Supposing and learning: a unified framework for belief revision

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Abstract

Consider two possible scenarios for belief revision. Initially the agent either believes that ϕ is not the case (that is, believes $\neg\phi$) or suspends belief about ϕ. In one scenario she receives reliable information that, as a matter of fact, ϕ *is* the case; call this scenario "learning that ϕ". In the other scenario she reasons about what she believes would be the case if ϕ were the case; call this scenario "supposing that ϕ". We argue that there are important differences between the two scenarios. As shown in Bonanno (2025a;b), it is possible to view the AGM theory of belief revision (Alchourrón et al. (1985)) as a theory of hypothetical, or suppositional, reasoning, rather than a theory of actual belief change in response to new information. By making an addition to the semantics considered in Bonanno (2025a), we (1) provide a unified framework for the analysis of both suppositional beliefs and information-driven belief change, (2) argue that some of the AGM axioms are not appropriate for the latter and (3) provide a list of axioms that seem appropriate for belief change in response to new information.

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1 Introduction

Consider two possible scenarios of belief revision for an agent who, initially, either believes that ϕ is not the case (that is, believes $\neg\phi$) or suspends judgment on ϕ (that is, considers both ϕ and $\neg\phi$ possible). In one scenario she receives reliable information that, as a matter of fact, ϕ *is* the case; call this scenario *"learning that ϕ"*. In the other scenario she reasons about what she believes would be (or is) the case if ϕ were (or is) the case;[1] call this scenario *"supposing that ϕ"*. We argue that (1) there are significant differences between the two scenarios, (2) contrary to the prevalent view, the AGM theory of belief revision (Alchourrón et al. (1985)) is best be viewed as a theory of suppositional beliefs, rather than information-driven belief change, and (3) some of the AGM postulates, while appropriate for modeling suppositional belief, are not fitting for modeling belief change prompted by reliable information. From now on, the expressions 'belief change prompted by reliable information' and 'information-driven belief change' will be shortened to 'belief change'.

By augmenting the structures introduced in Bonanno (2025a;b) we provide a unified framework where suppositional beliefs and belief change are modeled side by side. The proposed semantics characterizes both an account in terms of belief functions as well as in terms of modal logic, thus extending the analysis of Bonanno (2025b) by adding belief change to suppositional beliefs.

2 Supposing versus learning

Several authors have pointed out that there is a significant difference between supposing that ϕ and learning that ϕ:

> "Merely suppositional change is essentially different from "genuine" change due to new information." (Rott 1999, p. 410)
>
> "Supposing is like pretending, or making believe, in that suppositions do not call for justification in the way that beliefs do. We make them for the sake of argument." (Morreau 1998, p. 540)
>
> "In none of these contexts is supposing that ϕ equivalent to be-

[1] Note that we allow for both the subjunctive and the indicative conditional. The subjunctive form (if ϕ *were* the case then ψ *would be* the case) seems to be more appropriate when the agent initially believes $\neg\phi$, while the indicative form (if ϕ *is* the case then ψ *is* the case) seems to be more appropriate when the agent initially considers ϕ possible.

> lieving that ϕ... *Changing full beliefs calls for some sort of accounting or justification. Supposition does not...*" (Levi 1996, p. 5, emphasis added)

> "There seems to be a need to distinguish actual belief revision from belief revision that is merely hypothetical. [...] Ordinary theories of belief change do not seem suited to handle the sort of hypothetical belief change that goes on, for example, in debates where the participants agree, "for the sake of argument", on a certain common ground on which possibilities can be explored and disagreements can be aired. *One need not actually believe what one accepts in this way.*" (Segerberg 1998, p. 1, emphasis added)

There is also empirical evidence that, even in the case where what is being supposed or learned is compatible with the initial beliefs, people treat supposition and information differently: Zhao et al. (2012) found that there are

> "substantial differences between the conditional probability of an event A supposing an event B, compared to the probability of A after having learned B. Specifically, supposing B appears to have less impact on the credibility of A than learning that B is true." (Zhao et al. 2012, p. 373)

As an illustration of the difference between supposing and learning, consider the following example discussed by Stalnaker.

> I initially believe the following three things: first, General Smith is a shrewd judge of character – he knows (better than I) who is brave and who is not. Second, the general sends only brave men into battle. Third, Private Jones is cowardly. It follows from these three propositions that Jones will not be sent into battle, so I also initially believe that. Let us assume that someone is cowardly if he would run away under fire. So I believe that Private Jones would run away if he were to be sent into battle (which, for that reason, he won't be). (Stalnaker 1998, p. 46)

Thus, I initially believe that "Private Jones will not be sent into battle" and also that "Private Jones would run away if he were to be sent into battle"; we interpret the latter belief as a suppositional belief: on the supposition that Private Jones is sent into battle I believe that he would run away. In the process of entertaining the supposition, I maintain my belief that Private Jones is cowardly, and running away is what a cowardly person would do. But how would I react to the *information* that Private Jones was in fact sent into battle? In this case, Stalnaker suggests that

> since I think the general is a better judge of character than I, I would revise [my initial beliefs] by giving up my belief that Jones is cowardly. Of the three beliefs mentioned above, the first two are more robust than the third. (Stalnaker 1998, ibid.)

While Stalnaker offers this example as an illustration of the difference between causal counterfactuals and epistemic counterfactuals,[2] we propose a different interpretation: one that pertains to the difference between supposing and learning. Both 'ifs' in the above example are epistemic 'ifs' in that they relate to my beliefs. The statement "I believe that if Private Jones were sent into battle, he would run away" represents suppositional reasoning, in which I *express*, or elucidate, my initial beliefs, reiterating my belief that he is cowardly. The statement "if Private Jones *is* sent into battle, then I believe that he won't run away" represents my reaction to learning that Private Jones was in fact sent into battle, and my reaction is to *change* my beliefs to accommodate, and account for, what I just learned.

What is the crucial difference between learning ϕ and supposing ϕ? In our view, it can be found in Levi's observation that *a supposition requires no explanation*:

> If h is inconsistent with the corpus K we need to remove the offending $\neg h$ from K. We must contract by removing $\neg h$. Observe that $\neg h$ is not being removed because the inquirer has good reason to do so. Indeed, the contraction is not in earnest so that no such reason need be given. It is stipulated for the sake of the argument that $\neg h$ is to be given up. (Levi 1996, p. 28)

Indeed, it is common to state that a supposition is entertained "*for the sake of argument*"; similar expressions are: "suppose that, f*or whatever reason*, ϕ is the case ..." or "suppose that, *somehow*, ϕ were the case . . . ". In the above example, if I suppose that Private Jones is sent into battle, I am not required to come up with an explanation for why that would be the case; indeed, there may be

[2] Stalnaker views the 'if' in the expression "if Private Jones were sent into battle, he would run away" as a *causal* 'if', that is, an 'if' used to express Private Jones' disposition to act in a situation that I believe will not arise. On the other hand, he views the 'if' in the expression "if Private Jones *is* sent into battle, then he won't run away" as an *epistemic* 'if': it concerns how I would revise my beliefs if I were to *learn* that Private Jones was in fact sent into battle.

a number of plausible circumstances under which it could happen: perhaps it was a mistake, or not enough brave soldiers were available, etc. Since *no background scenario needs to be provided to account for the supposition*, I am justified in maintaining – to the extent possible – my initial beliefs, in particular, that Private Jones is cowardly; hence my suppositional belief that he would run away. On the other hand, reliable *information that contradicts my initial beliefs calls for an understanding of, or explanation for, why my initial beliefs were wrong*.

In the next section we review the suppositional account of AGM belief revision provided in Bonanno (2025a;b) and in Section 4 we extend the framework to incorporate belief change in response to reliable information and discuss a number of properties that seem plausible for belief change.

3 Kripke-Lewis semantics for AGM belief revision

In Bonanno (2025a) a Kripke-Lewis semantics was introduced to model AGM belief revision interpreted as a theory of suppositional beliefs and Bonanno (2025b) provided a translation of the AGM postulates into modal-logic axioms. In this section we review this approach. Recall that the AGM approach takes as starting point a set K of propositional formulas, interpreted as the initial beliefs; furthermore, for every propositional formula ϕ, $K * \phi$ denotes the change in K prompted by the input ϕ, the latter usually being interpreted as new information. As explained below, we instead interpret $\psi \in K * \phi$ as the proposition "the agent believes that if ϕ were (or is) the case then ψ would be (or is) the case".

3.1 Belief functions

We consider a propositional logic based on a countable set `At` of atomic formulas. Φ_0 denotes the set of Boolean formulas constructed from `At` as follows: $\texttt{At} \subset \Phi_0$ and if $\phi, \psi \in \Phi_0$ then $\neg\phi$ and $\phi \vee \psi$ belong to Φ_0. Define $\phi \to \psi$, $\phi \wedge \psi$, and $\phi \leftrightarrow \psi$ in terms of $\neg$ and $\vee$ in the usual way (e.g. $\phi \to \psi$ is a shorthand for $\neg\phi \vee \psi$).

Given a subset K of Φ_0, its deductive closure $Cn(K) \subseteq \Phi_0$ is defined as follows: $\psi \in Cn(K)$ if and only if there exist $\phi_1, ..., \phi_n \in K$ (with $n \geq 0$) such that $(\phi_1 \wedge ... \wedge \phi_n) \to \psi$ is a tautology. A set $K \subseteq \Phi_0$ is *consistent* if $Cn(K) \neq \Phi_0$; it is *deductively closed* if $K = Cn(K)$. Given a set $K \subseteq \Phi_0$ and a formula $\phi \in \Phi_0$, the *expansion* of K by ϕ, denoted by $K+\phi$, is defined as follows: $K+\phi = Cn\left(K \cup \{\phi\}\right)$.

Let $K \subseteq \Phi_0$ be a *consistent and deductively closed* set representing the agent's initial beliefs and let $\Psi \subseteq \Phi_0$ be a set of formulas representing possible informational or suppositional inputs. A *belief change function* based on Ψ and K is a function $\circ : \Psi \to 2^{\Phi_0}$ (2^{Φ_0} denotes the set of subsets of Φ_0) that associates with every formula $\phi \in \Psi$ a set $K \circ \phi \subseteq \Phi_0$, interpreted as the change in K prompted by the input ϕ. We follow the common practice of writing $K \circ \phi$ instead of $\circ(\phi)$ which has the advantage of making it clear that the belief change function refers to a given, *fixed*, K. If $\Psi \neq \Phi_0$ then $\circ$ is called a *partial* belief change function, while if $\Psi = \Phi_0$ then $\circ$ is called a *full-domain* belief change function.

3.2 Kripke-Lewis semantics

Definition 3.1. A *Kripke-Lewis frame* is a triple $\langle S, \mathcal{B}, f \rangle$ where

1. S is a set of *states*; subsets of S are called *events*.

2. $\mathcal{B} \subseteq S \times S$ is a binary relation on S which is serial: $\forall s \in S, \exists s' \in S$, such that $s\mathcal{B}s'$ ($s\mathcal{B}s'$ is an alternative notation for $(s, s') \in \mathcal{B}$). We denote by $\mathcal{B}(s)$ the set of states that are reachable from s by $\mathcal{B}$: $\mathcal{B}(s) = \{s' \in S : s\mathcal{B}s'\}$. $\mathcal{B}(s)$ is interpreted as the set of states that initially the agent considers doxastically possible at state s.

3. $f : S \times (2^S \setminus \varnothing) \to 2^S$ is a selection function that associates with every state-event pair (s, E) (with $E \neq \varnothing$) a set of states $f(s, E) \subseteq S$. $f(s, E)$ is interpreted as the set of states that are closest (or most similar) to s, conditional on event E.

We require seriality of the belief relation because it ensures that the initial beliefs at state s, represented by the non-empty set $\mathcal{B}(s)$, are consistent. Similarly, the requirement that $f(s, E)$ is defined if and only if $E \neq \varnothing$ ensures that the informational or suppositional input is consistent.

Adding a valuation to a frame yields a model. Thus a *model* is a tuple $\langle S, \mathcal{B}, f, V \rangle$ where $\langle S, \mathcal{B}, f \rangle$ is a frame and $V : \mathtt{At} \to 2^S$ is a valuation that assigns to every atomic formula $p \in \mathtt{At}$ the set of states where p is true.

Definition 3.2. Given a model $M = \langle S, \mathcal{B}, f, V \rangle$ define truth of a Boolean formula $\phi \in \Phi_0$ at a state $s \in S$ in model M, denoted by $s \models_M \phi$, as follows:

1. if $p \in \mathtt{At}$ then $s \models_M p$ if and only if $s \in V(p)$,

2. $s \models_M \neg\phi$ if and only if $s \not\models_M \phi$,

3. $s \models_M (\phi \vee \psi)$ if and only if $s \models_M \phi$ or $s \models_M \psi$ (or both).

We denote by $||\phi||_M$ the truth set of formula ϕ in model M: $||\phi||_M = \{s \in S : s \models_M \phi\}$.

Given a model $M = \langle S, \mathcal{B}, f, V\rangle$ and a state $s \in S$, let $K_{s,M} = \{\phi \in \Phi_0 : \mathcal{B}(s) \subseteq ||\phi||_M\}$; thus a Boolean formula ϕ belongs to $K_{s,M}$ if and only if at state s the agent believes ϕ, in the sense that ϕ is true at every state that the agent considers doxastically possible at state s. We identify $K_{s,M}$ with the agent's *initial beliefs at state s*.[3]

Next, given a model $M = \langle S, \mathcal{B}, f, V\rangle$ and a state $s \in S$, let $\Psi_M = \{\phi \in \Phi_0 : ||\phi||_M \neq \varnothing\}$[4] and define the following *partial* belief change function $\circ : \Psi_M \to 2^{\Phi_0}$ based on $K_{s,M}$ and Ψ_M:

$$\begin{aligned} &\psi \in K_{s,M} \circ \phi \;\text{ if and only if, }\; \forall s' \in \mathcal{B}(s),\, f\left(s', ||\phi||_M\right) \subseteq ||\psi||_M \\ &\text{or, equivalently, } \bigcup_{s' \in \mathcal{B}(s)} f\left(s', ||\phi||_M\right) \subseteq ||\psi||_M \end{aligned} \tag{RI}$$

Given the customary interpretation of selection functions in terms of conditionals, (RI) can be interpreted as stating that $\psi \in K_{s,M} \circ \phi$ if and only if at state s the agent believes that "if ϕ is (were) the case then ψ is (would be) the case". This interpretation will be made explicit in the modal logic considered in Section 3.3.

Definition 3.3. **(A)** An axiom A of belief change functions is *valid on a frame F* if, for every model based on that frame and for every state s in that model, the partial belief change function defined by (RI) satisfies A; it is *valid on a set of frames* $\mathcal{F}$ if it is valid on every frame $F \in \mathcal{F}$.

(B) An axiom A of belief change functions *is characterized by*, or *corresponds to*, or *characterizes*, a property P of frames if the following is true:

(1) A is valid on the class of frames that satisfy property P, and

(2) if a frame does not satisfy property P then A is not valid on that frame, that is, there is a model based on that frame and a state in that model where the partial belief change function defined by (RI) violates axiom A.

[3] It is shown in Bonanno (2025a) that the set $K_{s,M} \subseteq \Phi_0$ so defined is deductively closed and consistent.

[4] Since, in any given model, there are formulas ϕ such that $||\phi||_M = \varnothing$ (at the very least all the contradictions), Ψ_M is a proper subset of Φ_0.

AGM axiom	Frame property
$(K*1)$ $K*\phi = Cn(K*\phi)$	No additional property
$(K*2)$ $\phi \in K*\phi$	$(P*2)$ $\forall s \in S, \forall E \in 2^S \setminus \varnothing$, $\bigcup_{s' \in \mathcal{B}(s)} f(s', E) \subseteq E$
$(K*3)$ $K*\phi \subseteq K+\phi$	$(P*3)$ $\forall s \in S, \forall E \in 2^S \setminus \varnothing, \forall s' \in \mathcal{B}(s)$, if $s' \in E$ then $s' \in \bigcup_{x \in \mathcal{B}(s)} f(x, E)$
$(K*4)$ if $\neg\phi \notin K$ then $K+\phi \subseteq K*\phi$	$(P*4)$ $\forall s \in S, \forall E, F \in 2^S \setminus \varnothing$, if $\mathcal{B}(s) \cap E \neq \varnothing$ and $\mathcal{B}(s) \subseteq (S \setminus E) \cup F$ then $\bigcup_{s' \in \mathcal{B}(s)} f(s', E) \subseteq F$
$(K*5b)$ If $\neg\phi$ is not a tautology then $K*\phi \neq \Phi_0$	$(P*5b)$ $\forall s \in S, \forall E \in 2^S \setminus \varnothing, \exists s' \in \mathcal{B}(s)$ such that $f(s', E) \neq \varnothing$
$(K*6)$ if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$	No additional property
$(K*7)$ $K*(\phi \wedge \psi) \subseteq (K*\phi)+\psi$	$(P*7)$ $\forall s \in S, \forall E, F \in 2^S$ with $E \cap F \neq \varnothing$, $\bigcup_{s' \in \mathcal{B}(s)} (f(s', E) \cap F) \subseteq \bigcup_{s' \in \mathcal{B}(s)} f(s', E \cap F)$
$(K*8)$ If $\neg\psi \notin K*\phi$, then $(K*\phi)+\psi \subseteq K*(\phi \wedge \psi)$	$(P*8)$ $\forall s \in S, \forall E, F \in 2^S \setminus \varnothing$ if $\exists \hat{s} \in \mathcal{B}(s)$ such that $f(\hat{s}, E) \cap F \neq \varnothing$ then $\bigcup_{s' \in \mathcal{B}(s)} f(s', E \cap F) \subseteq \bigcup_{s' \in \mathcal{B}(s)} (f(s', E) \cap F)$

Figure 1: Semantic characterization of the AGM axioms.

The table in Figure 1, adapted from Bonanno (2025b), shows the list of AGM axioms for belief revision and, for every AGM axiom, the characterizing property of frames.[5]

3.3 A modal-logic translation of the AGM axioms

We now turn to the modal language considered in Bonanno (2025b), which contains three operators: a unimodal belief operator B, a bimodal conditional operator $>$ and the unimodal necessity operator $\Box$. The interpretation of $B\phi$ is "the agent believes ϕ", the interpretation of $\phi > \psi$ is "if ϕ is (or were) the case then ψ is (or would be) the case" and the interpretation of $\Box\phi$ is "ϕ is necessarily true".
The set Φ of formulas in the language is defined as follows:

- $\Phi_0 \subseteq \Phi$ (recall that Φ_0 is the set of Boolean formulas built on the countable set of atomic sentences `At`),
- if $\alpha, \beta \in \Phi$ then all of the following belong to Φ: $\Box\alpha$, $B\alpha$, $\alpha > \beta$ and all their Boolean combinations.

We focus on the basic normal logic, denoted by $\mathcal{L}$, consisting of the following axioms and rules of inference.[6] We denote general formulas by α, β and γ, while ϕ, ψ and χ are reserved for *Boolean* formulas (e.g. in Figure 2).

- Every formula that has the form of a classical tautology is a theorem.
- The consistency axiom D for B:

$$(D_B) \qquad B\alpha \rightarrow \neg B\neg\alpha.$$

- The conjunction axiom C for $\Box$, B and $>$:

$$(C_\Box) \qquad \Box\alpha \wedge \Box\beta \rightarrow \Box(\alpha \wedge \beta)$$

$$(C_B) \qquad B\alpha \wedge B\beta \rightarrow B(\alpha \wedge \beta)$$

$$(C_>) \qquad (\gamma > \alpha) \wedge (\gamma > \beta) \rightarrow (\gamma > (\alpha \wedge \beta))$$

- The necessity-to-belief axiom: $(NB) \quad \Box\alpha \rightarrow B\alpha$

[5] The reason for the absence from the table of the other half of AGM axiom $(K*5)$ (namely, if ϕ is a contradiction then $K * \phi = \Phi_0$) will become clear in the Section 3.3.

[6] We follow the nomenclature in (Chellas 1984, p. 115).

- The rule of inference *Modus Ponens*: $(MP)\quad \dfrac{\alpha\,,\, \alpha \to \beta}{\beta}$

- The rule of inference *Necessitation* for $\Box$ and $>$:

$$(N_{\Box})\quad \frac{\alpha}{\Box\alpha} \qquad\qquad (N_{>})\quad \frac{\beta}{\alpha > \beta}$$

- The rule of inference *RM* for $\Box$, B and $>$:

$$(RM_{\Box})\quad \frac{\alpha \to \beta}{\Box\alpha \to \Box\beta} \qquad\qquad (RM_{B})\quad \frac{\alpha \to \beta}{B\alpha \to B\beta}$$

$$(RM_{>})\quad \frac{\alpha \to \beta}{(\gamma > \alpha) \to (\gamma > \beta)}$$

As semantics for this modal logic we take the Kripke-Lewis frames of Definition 3.1. A model based on a frame is obtained, as before, by adding a valuation $V : \mathtt{At} \to 2^S$. The following definition expands Definition 3.2 by adding validation rules for formulas of the form $\Box\alpha$, $\alpha > \beta$ and $B\alpha$.

Definition 3.4. Truth of a formula $\alpha \in \Phi$ at state s in model M (denoted by $s \models_M \alpha$) is defined as follows:

1. if $p \in \mathtt{At}$ then $s \models_M p$ if and only if $s \in V(p)$.
2. $s \models_M \neg\alpha$ if and only if $s \not\models_M \alpha$.
3. $s \models_M (\alpha \vee \beta)$ if and only if $s \models_M \alpha$ or $s \models_M \beta$ (or both).
4. $s \models_M \Box\alpha$ if and only if, $\forall s' \in S, s' \models_M \alpha$ (that is, $||\alpha||_M = S$; thus, $s \models_M \neg\Box\neg\alpha$ if and only if, for some $s' \in S$, $s' \models_M \alpha$, that is, $||\alpha||_M \neq \varnothing$).
5. $s \models_M (\alpha > \beta)$ if and only if, either
   (a) $s \models_M \Box\neg\alpha$ (that is, $||\alpha||_M = \varnothing$), or
   (b) $s \models_M \neg\Box\neg\alpha$ (that is, $||\alpha||_M \neq \varnothing$) and, for every $s' \in f(s, ||\alpha||_M)$, $s' \models_M \beta$ (that is, $f(s, ||\alpha||_M) \subseteq ||\beta||_M$).[7]
6. $s \models_M B\alpha$ if and only if, $\forall s' \in \mathcal{B}(s)$, $s' \models_M \alpha$ (that is, $\mathcal{B}(s) \subseteq ||\alpha||_M$).

[7] Recall that, by definition of frame, $f(s, E)$ is defined only if $E \neq \varnothing$.

Definition 3.5. **(A)** A formula $\alpha \in \Phi$ is *valid on a frame F* if, for every model M based on that frame and for every state s in that model, $s \models_M \alpha$. A formula $\alpha \in \Phi$ is *valid on a set of frames* $\mathcal{F}$ if it is valid on every frame $F \in \mathcal{F}$.

(B) A formula $\alpha \in \Phi$ *is characterized by*, or *corresponds to*, or *characterizes*, a property P of frames if the following is true:

(1) α is valid on the class of frames that satisfy property P, and

(2) if a frame does not satisfy property P then α is not valid on that frame.

Figure 2 shows, for every AGM axiom, its translation into a corresponding modal formula. The translation is done as follows: for every AGM axiom $(K*i)$, Figure 1 provides the frame property $(P*i)$ that characterizes $(K*i)$ and, in turn, property $(P*i)$ is shown to characterize modal axiom $(A*i)$. The proofs are given in Bonanno (2025b).

Logic $\mathcal{L}$ makes the interpretation of $\psi \in K * \phi$ (that is, ψ belongs to the revised belief set $K * \phi$) transparent: the assertion that $\psi \in K * \phi$ corresponds to the statement that $B(\phi > \psi)$, that is, that the "agent believes that if ϕ were (or is) the case then ψ would be (or is) the case". *We take this to be a formal account of suppositional belief: the statement "on the supposition that ϕ the agent believes that ψ" is expressed by the formula $B(\phi > \psi)$.*

While the dominant interpretation of AGM belief revision is in terms of reaction to reliable new information – so that $K * \phi$ is interpreted as the revised belief set after the information represented by the formula ϕ has been made compatible with the initial belief set K – some authors have argued that the AGM axioms for belief revision are suitable for modeling suppositional beliefs but not for belief change in response to new information. For example, Levi writes

> "the contribution of Alchourrón, Gärdenfors and Makinson is best seen as a contribution to an account of reasoning for the sake of the argument and not as an account of the logic of belief change". (Levi 1996, p. 117)

The characterization of AGM belief revision in terms of the Kripke-Lewis semantics reviewed in this section shows that AGM belief revision can indeed be given a precise and consistent interpretation in terms of supposition rather than information.

In the next section we make an addition to the Kripke-Lewis semantics and, correspondingly, to logic $\mathcal{L}$, which enables us to capture information-driven belief change and to study its properties.

AGM axiom	Modal axiom/Rule of Inference (for $\phi, \psi, \chi \in \Phi_0$)
$(K*1)$ $K*\phi = Cn(K*\phi)$	$(A*1)$ $B(\phi > \psi) \wedge B(\phi > (\psi \to \chi)) \to B(\phi > \chi)$
$(K*2)$ $\phi \in K*\phi$	$(A*2)$ $B(\phi > \phi)$
$(K*3)$ $K*\phi \subseteq K+\phi$	$(A*3)$ $\big(\neg\Box\neg\phi \wedge B(\phi > \psi\big) \to B(\phi \to \psi)$
$(K*4)$ if $\neg\phi \notin K$ then $K+\phi \subseteq K*\phi$	$(A*4)$ $\big(\neg B\neg\phi \wedge B(\phi \to \psi)\big) \to B(\phi > \psi)$
$(K*5a)$ If $\neg\phi$ is a tautology, then $K*\phi = \Phi_0$	rule of inference: $(R*5a)$ $\dfrac{\neg\phi}{B(\phi > \psi)}$
$(K*5b)$ If $\neg\phi$ is not a tautology then $K*\phi \neq \Phi_0$	$(A*5b)$ $\big(\neg\Box\neg\phi \wedge B(\phi > \psi)\big) \to \neg B(\phi > \neg\psi)$
$(K*6)$ if $\phi \leftrightarrow \psi$ is a tautology then $K*\phi = K*\psi$	rule of inference: $(R*6)$ $\dfrac{\phi \leftrightarrow \psi}{B(\phi > \chi) \leftrightarrow B(\psi > \chi)}$
$(K*7)$ $K*(\phi \wedge \psi) \subseteq (K*\phi)+\psi$	$(A*7)$ $\neg\Box\neg(\phi \wedge \psi) \wedge B((\phi \wedge \psi) > \chi) \to B\big(\phi > (\psi \to \chi)\big)$
$(K*8)$ If $\neg\psi \notin K*\phi$, then $(K*\phi)+\psi \subseteq K*(\phi \wedge \psi)$	$(A*8)$ $\neg B(\phi > \neg\psi) \wedge B(\phi > (\psi \to \chi)) \to B\big((\phi \wedge \psi) > (\psi \wedge \chi)\big)$

Figure 2: Translation of the AGM axioms into modal formulas.

4 Adding belief change

We now extend the analysis of the previous section by adding belief change prompted by reliable information. We carry out the analysis along two parallel tracks: the belief function approach of AGM and the modal-logic approach discussed in the previous section; the two approaches are bridged by an extension of the Kripke-Lewis semantics.

4.1 Kripke-Lewis-Stalnaker (KLS) semantics

To the Kripke-Lewis frames of Definition 3.1 we add a "Stalnaker-like" function $g : S \times 2^S \setminus \varnothing \to S$ that takes as input a state s (representing the initial belief state of the agent) and a non-empty event E (representing information) and gives as output a new state $s' = g(s, E)$ representing the new belief state.

Thus a state $s \in S$ in a model based on a KLS frame captures the following beliefs:

initial beliefs:	$K_s = \left\{\phi : \mathcal{B}(s) \subseteq \|\phi\|\right\}$
initial suppositional beliefs: (ϕ is the supposition)	$\left\{\psi : \bigcup_{s' \in \mathcal{B}(s)} f(s', \|\phi\|) \subseteq \|\psi\|\right\}$
new beliefs after learning ϕ :	$\left\{\psi : \mathcal{B}(g(s, \|\phi\|)) \subseteq \|\psi\|\right\}$
new suppositional beliefs after learning ϕ : (ψ is the supposition)	$\left\{\chi : \bigcup_{s' \in \mathcal{B}(g(s, \|\phi\|))} f(s', \|\psi\|) \subseteq \|\chi\|\right\}$

In terms of belief functions, we denote by $K \odot \phi$ the new belief set after accommodating information ϕ in the initial belief set K. Given a model M based on a KLS frame we define (for $\|\phi\|_M \neq \varnothing$)

$$K \odot \phi = \left\{\psi : \mathcal{B}(g(s, \|\phi\|_M)) \subseteq \|\psi\|_M\right\}$$

Since the analysis of suppositional beliefs (that is, of beliefs of the form $B(\phi > \psi)$) was developed in the previous section, here we focus on the new element, namely the function g. We associate with it a new bimodal operator $\gg$, with the interpretation of $\phi \gg \psi$ as "if the agent is informed that ϕ then ψ

is the case". We will be mostly interested in formulas of the form $\phi > B\psi$: "if informed that ϕ, the agent believes that ψ".[8]

Thus, we extend the logic $\mathcal{L}$ of the previous section by adding the bimodal operator $>$ and, correspondingly,

- the conjunction axiom $(C_>)$ $(\gamma > \alpha) \wedge (\gamma > \beta) \rightarrow (\gamma > (\alpha \wedge \beta))$
- the rule of inference Necessitation: $(N_>)$ $\dfrac{\beta}{\alpha > \beta}$
- the rule of inference *RM*: $(RM_>)$ $\dfrac{\alpha \rightarrow \beta}{(\gamma > \alpha) \rightarrow (\gamma > \beta)}$

Call $\mathcal{L}^+$ this extension of $\mathcal{L}$.

Given a model M based on a KLS frame, a state s and formulas α and β, the validation rule for $>$ is as follows: $s \models_M (\alpha > \beta)$ if,

(a) either $s \models_M \Box\neg\alpha$ (that is, $\|\alpha\|_M = \varnothing$),

(b) or $s \models_M \neg\Box\neg\alpha$ (that is, $\|\alpha\|_M \neq \varnothing$) and, $g(s, \|\phi\|_M) \models_M \psi$; in particular, $s \models_M (\phi > B\psi)$ if $g(s, \|\phi\|_M)) \models_M B\psi$, that is, $\mathcal{B}(g(s, \|\phi\|_M)) \subseteq \|\psi\|_M$.[9]

4.2 Axioms for belief change

We list the proposed axioms both as properties of a belief function and as modal formulas, together with the KLS frame property that characterizes both. Proofs of the correspondence results are given in the Appendix. In all the expressions below, ϕ, ψ and χ are Boolean formulas (that is, $\phi, \psi, \chi \in \Phi_0$).

The first two axioms mirror AGM axioms $(K*1)$ and $(K*2)$.

Belief function:	$(K \odot 1)$	$K \odot \phi = Cn(K \odot \phi)$
Frame property:		no additional property
Modal formula:	$(A \odot 1)$	$\big(\phi > B\psi \ \wedge\ \phi > B(\psi \rightarrow \chi)\big) \rightarrow \phi > B\chi$

[8]Note, however, that having a non-doxastic formula on the right-hand-side might be useful, e.g. to express – if deemed necessary – the requirement that the information be correct: $\phi > \phi$.

[9]Recall that, by definition, $g(s, E)$ is defined only if $E \neq \varnothing$.

Belief function:	$(K \odot 2)$	$\phi \in K \odot \phi$
Frame property:	$(P \odot 2)$	$\forall s \in S, \forall E \in 2^S \setminus \varnothing,\ \mathcal{B}\big(g(s,E)\big) \subseteq E$
Modal formula:	$(A \odot 2)$	$\phi \gtrdot B\phi$

AGM axioms $(K*3)$ and $(K*4)$, which together require that if $\neg\phi \notin K$ then $K*\phi = K+\phi$, do not seem to be appropriate for belief change.[10] To see this, consider the following scenario:

> Yesterday my next-door neighbor hosted a shrimp-and-lobster party. I was unable to go, but I know that our common friend Steve did go. Based on what Steve told me in the past, I believe (though not strongly) that he is allergic to shellfish. I don't know if he ate or did not eat at the party: letting a be the atomic proposition "Steve ate at the party", $a \notin K$ and $\neg a \notin K$. Since I believe that Steve is allergic to shell fish, I believe that if he ate then he got sick: $(a \to b) \in K$, where b is the atomic proposition "Steve got sick"; thus, by AGM axioms $(K*3)$ and $(K*4)$, $b \in K*a$, that is, on the supposition that he ate, I believe that he got sick. On the other hand, if I *learn* that he actually ate at the party then I might abandon my belief that he is allergic to shellfish and *not* believe that he got sick, that is, $b \notin K \odot a$. In terms of our modal logic, if state s represents my initial state of mind, then it is possible to have $s \models B(a > b) \wedge (a \gtrdot \neg Bb)$ (or even the stronger $s \models B(a > b) \wedge (a \gtrdot B\neg b)$).

As another illustration of the difference between suppositional belief and actual belief change, consider Stalnaker's example described in Section 2. Let the atomic propositions b, c, j and r stand for "Private Jones is sent into **b**attle", "Private Jones is **c**owardly" "the general is a good **j**udge of character" and "Private Jones **r**uns away", respectively. Let state s represent my initial state of mind; then $s \models Bc \wedge Bj$, that is, I believe that Private Jones is cowardly and that the general is a good judge of character; because of this belief, $s \models B\neg b \wedge B(b > r)$, that is, I believe that Private Jones was not sent into battle and that if he had been sent into battle he would have run away. On the other hand, because $s \models Bj$, $s \models b \gtrdot (\neg Bc \wedge \neg Br)$, that is, if I learn that Private Jones was in fact sent into battle then I do not believe that he is cowardly and that he ran away.

The following two axioms (which are a weakening of AGM axioms $(K*3)$

[10] The corresponding frame property would be: if $\mathcal{B}(s) \cap E \neq \varnothing$ then $\mathcal{B}(g(s,E)) = \mathcal{B}(s) \cap E$.

and $(K*4)$) seem appropriate for belief change. The first states that if the agent learns something that she already believes, then her beliefs do not change. The second states that if, initially, she does not believe $\neg\phi$ and also believes $\phi \to \psi$ (that is, since K is deductively closed, $\psi \in K + \phi$), then upon learning that ϕ is in fact the case she does not believe $\neg\psi$; this second axiom captures a notion of some limited preservation of the initial beliefs (while being weaker than AGM axiom $(K*4)$, which would require the agent to believe ψ).

Belief function:	$(K \odot 3)$	If $\phi \in K$ then $K \odot \phi = K$
Frame property:	$(P \odot 3)$	$\forall s \in S, \forall E \in 2^S$, if $\mathcal{B}(s) \subseteq E$ then $\mathcal{B}\big(g(s,E)\big) = \mathcal{B}(s)$
Modal formula:	$(A \odot 3)$	$B\phi \to \big(B\psi \leftrightarrow (\phi > B\psi)\big)$

Belief function:	$(K \odot 4)$	If $\neg\phi \notin K$ and $\psi \in K + \phi$ then $\neg\psi \notin K \odot \phi$
Frame property:	$(P \odot 4)$	$\forall s \in S, \forall E, F \in 2^S$, if $\mathcal{B}(s) \cap E \neq \varnothing$ and $\mathcal{B}(s) \subseteq (S \setminus E) \cup F$ then $\mathcal{B}\big(g(s,E)\big) \cap F \neq \varnothing$
Modal formula:	$(A \odot 4)$	$\big(\neg B\neg\phi \wedge B(\phi \to \psi)\big) \to (\phi > \neg B\neg\psi)$

The next rule of inference and axiom mirror AGM axiom $(K*5)$ (which we split into two parts).

Belief function:	$(K \odot 5a)$	If ϕ is a contradiction, then $K \odot \phi = \Phi_0$
Frame property:		no additional property
Rule of inference:	$(R \odot 5a)$	$\dfrac{\neg\phi}{\phi > B\psi}$

Belief function:	$(K \odot 5b)$	If ϕ is not a contradiction, then $K \odot \phi \neq \Phi_0$
Frame property:		no additional property, (follows from seriality of $\mathcal{B}$)
Modal formula:	$(A \odot 5b)$	$\big(\neg\Box\neg\phi \;\wedge\; (\phi > B\psi)\big) \rightarrow (\phi > \neg B\neg\psi)$

The next rule of inference mirrors AGM axiom $(K * 6)$.

Belief function:	$(K \odot 6)$	If $\phi \leftrightarrow \psi$ is a tautology, then $K \odot \phi = K \odot \psi$
Frame property:		no additional property
Rule of inference:	$(R \odot 6)$	$\dfrac{\phi \leftrightarrow \psi}{(\phi > B\chi) \;\leftrightarrow\; (\psi > B\chi)}$

Thus, of the six basic AGM postulates $(K * 1)$-$(K * 6)$, only $(K * 1)$, $(K * 2)$, $(K * 5)$ and $(K * 6)$ seem to be appropriate for "genuine" belief change.

The supplementary AGM axioms $(K * 7)$ and $(K * 8)$, which together require that if $\neg\psi \notin K * \phi$ then $K * (\phi \wedge \psi) = K * \phi + \psi$, are also not appropriate for belief change. This reflects to the fact that $(K * 7)$ and $(K * 8)$ are a generalization of $(K * 3)$ and $(K * 4)$, respectively. Indeed, a simple modification of the above example of the shrimp-and-lobster party illustrates why AGM axioms $(K * 7)$ and $(K * 8)$ cannot in general be required for belief change. Modify the story by replacing "I know that Steve went to the party" with "I do not know if Steve went to the party", while leaving everything else unchanged. Let p denote the atomic proposition "Steve went to the party" (thus $p \notin K$ and $\neg p \notin K$). Then, on the supposition that Steve went to the party, I believe that if he ate he got sick: $(a \rightarrow b) \in K * p$, so that $b \in K * p + a$; thus, by AGM axioms $(K * 8)$, $b \in K * (p \wedge a)$. However, as explained above, if I *learn* that Steve went to the party and ate then I might abandon my belief that he is allergic to shell fish and not believe that he got sick: $b \notin K \odot (p \wedge a)$.

A further axiom that seems natural for belief change is the following, which says that if the agent believes χ upon being informed that ϕ and also upon being informed that ψ, then she should believe χ upon being informed that

$\phi \vee \psi$:

Belief function:	$(K \odot 7)$	$K \odot \phi \cap K \odot \psi \subseteq K \odot (\phi \vee \psi)$
Frame property:	$(P \odot 7)$	$\forall s \in S, \forall E, F \in 2^S \setminus \varnothing$ $\mathcal{B}(g(s, E \cup F)) \subseteq \mathcal{B}(g(s, E)) \cup \mathcal{B}(g(s, F))$
Modal formula:	$(A \odot 7)$	$\neg\Box\neg\phi \wedge \neg\Box\neg\psi \wedge (\phi > B\chi) \wedge (\psi > B\chi) \rightarrow (\phi \vee \psi) > B\chi$

5 Conclusion

As noted above, the dominant interpretation of the AGM theory is in terms of belief change in response to reliable information, so that $K * \phi$ is understood as the modified belief set after the information represented by the formula ϕ has been made compatible with the initial belief set K. This interpretation is apparent, for example, in the way in which the Success Axiom (AGM axiom $(K * 2)$: $\phi \in K * \phi$) is described or criticized in the literature:[11]

> "The [AGM] Success postulate says that the new information ϕ should always be included in the new belief set. [It] places enormous faith on the reliability of ϕ. The new information is perceived to be so reliable that it prevails over all previous conflicting beliefs, no matter what these beliefs might be." (Peppas 2008, p. 319)

> "In AGM revision, new information has primacy. This is mirrored in the Success postulate for revision. At each stage the system has total trust in the input information, and previous beliefs are discarded whenever that is needed to consistently incorporate the new information. This is an unrealistic feature since in real life, cognitive agents sometimes do not accept the new information that they receive."(Fermé and Hansson 2018, p. 65)

> "A system obeying [the Success axiom] is totally trusting at each stage about the input information; it is willing to give up whatever

[11] To address these criticisms, a more recent literature (Bonanno (2022), Booth et al. (2012), Garapa (2020), Hansson (1999), Hansson et al. (2001)) has dropped the Success Axiom by allowing the agent to discard some pieces of information as not credible or to accept the information only in a limited way.

> elements of the background theory must be abandoned to render it consistent with the new information. Once this information has been incorporated, however, it is at once as susceptible to revision as anything else in the current theory. Such a rule of revision seems to place an inordinate value on novelty, and its behaviour towards what it learns seems capricious." (Cross and Thomason 1992, p. 251)

We argued that an alternative interpretation of the AGM postulates is in terms of suppositional beliefs. Within that interpretation, the Success Axiom becomes entirely trivial : it merely says that the agent believes that if ϕ were (or is) the case then ϕ would be (or is) the case.

We extended the Kripke-Lewis semantics introduced in Bonanno (2025a) by adding a Stalnaker selection function yielding the change in the belief state of the agent upon receiving reliable information that ϕ is the case. In the resulting Kripke-Lewis-Stalnaker (KLS) semantics one can model suppositional beliefs and "genuine" belief change side by side. We argued that some of the AGM postulates, while plausible for suppositional beliefs, do not seem to be appropriate for belief change based on reliable information. We provided some axioms that seem to be appropriate for belief change. Further work needs to be done to provide a full account of "rational" belief change. We believe that further investigation needs to be based not just on properties or axioms that appear plausible and appealing at the abstract level, but on concrete contexts and applications.

Starting with Hintikka's seminal contribution (Hintikka (1962)), the notion of belief has been studied within the context of modal logic, which allows one to express properties such as positive introspection of beliefs ($B\phi \to BB\phi$), negative introspection of beliefs ($\neg B\phi \to B\neg B\phi$), the relationship between knowledge and belief, etc. A desirable feature of the framework put forward in this paper is that the postulates for suppositional beliefs can be translated into formulas in a modal logic and the same was done with the proposed axioms for belief change, thereby unifying the treatment of belief, suppositional belief revision and belief change under the umbrella of modal logic.

A Proofs

Proposition 1. *Belief function property:*

$$(K \odot 1) \qquad K \odot \phi = Cn(K \odot \phi)$$

and modal formula

$$(A \odot 1) \qquad \big(\phi > B\psi \wedge \phi > B(\psi \to \chi)\big) \to \phi > B\chi$$

are valid on every KLS frame.

Proof. Fix a model based on a KLS frame, a state $s \in S$ and let $K_s = \{\psi \in \Phi_0 : \mathcal{B}(s) \subseteq \|\psi\|\}$ be the initial beliefs at state s.[12] Let $\Psi = \{\phi \in \Phi_0 : \|\phi\| \neq \varnothing\}$ and, for every $\phi \in \Psi$, let $K_s \odot \phi \subseteq \Phi_0$ be the new beliefs (in response to input ϕ) defined by

$$\psi \in K_s \odot \phi \text{ if and only if } \mathcal{B}\big((g(s, \|\phi\|)\big) \subseteq \|\psi\| \qquad (RI^{\odot})$$

First we show that $K_s \odot \phi$ is deductively closed, that is, $K_s \odot \phi = Cn\big(K_s \odot \phi\big)$. The inclusion $K_s \odot \phi \subseteq Cn\big(K_s \odot \phi\big)$ follows from the fact that, for every $\psi \in K_s \odot \phi$, $\psi \to \psi$ is a tautology. Next we show that $Cn\big(K_s \odot \phi\big) \subseteq K_s \odot \phi$. Since, by hypothesis, $\|\phi\| \neq \varnothing$, $g(s, \|\phi\|)$ is well defined. Fix an arbitrary $\psi \in Cn\big(K_s \odot \phi\big)$; then there exist $\phi_1, \ldots, \phi_n \in K_s \odot \phi$ ($n \geq 0$) such that $(\phi_1 \wedge \cdots \wedge \phi_n) \to \psi$ is a tautology, so that $\|(\phi_1 \wedge \cdots \wedge \phi_n) \to \psi\| = S$, that is, $\|\phi_1 \wedge \cdots \wedge \phi_n\| \subseteq \|\psi\|$. Fix an arbitrary $i \in \{1, \ldots, n\}$; since $\phi_i \in K_s \odot \phi$, by ($RI^{\odot}$) $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\phi_i\|$. Hence $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\phi_1\| \cap \cdots \cap \|\phi_n\| = \|\phi_1 \wedge \cdots \wedge \phi_n\|$. Thus, since $\|\phi_1 \wedge \cdots \wedge \phi_n\| \subseteq \|\psi\|$, it follows that $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\psi\|$, that is, by ($RI^{\odot}$), $\psi \in K_s \odot \phi$

Next we show that $(A \odot 1)$ is valid on every frame. Fix a model based on a KLS frame, a state $s \in S$ and arbitrary formulas $\phi, \psi, \chi \in \Phi_0$ and suppose that $s \models \big(\phi > B\psi \wedge \phi > B(\psi \to \chi)\big)$. If $\|\phi\| = \varnothing$ then, by the validation rule, $\phi > B\chi$, for every $\chi \in \Phi_0$. Suppose, therefore that $\|\phi\| \neq \varnothing$, so that $g(s, \|\phi\|)$ is well defined. Then, since $s \models \phi > B\psi$, $g(s, \|\phi\|) \models B\psi$, that is,

$$\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\psi\| \qquad (1)$$

Since $s \models \phi > B(\psi \to \chi)$, $g(s, \|\phi\|) \models B(\psi \to \chi)$, that is,

$$\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\psi \to \chi\| = \|\neg\psi\| \cup \|\chi\| = (S \setminus \|\psi\|) \cup \|\chi\| \qquad (2)$$

It follows from (1) and (2), that

$$\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\psi\| \cap \big((S \setminus \|\psi\|) \cup \|\chi\|\big) = \varnothing \cup (\|\psi\| \cap \|\chi\|) \subseteq \|\chi\| \qquad (3)$$

Thus, $g(s, \|\phi\|) \models B\chi$ and, therefore, $s \models \phi > B\chi$. □

[12] It is shown in (Bonanno 2025a, Lemma 2) that K_s is consistent and deductively closed.

Proposition 2. *Belief function property:*

$$(K \odot 2) \qquad \phi \in K \odot \phi$$

and modal formula

$$(A \odot 2) \qquad \phi > B\phi$$

are characterized by KLS frame property

$$(P \odot 2) \qquad \forall s \in S, \forall E \in 2^S \setminus \varnothing, \ \mathcal{B}\big(g(s, E)\big) \subseteq E$$

Proof. First we show that $(K \odot 2)$ is characterized by $(P \odot 2)$. Fix an arbitrary model based on a frame that satisfies $(P \odot 2)$, an arbitrary state s and let $\odot$ be the partial belief change function defined by $(RI^{\odot})$. Let $\phi \in \Phi_0$ be in the domain of $\odot$, that is, $\|\phi\| \neq \varnothing$. Then $g(s, \|\phi\|)$ is well defined and, by $(P \odot 2)$, $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\phi\|$, so that, by $(RI^{\odot})$, $\phi \in K_s \odot \phi$. Conversely, consider a frame that violates $(P \odot 2)$. Then there exist $s \in S$ and $E \in 2^S \setminus \varnothing$ such that $\mathcal{B}\big(g(s, E)\big) \nsubseteq E$. Let $p \in \texttt{At}$ be an atomic proposition and define a model based on this frame where $\|p\| = E$. Then, since $\mathcal{B}\big(g(s, \|p\|)\big) \nsubseteq \|p\|$, $p \notin K_s \odot p$, violating $(K \odot 2)$.

Next we show that $(A \odot 2)$ is characterized by $(P \odot 2)$. Fix an arbitrary model based on a frame that satisfies $(P \odot 2)$, an arbitrary state s and an arbitrary $\phi \in \Phi_0$. We need to show that $s \models \phi > B\phi$. If $\|\phi\| = \varnothing$ this is true by the validation rule. Suppose, therefore, that $\|\phi\| \neq \varnothing$ so that $g(s, \|\phi\|)$ is well defined. By $(P \odot 2)$, $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\phi\|$ and thus $s \models \phi > B\phi$. Conversely, consider a frame that violates $(P \odot 2)$. Then there exist $s \in S$ and $E \in 2^S \setminus \varnothing$ such that $\mathcal{B}\big(g(s, E)\big) \nsubseteq E$. Let $p \in \texttt{At}$ be an atomic proposition and define a model based on this frame where $\|p\| = E$. Then, since $\mathcal{B}\big(g(s, \|p\|)\big) \nsubseteq \|p\|$, $s \not\models p > Bp$, violating $(A \odot 2)$. □

Proposition 3. *Belief function property:*

$$(K \odot 3) \qquad \textit{If } \phi \in K \textit{ then } K \odot \phi = K$$

and modal formula

$$(A \odot 3) \qquad B\phi \to \big(B\psi \leftrightarrow (\phi > B\psi)\big)$$

are characterized by KLS frame property

$$(P \odot 3) \qquad \forall s \in S, \forall E \in 2^S, \ \textit{if } \mathcal{B}(s) \subseteq E \textit{ then } \mathcal{B}\big(g(s, E)\big) = \mathcal{B}(s)$$

Proof. First we show that $(K\odot 3)$ is characterized by $(P\odot 3)$. Fix an arbitrary model based on a frame that satisfies $(P\odot 3)$, an arbitrary state s and let $\odot$ be the partial belief change function defined by $(RI^{\odot})$. Let $\phi \in \Phi_0$ be in the domain of $\odot$, that is, $\|\phi\| \neq \varnothing$. Then $g(s,\|\phi\|)$ is well defined. Suppose that $\phi \in K_s$, that is, $\mathcal{B}(s) \subseteq \|\phi\|$. Then, by $(P\odot 3)$ (with $E = \|\phi\|$), $\mathcal{B}\big(g(s,\|\phi\|)\big) = \mathcal{B}(s)$, so that, for every $\psi \in \Phi_0$, $\mathcal{B}(s) \subseteq \|\psi\|$ – that is, $\psi \in K_s$ – if and only if $\mathcal{B}\big(g(s,\|\phi\|)\big) \subseteq \|\psi\|$ – that is, by $(RI^{\odot})$, $\psi \in K_s \odot \phi$. Hence $K_s = K_s \odot \phi$. Conversely, consider a frame that violates $(P\odot 3)$. Then there exist $s \in S$ and $E \in 2^S \setminus \varnothing$ such that $\mathcal{B}(s) \subseteq E$ and $\mathcal{B}\big(g(s,E)\big) \neq \mathcal{B}(s)$. Two cases are possible.

CASE 1: $\mathcal{B}\big(g(s,E)\big) \nsubseteq \mathcal{B}(s)$. Let $p,q \in \mathtt{At}$ be atomic formulas and construct a model where $\|p\| = E$ and $\|q\| = \mathcal{B}(s)$. Then, since $\mathcal{B}(s) \subseteq \|p\|$ and $\mathcal{B}(s) \subseteq \|q\|$, $p \in K_s$ and $q \in K_s$ but, since $\mathcal{B}\big(g(s,\|p\|)\big) \nsubseteq \|q\|$, $q \notin K_s \odot p$; hence $K_s \neq K_s \odot p$.

CASE 2: $\mathcal{B}(s) \nsubseteq \mathcal{B}\big(g(s,E)\big)$. Let $p,q \in \mathtt{At}$ be atomic formulas and construct a model where $\|p\| = E$ and $\|q\| = \mathcal{B}\big(g(s,E)\big) = \mathcal{B}\big(g(s,\|p\|)\big)$. Then, $p \in K_s$ and $q \in K_s \odot p$ but (since $\mathcal{B}(s) \nsubseteq \|q\|$) $q \notin K_s$, so that $K_s \neq K_s \odot p$.

Next we show that $(A\odot 3)$ is characterized by $(P\odot 3)$. Fix an arbitrary model based on a frame that satisfies $(P\odot 3)$, arbitrary state s and formula $\phi \in \Phi_0$ and suppose that $s \models B\phi$, that is, $\mathcal{B}(s) \subseteq \|\phi\|$. By seriality of $\mathcal{B}$, $\|\phi\| \neq \varnothing$ so that $g(s,\|\phi\|)$ is well defined. By $(P\odot 3)$, $\mathcal{B}\big(g(s,\|\phi\|)\big) = \mathcal{B}(s)$. Thus, for every formula $\psi \in \Phi_0$, $\mathcal{B}(s) \subseteq \|\psi\|$ – that is, $s \models B\psi$ – if and only if $\mathcal{B}\big(g(s,\|\phi\|)\big) \subseteq \|\psi\|$ – that is, $s \models \phi > B\psi$. Hence $s \models B\psi \leftrightarrow (\phi > B\psi)$. Conversely, consider a frame that violates $(P\odot 3)$. Then there exist $s \in S$ and $E \in 2^S \setminus \varnothing$ such that $\mathcal{B}(s) \subseteq E$ and $\mathcal{B}\big(g(s,E)\big) \neq \mathcal{B}(s)$. Two cases are possible.

CASE 1: $\mathcal{B}\big(g(s,E)\big) \nsubseteq \mathcal{B}(s)$. Let $p,q \in \mathtt{At}$ be atomic formulas and construct a model where $\|p\| = E$ and $\|q\| = \mathcal{B}(s)$. Then, since $\mathcal{B}(s) \subseteq \|p\|$ and $\mathcal{B}(s) \subseteq \|q\|$, $s \models Bp \wedge Bq$ but, since $\mathcal{B}\big(g(s,\|p\|)\big) \nsubseteq \|q\|$, $s \not\models p > Bq$, so that, $s \not\models Bq \leftrightarrow (p > Bq)$, yielding a violation of $(A\odot 3)$.

CASE 2: $\mathcal{B}(s) \nsubseteq \mathcal{B}\big(g(s,E)\big)$. Let $p,q \in \mathtt{At}$ be atomic formulas and construct a model where $\|p\| = E$ and $\|q\| = \mathcal{B}\big(g(s,E)\big)$. Then, $s \models Bp \wedge (p > Bq)$ but $s \not\models Bq$, so that $s \not\models Bq \leftrightarrow (p > Bq)$. □

Proposition 4. *Belief function property:*

$$(K\odot 4) \qquad \textit{If } \neg\phi \notin K \textit{ and } \psi \in K + \phi \textit{ then } \neg\psi \notin K \odot \phi$$

and modal formula

$$(A\odot 3) \qquad \big(\neg B\neg\phi \wedge B(\phi\to\psi)\big) \to (\phi > \neg B\neg\psi)$$

are characterized by KLS frame property

$$(P\odot 3) \quad \forall s\in S, \forall E,F\in 2^S,$$
$$\textit{if } \mathcal{B}(s)\cap E\neq\varnothing \textit{ and } \mathcal{B}(s)\subseteq (S\setminus E)\cup F \textit{ then } \mathcal{B}\big(g(s,E)\big)\cap F\neq\varnothing$$

Proof. First we show that $(K\odot 4)$ is characterized by $(P\odot 4)$. Fix an arbitrary model based on a frame that satisfies $(P\odot 4)$, an arbitrary state s and let $\odot$ be the partial belief change function defined by $(RI^{\odot})$. Let $\phi\in\Phi_0$ and suppose that $\neg\phi\notin K_s$, that is $\mathcal{B}(s)\cap\|\phi\|\neq\varnothing$ (so that $\|\phi\|\neq\varnothing$ and thus $g(s,\|\phi\|)$ is well defined). Fix an arbitrary $\psi\in\Phi_0$ and suppose that $\psi\in K_s+\phi$, that is (since K_s is deductively closed), $(\phi\to\psi)\in K_s$, i.e. $\mathcal{B}(s)\subseteq\|\phi\to\psi\| = (S\setminus\|\phi\|)\cup\|\psi\|$. Then by $(P\odot 4)$ (with $E=\|\phi\|$ and $F=\|\psi\|$), $\mathcal{B}\big(g(s,\|\phi\|)\big)\cap\|\psi\|\neq\varnothing$ and thus, by $(RI^{\odot})$, $\neg\psi\notin K_s\odot\phi$. Conversely, consider a frame that violates $(P\odot 4)$. Then there exist $s\in S$ and $E,F\in 2^S\setminus\varnothing$ such that $\mathcal{B}(s)\cap E\neq\varnothing$ and $\mathcal{B}(s)\subseteq(S\setminus E)\cup F$ but $\mathcal{B}\big(g(s,E)\big)\cap F=\varnothing$, that is, $\mathcal{B}\big(g(s,E)\big)\subseteq S\setminus F$. Let $p,q\in \texttt{At}$ be atomic formulas and construct a model where $\|p\|=E$ and $\|q\|=F$. Then, since $\mathcal{B}(s)\cap\|p\|\neq\varnothing$, $\neg p\notin K_s$; furthermore, since $\mathcal{B}(s)\subseteq(S\setminus\|p\|)\cup\|q\| = \|p\to q\|$, $(p\to q)\in K_s$. However, since $\mathcal{B}\big(g(s,\|p\|)\big)\subseteq S\setminus\|q\|) = \|\neg q\|$, $\neg q\in K_s\odot p$, yielding a violation of $(K\odot 4)$.

Next we show that $(A\odot 4)$ is characterized by $(P\odot 4)$. Fix an arbitrary model based on a frame that satisfies $(P\odot 4)$, arbitrary state s and formulas $\phi,\psi\in\Phi_0$ and suppose that $s\models\neg B\neg\phi\wedge B(\phi\to\psi)$, that is, $\mathcal{B}(s)\cap\|\phi\|\neq\varnothing$ and $\mathcal{B}(s)\subseteq(S\setminus\|\phi\|)\cup\|\psi\|$. Then, by $(P\odot 4)$, $\mathcal{B}\big(g(s,\|\phi\|)\big)\cap\|\psi\|\neq\varnothing$, that is, $g(s,\|\phi\|)\models\neg B\neg\psi$ and thus $s\models\phi>\neg B\neg\psi$. Conversely, consider a frame that violates $(P\odot 4)$. Then there exist $s\in S$ and $E,F\in 2^S\setminus\varnothing$ such that $\mathcal{B}(s)\cap E\neq\varnothing$ and $\mathcal{B}(s)\subseteq(S\setminus E)\cup F$ but $\mathcal{B}\big(g(s,E)\big)\cap F=\varnothing$, that is, $\mathcal{B}\big(g(s,E)\big)\subseteq S\setminus F$. Let $p,q\in\texttt{At}$ be atomic formulas and construct a model where $\|p\|=E$ and $\|q\|=F$. Then, since $\mathcal{B}(s)\cap\|p\|\neq\varnothing$, $s\models\neg B\neg p$; furthermore, since $\mathcal{B}(s)\subseteq(S\setminus\|p\|)\cup\|q\|=\|p\to q\|$, $s\models B(p\to q)$. However, since $\mathcal{B}\big(g(s,\|p\|)\big)\subseteq S\setminus\|q\|) = \|\neg q\|$, $g(s,\|p\|)\models B\neg q$ so that $g(s,\|p\|)\not\models\neg B\neg q$ and thus $s\not\models p>\neg B\neg q$, yielding a violation of $(A\odot 4)$. □

Remark 1. *Concerning $(K\odot 5a)$, note that if ϕ is a contradiction then, in any model, $\|\phi\|=\varnothing$ and thus ϕ is not in the domain of the function $\odot$ defined at an arbitrary state*

s by ($RI^{\odot}$). Thus one can extend any such partial belief function by postulating that if ϕ is a contradiction then $K_s \odot \phi = \Phi_0$.
Concerning ($A \odot 5a$), if ϕ is a contradiction then, in any model, $\|\phi\| = \varnothing$ and thus by the validation rule, for every formula ψ and every state s, $s \models \phi > B\psi$. This fact is reflected in the rule of inference ($R \odot 5a$).

Remark 2. *Concerning ($K \odot 5b$), fix an arbitrary $\phi \in \Phi_0$ which is not a contradiction. Fix an arbitrary model and an arbitrary state s and let $\odot$ be the partial belief function defined by ($RI^{\odot}$). If, in that model, $\|\phi\| \neq \varnothing$ then $\|\phi\|$ is in the domain of $\odot$ and $g(s, \|\phi\|)$ is well defined. By seriality of $\mathcal{B}$, $\mathcal{B}\big(g(s, \|\phi\|)\big) \neq \varnothing$. Let p be an atomic formula; then $\|p \wedge \neg p\| = \varnothing$ and thus $\mathcal{B}\big(g(s, \|\phi\|)\big) \nsubseteq \|p \wedge \neg p\|$ so that $(p \wedge \neg p) \notin K_s \odot \phi$ and thus $K_s \odot \phi \neq \Phi_0$. On the other hand, if in that model $\|\phi\| = \varnothing$ then ϕ is not in the domain of $\odot$.*
Next we prove validity of ($A \odot 5b$) on every frame. Fix an arbitrary model, an arbitrary state s and arbitrary $\phi, \psi \in \Phi_0$ and suppose that $s \models \neg\Box\neg\phi \wedge (\phi > B\psi)$. Since $s \models \neg\Box\neg\phi$, $\|\phi\| \neq \varnothing$ and thus $g(s, \|\phi\|)$ is well defined. Since $s \models \phi > B\psi$, $g(s, \|\phi\|) \models B\psi$. By axiom D_B (reflecting seriality of $\mathcal{B}$), $g(s, \|\phi\|) \models (B\psi \to \neg B\neg\psi)$ so that $g(s, \|\phi\|) \models \neg B\neg\psi$ and thus $s \models \phi > \neg B\neg\psi$.

Remark 3. *The validity of ($K \odot 6$) and ($A \odot 6$) on every frame is a trivial consequence of the fact that if $\phi \leftrightarrow \psi$ is a tautology then, in every model, $\|\phi \leftrightarrow \psi\| = S$ which is equivalent to $\|\phi\| = \|\psi\|$.*

Proposition 5. *Belief function property:*

$$(K \odot 7) \qquad K \odot \phi \,\cap\, K \odot \psi \subseteq K \odot (\phi \vee \psi)$$

and modal formula

$$(A \odot 7) \qquad \big(\neg\Box\neg\phi \wedge \neg\Box\neg\psi \wedge (\phi > B\chi) \wedge (\psi > B\chi)\big) \to (\phi \vee \psi) > B\chi$$

are characterized by KLS frame property

$$(P \odot 7) \qquad \forall s \in S, \forall E, F \in 2^S,\ \mathcal{B}\big(g(s, E \cup F)\big) \subseteq \mathcal{B}\big(g(s, E)\big) \cup \mathcal{B}\big(g(s, F)\big)$$

Proof. First we prove that ($K \odot 7$) is characterized by ($P \odot 7$). Fix a frame that satisfies property ($P \odot 7$), an arbitrary model based on it, an arbitrary state $s \in S$ and let $\odot$ be the partial belief change function based on K_s defined by ($RI^{\odot}$). Let

$\phi, \psi \in \Phi_0$ be in the domain of $\odot$ (thus $\|\phi\| \neq \varnothing$ and $\|\psi\| \neq \varnothing$) and fix an arbitrary $\chi \in (K_s \odot \phi) \cap (K_s \odot \psi)$. Then, by ($RI^{\odot}$), $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\chi\|$ and $\mathcal{B}\big(g(s, \|\psi\|)\big) \subseteq \|\chi\|$ and thus, by Property ($P \odot 7$) (with $E = \|\phi\|$ and $F = \|\psi\|$) and the fact that $\|\phi\| \cup \|\psi\| = \|\phi \vee \psi\|$, $\mathcal{B}\big(g(s, \|\phi \vee \psi\|)\big) \subseteq \|\chi\|$, that is, by ($RI^{\odot}$), $\chi \in K_s \odot (\phi \vee \psi)$. Conversely, fix a frame that violates property ($P \odot 7$). Then there exist $s \in S$ and $E, F \in 2^S \setminus \varnothing$ such that

$$\mathcal{B}\big(g(s, E \cup F\big) \not\subseteq \mathcal{B}\big(g(s, E)\big) \cup \mathcal{B}\big(g(s, F)\big) \tag{4}$$

Let $p, q, r \in \texttt{At}$ be atomic formulas and construct a model where $\|p\| = E$, $\|q\| = F$ and $\|r\| = \mathcal{B}\big(g(s, \|\phi\|)\big) \cup \mathcal{B}\big(g(s, \psi\|)\big)$. Then, by (4) (since $\|p\| \cup \|q\| = \|p \vee q\|$), $\mathcal{B}\big(g(s, \|p \vee q\|)\big) \not\subseteq \|r\|$ and thus, by ($RI^{\odot}$), $r \notin K_s \odot (p \vee q)$. On the other hand, since $\mathcal{B}\big(g(s, \|p\|)\big) \subseteq \|r\|$ and $\mathcal{B}\big(g(s, \|q\|)\big) \subseteq \|r\|$, $r \in K_s \odot p$ and $r \in K_s \odot q$, yielding a violation of ($K \odot 7$).

Next we show that ($A \odot 7$) is characterized by ($P \odot 7$). Fix a frame that satisfies property ($P \odot 7$), an arbitrary model based on it, an arbitrary state $s \in S$ and arbitrary $\phi, \psi, \chi \in \Phi_0$ and assume that $s \models \neg\Box\neg\phi \wedge \neg\Box\neg\psi \wedge \phi > B\chi \wedge \psi > B\chi$. Then $\|\phi\| \neq \varnothing$, $\|\psi\| \neq \varnothing$, $\mathcal{B}\big(g(s, \|\phi\|)\big) \subseteq \|\chi\|$ and $\mathcal{B}\big(g(s, \|\psi\|)\big) \subseteq \|\chi\|$. Thus, by ($P \odot 7$) (with $E = \|\phi\|$ and $F = \|\psi\|$ and using the fact that $\|\phi\| \cup \|\psi\| = \|\phi \vee \psi\|$), $\mathcal{B}\big(g(s, \|\phi \vee \psi\|)\big) \subseteq \|\chi\|$, so that $s \models (\phi \vee \psi) > B\chi$. Conversely, fix a frame that violates property ($P \odot 7$). Then there exist $s, \in S$ and $E, F \in 2^S \setminus \varnothing$ such that (4) holds. Let $p, q, r \in \texttt{At}$ be atomic formulas and construct a model where $\|p\| = E$, $\|q\| = F$ and $\|r\| = \mathcal{B}\big(g(s, \|\phi\|)\big) \cup \mathcal{B}\big(g(s, \psi\|)\big)$. Then, since $\|p\| \neq \varnothing$ and $\|q\| \neq \varnothing$, $s \models \neg\Box\neg p \wedge \neg\Box\neg q$; furthermore, $s \models (p > Br) \wedge (q > Br)$. On the other hand (noting that $\|p\| \cup \|q\| = \|p \vee q\|$), since $\mathcal{B}\big(g(s, \|p \vee q\|)\big) \not\subseteq \|r\|$, $s \not\models (p \vee q) > Br$, yielding a violation of axiom ($A \odot 7$). $\square$

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