat{s} \in \mathcal{B}(s)$) that

$$s \models \neg B\big(p > (q \to r)\big). \tag{13}$$

Thus, by (11), (12) and (13), axiom (A7) is not valid on the given frame. □

Proposition 6. *The modal axiom*

$$\big(\neg B(\phi > \neg\psi) \wedge B(\phi > (\psi \to \chi)\big) \to B\big((\phi \wedge \psi) > (\psi \wedge \chi)\big) \quad (\phi, \psi, \chi \in \Phi_0) \tag{A8}$$

is characterized by the following property of frames:

$$\begin{aligned} &\forall s \in S, \forall E, F \in 2^S \setminus \{\varnothing\}, \\ &\textit{if } \exists \hat{s} \in \mathcal{B}(s) \textit{ such that } f(\hat{s}, E) \cap F \neq \varnothing \textit{ then}, \\ &\forall s' \in \mathcal{B}(s), f(s', E \cap F) \subseteq \bigcup_{x \in \mathcal{B}(s)} (f(x, E) \cap F). \end{aligned} \tag{$P*8$}$$

Proof. First we show that Axiom (A8) is valid on every frame that satisfies Property ($P*8$). Fix a model based on a frame that satisfies Property ($P*8$), an arbitrary $s \in S$ and arbitrary $\phi, \psi, \chi \in \Phi_0$ and suppose that $s \models \neg B(\phi > \neg\psi) \wedge B(\phi > (\psi \to \chi))$. We need to show that $s \models B\big((\phi \wedge \psi) > (\psi \wedge \chi)\big)$, that is, that, $\forall s' \in \mathcal{B}(s), s' \models (\phi \wedge \psi) > (\psi \wedge \chi)$. Since $s \models \neg B(\phi > \neg\psi)$, there exists an $\hat{s} \in \mathcal{B}(s)$ such that $\hat{s} \not\models \phi > \neg\psi$, that is, $||\phi|| \neq \varnothing$ and $f(\hat{s}, ||\phi||) \nsubseteq ||\neg\psi||$, that is, $f(\hat{s}, ||\phi||) \cap ||\psi|| \neq \varnothing$, so that, by Property ($P*8$) (with $E = ||\phi||$ and $F = ||\psi||$ and noting that $||\phi|| \cap ||\psi|| = ||\phi \wedge \psi||$),

$$\forall s' \in \mathcal{B}(s),\ f(s', ||\phi \wedge \psi||) \subseteq \bigcup_{x \in \mathcal{B}(s)} \big(f(x, ||\phi||) \cap ||\psi||\big) \subseteq ||\psi||. \tag{14}$$

Since $s \models B\big(\phi > (\psi \to \chi)\big)$, $\forall s' \in \mathcal{B}(s), f(s', \|\phi\|) \subseteq \|\psi \to \chi\|$. It follows from this and (14) that

$$\begin{aligned}&\forall s' \in \mathcal{B}(s), f(s', \|\phi \wedge \psi\|) \subseteq \|\psi \to \chi\| \cap \|\psi\| = \|\psi \wedge \chi\|,\\ &\text{that is, } s' \models (\phi \wedge \psi) > (\psi \wedge \chi).\end{aligned} \tag{15}$$

Next we show that Axiom (A8) is not valid on a frame that violates Property ($P*8$). Fix such a frame, that is, a frame where there exist $s \in S$, $\hat{s}, \tilde{s} \in \mathcal{B}(s)$ and $E, F \in 2^S \setminus \{\varnothing\}$ such that

$$f(\hat{s}, E) \cap F \neq \varnothing \quad \text{and} \quad f(\tilde{s}, E \cap F) \nsubseteq \bigcup_{x \in \mathcal{B}(s)} (f(x, E) \cap F). \tag{16}$$

Let $p, q, r \in \mathtt{At}$ and construct a model based on this frame where $\|p\| = E$, $\|q\| = F$ and $\|r\| = \bigcup_{s' \in \mathcal{B}(s)} f(s', E)$. Then, for all $s' \in \mathcal{B}(s)$, $f(s', \|p\|) \subseteq \|r\| \subseteq \|\neg q\| \cup \|r\| = \|q \to r\|$, that is, $s' \models p > (q \to r)$ and thus

$$s \models B\big(p > (q \to r)\big). \tag{17}$$

Since, by hypothesis, $f(\hat{s}, E) \cap F \neq \varnothing$ (that is, $f(\hat{s}, \|p\|) \nsubseteq \|\neg q\|$, which implies that $\hat{s} \not\models (p > \neg q)$, $s \not\models B(p > \neg q)$, that is,

$$s \models \neg B(p > \neg q). \tag{18}$$

Furthermore,

$$\bigcup_{s' \in \mathcal{B}(s)} \big(f(s', E) \cap F\big) = \left(\bigcup_{s' \in \mathcal{B}(s)} f(s', E)\right) \cap F = \|r\| \cap \|q\| = \|q \wedge r\|. \tag{19}$$

By hypothesis, $f(\tilde{s}, \|p \wedge q\|) = f(\tilde{s}, E \cap F) \nsubseteq \bigcup_{x \in \mathcal{B}(s)} (f(x, E) \cap F)$ so that, by (19), $f(\tilde{s}, \|p \wedge q\|) \nsubseteq \|q \wedge r\|$, that is, $\tilde{s} \not\models (p \wedge q) > (q \wedge r)$ and thus $s \not\models B\big((p \wedge q) > (q \wedge r)\big)$, that is

$$s \models \neg B\big((p \wedge q) > (q \wedge r)\big). \tag{20}$$

From (17), (18) and (20) we get a violation of Axiom (A8). □

References

C. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: partial meet contraction and revision functions. *The Journal of Symbolic Logic*, 50:510–530, 1985.

G. Bonanno. A Kripke-Lewis semantics for belief update and belief revision. *Artificial Intelligence*, 339, 2025. doi: 10.1016/j.artint.2024.104259.

E. Fermé and S. O. Hansson. *Belief change: introduction and overview*. Springer, 2018.

P. Gärdenfors. *Knowledge in flux: modeling the dynamics of epistemic states*. MIT Press, 1988.

L. Giordano, V. Gliozzi, and N. Olivetti. A conditional logic for belief revision. In J. Dix, L. F. n. del Cerro, and U. Furbach, editors, *Logics in Artificial Intelligence*, pages 294–308, Berlin, Heidelberg, 1998. Springer. URL https://link.springer.com/chapter/10.1007/3-540-49545-2_20.

L. Giordano, V. Gliozzi, and N. Olivetti. Belief revision and the Ramsey test: A solution. In F. Esposito, editor, *AI*IA 2001: Advances in Artificial Intelligence*, pages 165–175, Berlin, Heidelberg, 2001. Springer. doi: https://doi.org/10.2307/2185133.

L. Giordano, V. Gliozzi, and N. Olivetti. Weak AGM postulates and strong Ramsey Test: A logical formalization. *Artificial intelligence*, 168:1–37, 2005.

M. Günther and C. Sisti. Ramsey's conditionals. *Synthese*, 200(2):165–196, 2022. doi: 10.1007/s11229-022-03586-1.