COMMON BELIEF WITH THE LOGIC OF INDIVIDUAL BELIEF

Giacomo Bonanno and Klaus Nehring
Department of Economics,
University of California,
Davis, CA 95616-8578 USA
gfbonanno@ucdavis.edu kdnehring@ucdavis.edu

March 1999

Abstract

The logic of common belief does not always reflect that of individual beliefs. In particular, even when the individual belief operators satisfy the KD45 logic, the common belief operator may fail to satisfy axiom 5. That is, it can happen that neither is A commonly believed nor is it common belief that A is not commonly believed. We identify the intersubjective restrictions on individual beliefs that are incorporated in axiom 5 for common belief.

Mathematics Subject Classification: 03B45, 68T27.
Keywords: Multi-modal logic, Common belief, Negative introspection, Axiom 5.

1. Introduction

Since Lewis's (1969) and Aumann's (1976) pioneering contributions, the concepts of common knowledge and common belief have been discussed extensively in the literature, both syntactically and semantically[1]. At the individual level the difference between knowledge and belief is usually identified with the presence or absence of the Truth Axiom ($\square_i A \rightarrow A$), which is interpreted as "if individual i believes that A, then A is true". In such a case the individual is often said to know that A (thus it is possible for an individual to believe a false proposition but she cannot know a false proposition). While at the individual level the Truth Axiom captures merely a relationship between the individual's beliefs and the external world, at the interpersonal level it has very strong implications. For example, the following is a consequence of the Truth Axiom: $\square_i \square_j A \rightarrow \square_i A$, that is, if individual i believes that individual j believes that A, then individual i herself believes that A. The reason why the Truth Axiom is much stronger in an interpersonal context than appears at first glance is that it amounts to assuming that agreement of any individual's belief with the truth is common knowledge. Given its logical force, it is not surprising to find that the Truth Axiom has strong implications for the logic of common knowledge. In particular, if each individual's beliefs satisfy the strongest logic of knowledge (namely S5 or KT5), the associated common knowledge operator satisfies this logic too. On the other hand, it is well-known[2] that such is not the case for belief: bereft of the Truth Axiom, even the strongest logic for individual belief (KD45 or weak S5) is insufficient to ensure satisfaction of the "Negative Introspection" axiom for common belief:

$$5^* \quad \neg \square_* A \rightarrow \square_* \neg \square_* A$$

(where $\square_*$ denotes the common belief operator). That is to say, it can happen that neither is A commonly believed nor is it common belief that A is not commonly believed (see Example 1 in Section 3). In this note we identify the intersubjective restrictions on beliefs implied by 5^*. Let K_n^*D45 be the multimodal system where the beliefs of each individual satisfy the KD45 logic and common belief is defined in the usual way. We show that the system obtained by adding 5^* to K_n^*D45 coincides with the system obtained by adding to K_n^*D45 the following axiom

$$T^{CB} \quad \square_i \square_* A \rightarrow \square_* A$$

[1] See, fo example, Fagin et al (1995) and references therein.
[2] See, for example, Colombetti (1993) and Lismont and Mongin (1994).

which, in turn, is equivalent to the system obtained by adding to K_n^*D45 the following axiom

$$\text{SW} \quad \Box_i\Box_*A \to \Box_j\Box_*A.$$

In a system obtained by adding any of the above axioms to K_n^*D45, common belief obeys the same logic as individual beliefs.

2. Characterization of 5^*

We consider a multimodal system with $n + 1$ operators $\Box_1, \Box_2, \ldots, \Box_n, \Box_*$ where, for $i = 1, \ldots, n$, the interpretation of $\Box_i A$ is "individual i believes that A", while $\Box_* A$ is interpreted as "it is common belief that A". The basic system K_n^* is given by a suitable axiomatization of Propositional Calculus together with the following axiom schemata and rules of inference[3]:

K	$\Box(A \to B) \to (\Box A \to \Box B)$	$(\forall \Box \in \{\Box_1, \ldots, \Box_n, \Box_*\})$
CB1	$\Box_* A \to \Box_i A$	$(\forall i = 1, \ldots, n)$
CB2	$\Box_* A \to \Box_i \Box_* A$	$(\forall i = 1, \ldots, n)$
CB3	$\Box_*(A \to \Box_1 A \wedge \ldots \wedge \Box_n A) \to (\Box_1 A \wedge \ldots \wedge \Box_n A \to \Box_* A)$	
MP (modus ponens)	$\frac{A,\ A \to B}{B}$	
RN (necessitation)	$\frac{A}{\Box A}$	$(\forall \Box \in \{\Box_1, \ldots, \Box_n, \Box_*\})$

We are interested in the extension of K_n^*, denoted by K_n^*D45, obtained by adding the following axioms for individual beliefs ($i = 1, \ldots, n$):

D $\quad \Box_i A \to \neg\Box_i \neg A$

4 $\quad \Box_i A \to \Box_i \Box_i A$

5 $\quad \neg\Box_i A \to \Box_i \neg\Box_i A$

In general, the common belief operator does not inherit all the properties of the individuals' belief operators. In particular, as shown in the next section, 5^* is not provable in K_n^*D45. Thus axiom 5^* must involve restrictions on the beliefs of the individuals which presumably are intersubjective in nature. Such restrictions are

[3] See Bonanno (1996). See also Lismont (1993) and Lismont and Mongin (1994).

fully captured by axioms SW and T^{CB}, as the following theorem shows. In the proofs we make use of the fact that, as is well-known (see, for example, Lismont and Mongin, 1994), K_n^*D45 has the following theorem:

CB4 $\quad (\Box_1\Box_* A \wedge \cdots \wedge \Box_n\Box_* A) \to \Box_* A$

Theorem 2.1. $K_n^*D45 + 5^*$, $K_n^*D45 + SW$ and $K_n^*D45 + T^{CB}$ are the same system.

Proof. Proof that $K_n^*D45 + SW \vdash T^{CB}$:[4]

1.	$\Box_i\Box_* A \to \Box_1\Box_* A$	(SW)
...	$\cdots$	(SW)
n.	$\Box_i\Box_* A \to \Box_n\Box_* A$	(SW)
n+1.	$\Box_i\Box_* A \to (\Box_1\Box_* A \wedge \cdots \wedge \Box_n\Box_* A)$	(1,...,n,PL)
n+2.	$(\Box_1\Box_* A \wedge \cdots \wedge \Box_n\Box_* A) \to \Box_* A$	(CB4)
n+3.	$\Box_i\Box_* A \to \Box_* A$	(n+1, n+2, PL)

Proof that $K_n^*D45 + T^{CB} \vdash SW$:

1.	$\Box_i\Box_* A \to \Box_* A$	(T^{CB})
2.	$\Box_* A \to \Box_j\Box_* A$	(CB2)
3.	$\Box_i\Box_* A \to \Box_j\Box_* A$	(1,2,PL)

Proof that $K_n^*D45 + 5^* \vdash T^{CB}$:

1.	$\neg\Box_* A \to \Box_*\neg\Box_* A$	(5^*)
2.	$\Box_*\neg\Box_* A \to \Box_i\neg\Box_* A$	(CB1)
3.	$\Box_i\neg\Box_* A \to \neg\Box_i\Box_* A$	(D)
4.	$\neg\Box_* A \to \neg\Box_i\Box_* A$	(1,2,3,PL)
5.	$\Box_i\Box_* A \to \Box_* A$	(4,PL)

Proof that $K_n^*D45 + T^{CB} \vdash 5^*$:

[4] PL stands for "Propositional Logic" and RK is the following derived rule of inference, which, as is well-known (cf. Chellas, 1984), is satisfied by every normal modal operator: $\frac{A \to B}{\Box A \to \Box B}$ for every $\Box \in \{\Box_1, \ldots, \Box_n, \Box_*\}$.

1.	$\Box_i \Box_* A \rightarrow \Box_* A$	(T^{CB})
2.	$\neg\Box_* A \rightarrow \neg\Box_i \Box_* A$	(1,PL)
3.	$\neg\Box_i \Box_* A \rightarrow \Box_i \neg\Box_i \Box_* A$	(5)
4.	$\Box_* A \rightarrow \Box_i \Box_* A$	(CB2)
5.	$\neg\Box_i \Box_* A \rightarrow \neg\Box_* A$	(4,PL)
6.	$\Box_i \neg\Box_i \Box_* A \rightarrow \Box_i \neg\Box_* A$	(5,RK)
7.	$\neg\Box_* A \rightarrow \Box_i \neg\Box_* A$	(2,3,6,PL)

Now, a repetition of steps 1-7 for every $i = 1, \ldots, n$ leads to

8.	$\neg\Box_* A \rightarrow (\Box_1 \neg\Box_* A \wedge \ldots \wedge \Box_n \neg\Box_* A)$	(7,PL)
9.	$\Box_* (\neg\Box_* A \rightarrow (\Box_1 \neg\Box_* A \wedge \ldots \wedge \Box_n \neg\Box_* A))$	(8,RN)
10.	$\Box_* (\neg\Box_* A \rightarrow (\Box_1 \neg\Box_* A \wedge \ldots \wedge \Box_n \neg\Box_* A)) \rightarrow$ $((\Box_1 \neg\Box_* A \wedge \ldots \wedge \Box_n \neg\Box_* A) \rightarrow \Box_* \neg\Box_* A)$	(CB3)
11.	$(\Box_1 \neg\Box_* A \wedge \ldots \wedge \Box_n \neg\Box_* A) \rightarrow \Box_* \neg\Box_* A$	(9,10,MP)
12.	$\neg\Box_* A \rightarrow \Box_* \neg\Box_* A$	(8,11,PL)

3. 5^* is not provable in $K_n^* D45$

In order to show that 5^* is not provable in $K_n^* D45$, we need to introduce the semantics. A model is a tuple $M = \langle W, R_1, \ldots, R_n, R_*, V \rangle$ where W is a non-empty set of possible worlds, $R_1, \ldots, R_n, R_*$ are binary accessibility relations on W and V is a valuation, that is, a function that associates with every atomic proposition p the set of possible worlds where p is true. The valuation is extended to the set of formulas in the usual way; we denote the fact that formula A is true at world w in model M by $M, w \models A$. Thus, in particular, for $i = 1, \ldots, n$, $M, w \models \Box_i A$ if and only if $M, w' \models A$ for all w' such that $w R_i w'$. Similarly, $M, w \models \Box_* A$ if and only if $M, w' \models A$ for all w' such that $w R_* w'$.

The following result is well-known[5].

Theorem 3.1. (i) K_n^* is sound and complete with respect to the class of models where R_* is the transitive closure of $R_1 \cup \ldots \cup R_n$.[6] (ii) $K_n^* D45$ is sound and complete with respect to the class of models where, in addition, R_i $(i = 1, \ldots, n)$ is serial, transitive and euclidean.[7]

---

[5] See, for example, Bonanno (1996), Lismont (1993), Lismont and Mongin (1994), Fagin et al. (1995).

[6] That is, $a R_* b$ if and only if there are sequences $\langle w_1, \ldots, w_m \rangle$ and $\langle i_1, \ldots, i_m \rangle$ such that (1) $w_1 = a$, (2) $w_m = b$ and (3) for every $k = 1, \ldots, m - 1$, $w_k R_{i_k} w_{k+1}$.

[7] R_i is serial if $\forall w, \exists w' : w R_i w'$; it is transitive if $\forall w, w', w''$, if $w R_i w'$ and $w' R_i w''$ then $w R_i w''$; it is euclidean if $\forall w, w', w''$ if $w R_i w'$ and $w R_i w''$ then $w' R_i w''$.

A model where R_* is the transitive closure of $R_1 \cup \dots \cup R_n$ will be called a CB-model.

Example 1. Consider the following CB-model: $W = \{a, b\}$, $R_1 = \{(a, a), (b, b)\}$; $R_2 = \{(a, b), (b, b)\}$. Thus $R_* = \{(a, a), (a, b), (b, b)\}$. Let p be an atomic proposition which is true at b and false at a. Then the formula $(\neg \Box_* p \to \Box_* \neg \Box_* p)$, which is an instance of 5^*, is false at a. Since in this model both R_1 and R_2 are serial, transitive and euclidean, it follows from the soundness part of (ii) of Theorem 3.1 that 5^* is not provable in $K_n^* D45$.

References

[1] Aumann, Robert. 1976. Agreeing to disagree. Annals of Statistics, 4: 1236-1239.

[2] Bonanno, Giacomo. 1996. On the logic of common belief. Mathematical Logic Quarterly, 42: 305-311.

[3] Chellas, Brian. 1984. Modal logic: an introduction. Cambridge University Press.

[4] Colombetti, Marco. 1993. Formal semantics for mutual beliefs. Artificial intelligence, 62: 341-353.

[5] Fagin, Ronald, Joseph Halpern, Yoram Moses and Moshe Vardi. 1995, Reasoning about knowledge. MIT Press.

[6] Lewis, David. 1969. Convention: a philosophical study, Harvard University Press.

[7] Lismont, Luc. 1993. La connaissance commune en logique modale, Mathematical Logic Quarterly, 39: 115-130.

[8] Lismont, Luc and Philippe Mongin. 1994. On the logic of common belief and common knowledge. Theory and Decision, 37: 75-106.