INFORMATION, KNOWLEDGE AND BELIEF

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April 2000, revised October 2000

Keywords: information, signals, knowledge, belief, introspection, veridicality, updating.

Abstract

We model *information* as possibilities consistent with signals received from the environment. *Knowledge* is obtained by reasoning about the signals received as well as those that might have been received but were not. We use the term 'knowledge' to refer to those beliefs that are obtained by reasoning about the available information and *nothing else*. That is, one ought to be able to *fully* justify what one knows by means of the information that is available to her. We use the term 'belief' to refer to those beliefs that are based on information but not necessarily *only* on information. We investigate the relationship between information, knowledge and belief, as well as the issue of updating knowledge and belief in response to changes in information.

*An earlier version of this paper was presented at the Fourth Conference on Logic and the Foundations of Game and Decision Theory (LOFT4), Turin, July 2000. I benefited from comments by the conference participants, in particular Pierpaolo Battigalli, Johan van Benthem and Joe Halpern. I would also like to express my gratitude to an anonymous referee for several helpful suggestions.

1 Introduction

Starting with Hintikka's (1962) pioneering contribution, the notions of knowledge and belief have been studied extensively in the literature. The distinguishing feature of knowledge is the veridicality property: while it is possible for an individual to mistakenly believe something which is false, only true facts can be known. The literature has mainly focused on the notions of *rational* belief (or knowledge) as captured by the introspection properties. *Positive introspection* is the property that if the individual believes A then she believes that she believes A, while *negative introspection* requires that if the individual does not believe A then she believe that she does not believe A. Often, and especially in the economics literature, knowledge and belief are modeled together and their interaction is captured by two more rationality postulates: (1) beliefs must be compatible with knowledge, in the sense that whatever is known is also believed, and (2) the individual knows her own beliefs, in the sense that if she believes A then she knows that she believes A.

If beliefs are derived from knowledge, what is the source of one's knowledge? *Changes* in knowledge are typically thought of as prompted by the discovery of new evidence or the receipt of new information. Indeed, in the economics literature it is common to *identify* knowledge with information: one postulates an initial "information partition" and the receipt of new information is modeled as a refinement of this partition.

The objective of this paper is to explicitly model information and the process of deriving knowledge from information. We model *information* as possibilities consistent with signals received from the environment. *Knowledge* is obtained by reasoning about the signals received as well as those that are missing, that is, by eliminating those states that would have yielded further signals. This can be illustrated with the following example, adapted from Conan Doyle's *Silver Blaze* mystery. In the dead of night, someone removed the horse Silver Blaze from the stable in which he was kept. Footprints found outside the stable match those of two individuals, who therefore become the primary suspects. During the investigation, Scotland Yard Inspector Gregory asks Sherlock Holmes (Baring-Gould, 1967, Vol. 2, p. 277):

> Gregory: "Is there any other point to which you would wish to draw my attention?"

Holmes: "To the curious incident of the dog in the night-time."

Gregory: "The dog did nothing in the night-time."

Holmes: "That was the curious incident."

Holmes deduces from the fact that the dog did not bark (absence of a signal) that the thief must have been known to the dog and is therefore able to eliminate one of the two suspects on these grounds.[1]

We use the term 'knowledge' to refer to those beliefs that are obtained by reasoning about the available information and *nothing else*. That is, one ought to be able to *fully* justify what one knows by means of the information that is available to her. Thus two rational individuals who were exposed to the same information would necessarily end up sharing the same knowledge. We use the term 'belief' to refer to those beliefs that go beyond the information received and therefore involve an element of arbitrariness: they are based on information but not *only* on information. Beliefs might reflect some extra elements such as intuition, a hunch, a guess, etc. We investigate the relationship between information, knowledge and belief, as well as the issue of updating knowledge and belief in response to changes in information.

It is worth noting that, since our account of knowledge is one of "belief entirely based on information", knowledge need not satisfy the veridicality axiom (if the individual knows A then A is true). If information is accurate, it will. But, unfortunately, we often trust information which is incorrect; in such cases it is possible for us to mistakenly claim that we know something which, as a matter of fact, happens to be false.[2]

[1] This Sherlock Holmes example has been exploited by several authors, in different contexts. Dawid and Dickey (1977) use it to make a similar point to ours in the context of face-value likelihood and Bayesian inference. They stress the fact that the way in which probabilities should be updated after receipt of information may depend on what information might have been received, but wasn't. Dekel et al. (1998) use it in the context of how to model the notion of unawareness (see also Geanakoplos, 1989). It is important to stress that our interpretation of the Sherlock Holmes example *has nothing to do with unawareness*: we take it that both Sherlock Holmes and Inspector Gregory were aware of the fact that there was a dog in the stable and that the dog did nothing. The difference between them is that Sherlock Holmes, unlike Gregory, infers something from the absence of a potential signal (the dog barking) whereas Gregory only pays attention to the actual (or positive) signals (the footprints). What we mean by 'signal' is further explained in Section 2.

[2] A clear illustration of this can be found in a recent newspaper article (*The Sacramento Bee*, September 1, 2000): "Mark J. made a big bet in mid-August that Emulex shares

The paper is organized as follows. Section 2 deals with the notion of information as signals, while section 3 models knowledge as belief rationally arrived at on the basis of, and only of, information. Section 4 contains some remarks about the syntactic approach. Section 5 deals with the interaction of knowledge and belief. In Section 6 we discuss how knowledge and beliefs change in response to the receipt of new information. Section 7 concludes.

2 Information as signals

We will think of information as possibilities associated (or consistent) with signals received from the environment. Let Ω be a set of *states* and $\Omega_K \subseteq \Omega$ the set of *known* states. For example, Ω could be the set of all diseases and Ω_K the set of known diseases (e.g. the set of diseases in some database). Let Σ be a set of *signals* and $\sigma : \Omega \to 2^\Sigma$ (where 2^Σ denotes the set of subsets of Σ) be a function that associates with every state ω the set of signals produced by ω. In the example where states are identified with diseases, signals can be thought of as symptoms so that $\sigma(\omega)$ is the set of symptoms associated with disease ω.

It is worth stressing that by 'signal' we mean something that *alters the physical environment*, so that there is an objectively measurable difference between a situation where the signal is present and a situation where it absent. Thus a sound, a symptom, the presence of a physical clue such as a footprint, etc., are all signals. On the other hand, in our terminology, the *absence* of a signal (e.g. the dog not barking) is *not* itself a signal. This allows us to think of information as signals received and to represent knowledge as inference based on the signals that are present as well as those that are absent.[3]

would decline, federal prosecutors say. Instead they soared, leaving him with a paper loss of almost $100,000 in just a week. So J. took matters into his own hands. [...] On the evening of August 24, he sent a fake press release by e-mail to Internet Wire, a Los Angeles service where he had previously worked, warning that Emulex's chief executive had resigned and its earnings were overstated. The next morning, just as financial markets opened, Internet Wire distributed the damaging release to news organizations and Web sites. An hour later, shareholders in Emulex were $2.5 billion poorer. And J. would soon be $240,000 richer. [...] The hoax [...] was revealed within an hour of the first news report and Emulex stock recovered the same day. Still, investors who [believed the fake news release and] panicked and sold their shares, or had sell orders automatically executed at present prices, are unlikely to recover their losses".

[3] Thus we think of the individual as being aware of all the potential signals (the set Σ)

Definition 1 *The information function* $\mathcal{I} : \Omega \to 2^{\Omega}$ *is given by:*

$$\mathcal{I}(\omega) = \{\omega' \in \Omega_K : \sigma(\omega') \supseteq \sigma(\omega)\}.$$

Thus $\mathcal{I}(\omega)$ is the set of known states that are compatible with the signals produced by the true state ω, in the sense that those states would also have produced those signals (although they might have more signals associated with them). For example, one can imagine a database of known diseases and their associated symptoms and a (admittedly not very sophisticated) computer program which receives as input a patient's symptoms and gives as output the list of diseases in the database that manifest all those symptoms (although possibly more). The next step for a careful doctor would be to research each of the reported possible diseases and eliminate as true possibilities all those diseases that had extra symptoms not exhibited by the patient.[4] This step, corresponding to the notion of knowledge derived from information, will be the object of the next section. First we investigate the properties satisfied by the information function.

Remark 2 *The structures considered in this paper (which are common in the game theory and economics literature) consist of a set of states* Ω *and one or more functions (denoted by* $\mathcal{I}$ *for information,* $\mathcal{K}$ *for knowledge and* $\mathcal{B}$ *for belief) from* Ω *to the set of subsets of* Ω*. These structures are equivalent to Kripke structures (commonly used in computer science and philosophy) where, instead of functions, one postulates binary relations on* Ω*. Given a function* $F : \Omega \to 2^{\Omega}$*, the corresponding binary relation* R *is obtained as follows:* $\omega R \omega'$ *if and only if* $\omega' \in F(\omega)$*. Conversely, given a binary relation* R*, the corresponding function* $F : \Omega \to 2^{\Omega}$ *is defined by:* $\omega' \in F(\omega)$ *if and only if* $\omega R \omega'$*.*

Remark 3 *The information function does not necessarily satisfy reflexivity* $(\omega \in \mathcal{I}(\omega))$ *nor euclideanness (if* $\omega' \in \mathcal{I}(\omega)$ *then* $\mathcal{I}(\omega) \subseteq \mathcal{I}(\omega')$*). The possible lack of reflexivity is illustrated by the following example:* $\Omega = \{\alpha, \beta\}$*,* $\Omega_K = \{\beta\}$*,* $\Sigma = \{s\}$*,* $\sigma(\alpha) = \sigma(\beta) = \{s\}$*. Then* $\mathcal{I}(\alpha) = \{\beta\}$*. A possible interpretation is:* α *is the true, and as yet unknown, disease and it displays the*

as well as the relationship between states and signals (the function $\sigma : \Omega \to 2^{\Sigma}$).

[4] A referee pointed out that in the field of diagnosis further complexity arises in the presence of combinations of diseases with interacting symptoms, e.g. symptons that cancel each other out (see, for example, Stefik, 1995).

same symptom as the known disease β. Thus one would be led to mistakenly diagnose the patient as having disease β. The possible lack of euclideanness is shown in the following example: $\Omega = \Omega_K = \{\alpha, \beta\}$, $\Sigma = \{s_1, s_2\}$, $\sigma(\alpha) = \{s_1\}$, $\sigma(\beta) = \{s_1, s_2\}$. Then $\mathcal{I}(\alpha) = \{\alpha, \beta\}$ and $\mathcal{I}(\beta) = \{\beta\}$. A possible interpretation is: α and β are the two possible suspects whose footprints match the ones found near the stable (signal s_1). The actual thief is α who is known to the dog. The dog did not bark; he would have (signal s_2) if the thief had been β.

Proposition 4 *The information function satisfies the following properties:*

(1) Secondary reflexivity: $\forall \alpha, \beta \in \Omega$, if $\beta \in \mathcal{I}(\alpha)$ then $\beta \in \mathcal{I}(\beta)$.

(2) Transitivity: $\forall \alpha, \beta \in \Omega$, if $\beta \in \mathcal{I}(\alpha)$ then $\mathcal{I}(\beta) \subseteq \mathcal{I}(\alpha)$.

Proof. (1) If $\beta \in \mathcal{I}(\alpha)$ then $\beta \in \Omega_K$. Thus, since $\sigma(\beta) \supseteq \sigma(\beta)$, $\beta \in \mathcal{I}(\beta)$.

(2) Let $\beta \in \mathcal{I}(\alpha)$ and choose an arbitrary $\gamma \in \mathcal{I}(\beta)$. Then $\sigma(\beta) \supseteq \sigma(\alpha)$, $\gamma \in \Omega_K$ and $\sigma(\gamma) \supseteq \sigma(\beta)$. Thus $\sigma(\gamma) \supseteq \sigma(\alpha)$ and hence $\gamma \in \mathcal{I}(\alpha)$. ■

The following proposition shows that every information function which is secondary reflexive and transitive can be thought of as possibilities arising from signals.

Proposition 5 *Let $\mathcal{I} : \Omega \to 2^{\Omega}$ be secondary reflexive and transitive. Then there exists a subset Ω_K of Ω, a set Σ and a function $\sigma : \Omega \to 2^{\Sigma}$ such that $\mathcal{I}(\omega) = \{\omega' \in \Omega_K : \sigma(\omega') \supseteq \sigma(\omega)\}$.*

Proof. To simplify the notation, we shall give a proof for the case where Ω is countable.[5] Let $\{\omega_1, \omega_2, ...\}$ be an enumeration of Ω. Let $\Omega_K = \{\omega \in \Omega : \omega \in \mathcal{I}(\omega') \text{ for some } \omega' \in \Omega\}$. Let $\Sigma = \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers) and define $\sigma : \Omega \to 2^{\mathbb{N}}$ as follows: $\sigma(\omega_i) = \{i\} \cup \{j \in \mathbb{N} : \omega_i \in \mathcal{I}(\omega_j)\}$. First we show that if $\omega_i \in \mathcal{I}(\omega_j)$ (hence $\omega_i \in \Omega_K$) then $\sigma(\omega_i) \supseteq \sigma(\omega_j)$. By definition of σ, $j \in \sigma(\omega_i)$. Two cases are possible: (1) $\sigma(\omega_j) = \{j\}$ and (2) $\sigma(\omega_j) \supset \{j\}$. In case (1) there is nothing to prove, since $j \in \sigma(\omega_i)$. In case (2) fix an arbitrary $n \in \sigma(\omega_j)$ with $n \neq j$. By definition of σ, $\omega_j \in \mathcal{I}(\omega_n)$. This, in conjunction with $\omega_i \in \mathcal{I}(\omega_j)$, yields, by transitivity of $\mathcal{I}$, $\omega_i \in \mathcal{I}(\omega_n)$. Hence, by definition of σ, $n \in \sigma(\omega_i)$. Next we show that if

[5] The proof for the case where Ω is not countable is essentially the same, although notationally more complex. First choose a set of signals Σ with cardinality at least that of Ω and let $i : \Omega \to \Sigma$ be a one-to-one index function. Then define $\sigma(\omega) = \{i(\omega)\} \cup \{i(\omega') : \omega \in \mathcal{I}(\omega')\}$. The proof then becomes a straightforward adaptation of the one given here.

$\omega_i \in \Omega_K$ and $\sigma(\omega_i) \supseteq \sigma(\omega_j)$ then $\omega_i \in \mathcal{I}(\omega_j)$. Two cases are possible: (1) $i = j$, and (2) $i \neq j$. In case (1), since $\omega_i \in \Omega_K$ there exists an ω_n such that $\omega_i \in \mathcal{I}(\omega_n)$; hence, by secondary reflexivity of $\mathcal{I}$, $\omega_i \in \mathcal{I}(\omega_i)$. Now consider case (2). By definition of σ, $j \in \sigma(\omega_j)$. Thus, since $\sigma(\omega_i) \supseteq \sigma(\omega_j)$, $j \in \sigma(\omega_i)$. Hence, by definition of σ, $\omega_i \in \mathcal{I}(\omega_j)$. ■

Because of Proposition 5, from now on we shall take information functions as primitives and refer to the underlying interpretation in terms of signals only when needed to clarify matters.

3 Deriving knowledge from information

As the Sherlock Holmes example discussed in the Introduction shows, if information is thought of as possibilities consistent with signals received, a rational individual forming beliefs based on the available information needs to reason about not only the signals received but also those that might have been received and were not. The Sherlock Holmes example can be illustrated in the formalism of information functions as follows (for reasons that will become clear later on, we increase the number of suspects from two to three). Let $\Omega = \{\alpha, \beta, \gamma\}$, where each state identifies a particular suspect, and let $\mathcal{I}(\alpha) = \mathcal{I}(\beta) = \{\alpha, \beta, \gamma\}$ and $\mathcal{I}(\gamma) = \{\gamma\}$.[6] This is illustrated in Figure 1 where there is an arrow from ω to ω' if and only if $\omega' \in \mathcal{I}(\omega)$.

An alternative interpretation of this example is the following: a patient reports headache and vomiting to the doctor. In the doctor's experience, these symptoms arise in case of food poisoning (α), stomach flu (β) and meningitis (γ). Suppose that, as a matter of fact, the patient suffers from food poisoning (α). Based on the symptoms, the possible illnesses are these three: $\mathcal{I}(\alpha) = \{\alpha, \beta, \gamma\}$. However, further research or better recollection of past cases, would reveal to the doctor that meningitis also has neck pain as a symptom, leading to $\mathcal{I}(\gamma) = \{\gamma\}$. Thus a careful doctor, noticing (and confirming) the absence of neck pain, would have to narrow down the possibilities to food poisoning and stomach flu. This inferential step corresponds

[6] All three are suspects because of the signal (clue) represented by the footprints (they match those of each of them). The dog did not bark. Suspects α and β are known to the dog, while γ is not. If γ had been the thief, a further signal would have been observed, namely the dog's barking. Formally, $\Omega_K = \Omega$, $\Sigma = \{f, b\}$, where 'f ' stands for 'footprints' and 'b' for 'barking', and $\sigma : \Omega \to \Sigma$ is given by $\sigma(\alpha) = \sigma(\beta) = \{f\}$, $\sigma(\gamma) = \{f, b\}$. Using Definition 1 one obtains the above information function.

to the derivation of knowledge from information.

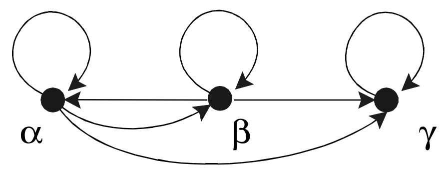

Figure 1

Let $\mathcal{K} : \Omega \to 2^{\Omega}$ be the *knowledge function*. $\mathcal{K}(\omega)$ is interpreted as the set of states that, based on what he knows, the individual cannot rule out when the true state is ω. We impose the following requirements:

(1) knowledge should be based on the information received,

(2) knowledge should reflect reasoning about the information, and

(3) knowledge should be derived *exclusively* from the available information.

The first requirement is captured by the following condition:

$$\forall \omega \in \Omega, \quad \mathcal{K}(\omega) \subseteq \mathcal{I}(\omega) \tag{R.1}$$

which says that, according to the individual's knowledge, a state is possible only if it is one of the states compatible with the information received.

An individual who is wondering whether a particular state suggested by the available information is a 'true possibility' should ask herself "if that state were indeed the true state, what information would I be looking at?". If the answer to that question is "information different from the one I actually have", then she should rule out that state. Thus the second requirement is captured by the following condition:

$$if \;\; \omega' \in \mathcal{K}(\omega) \;\; then \;\; \mathcal{I}(\omega') = \mathcal{I}(\omega). \tag{R.2}$$

Applying (R.2) to the example of Figure 1, we get that $\gamma \notin \mathcal{K}(\alpha)$, since $\mathcal{I}(\gamma) \neq \mathcal{I}(\alpha)$. This reflects Sherlock Holmes' reasoning that the thief cannot be γ because more signals would have been available (the dog's barking) if he had been. However, (R.1) and (R.2) together are consistent with, for example, $\mathcal{K}(\alpha) = \{\beta\}$, that is, with Sherlock Holmes' concluding that the thief must have been β. This seems to be an arbitrary conclusion, not justified by the available information. We want knowledge to reflect *all and only* the available information. This requirement is captured by the following property, which requires that every state that is consistent with the information and would have produced the same information be considered possible:

$$if\ \omega' \in \mathcal{I}(\omega)\ and\ \mathcal{I}(\omega') = \mathcal{I}(\omega)\ \ then\ \ \omega' \in \mathcal{K}(\omega). \tag{R.3}$$

Applying (R.1)-(R.3) to the above example we get: $\mathcal{K}(\alpha) = \mathcal{K}(\beta) = \{\alpha, \beta\}$ and $\mathcal{K}(\gamma) = \{\gamma\}$.

The following proposition lists two important properties of the knowledge function.

Proposition 6 *Let $\mathcal{K}$ be a knowledge function derived from information according to (R.1)-(R.3). Then $\mathcal{K}$ satisfies the following properties:*

(1) Transitivity: $\forall \alpha, \beta \in \Omega$, if $\beta \in \mathcal{K}(\alpha)$ then $\mathcal{K}(\beta) \subseteq \mathcal{K}(\alpha)$;

(2) Euclideanness: $\forall \alpha, \beta \in \Omega$, if $\beta \in \mathcal{K}(\alpha)$ then $\mathcal{K}(\alpha) \subseteq \mathcal{K}(\beta)$.

Proof. (1) Let $\beta \in \mathcal{K}(\alpha)$ and choose an arbitrary $\gamma \in \mathcal{K}(\beta)$. By (R.2), $\mathcal{I}(\alpha) = \mathcal{I}(\beta)$ and $\mathcal{I}(\beta) = \mathcal{I}(\gamma)$. Thus $\mathcal{I}(\gamma) = \mathcal{I}(\alpha)$. By (R.1), $\beta \in \mathcal{I}(\alpha)$ and $\gamma \in \mathcal{I}(\beta)$; hence, by transitivity of $\mathcal{I}$ (cf. Proposition 4), $\gamma \in \mathcal{I}(\alpha)$. By (R.3), this, together with $\mathcal{I}(\gamma) = \mathcal{I}(\alpha)$, yields $\gamma \in \mathcal{K}(\alpha)$.

(2) Let $\beta \in \mathcal{K}(\alpha)$ and choose an arbitrary $\gamma \in \mathcal{K}(\alpha)$. By (R.1), $\gamma \in \mathcal{I}(\alpha)$. By secondary reflexivity of $\mathcal{I}$ (cf. Proposition 4), $\gamma \in \mathcal{I}(\gamma)$. By (R.2) $\mathcal{I}(\alpha) = \mathcal{I}(\beta)$ and $\mathcal{I}(\alpha) = \mathcal{I}(\gamma)$. Thus $\mathcal{I}(\beta) = \mathcal{I}(\gamma)$ and therefore $\gamma \in \mathcal{I}(\beta)$. Hence, by (R.3), $\gamma \in \mathcal{K}(\beta)$. ■

Thus, in the analysis of this paper, the introspective properties of knowledge are not postulated as primitives but are *derived* properties.

Remark 7 *Since $\mathcal{I}$ in general is not reflexive (cf. Remark 3) it is clear that $\mathcal{K}$ need not be reflexive. For example, if $\Omega = \{\alpha, \beta\}$ and $\mathcal{I}(\alpha) = \mathcal{I}(\beta) = \{\beta\}$, then $\mathcal{K}(\alpha) = \{\beta\}$. Thus it is possible for somebody to "know" something which is false (as illustrated in Footnote 2). This, however, happens only if the information received is inaccurate, that is, only if $\mathcal{I}$ is not reflexive. It is an immediate consequence of (R.3) that if $\mathcal{I}$ is reflexive, then so is $\mathcal{K}$.*

Remark 8 *Another property which in general is not satisfied by* $\mathcal{K}$ *is seriality* ($\forall\omega \in \Omega$, $\mathcal{K}(\omega) \neq \emptyset$)*, as the following example shows:* $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{I}(\alpha) = \{\beta, \gamma\}$, $\mathcal{I}(\beta) = \{\beta\}$ *and* $\mathcal{I}(\gamma) = \{\gamma\}$. *By (R.1),* $\mathcal{K}(\alpha) \subseteq \{\beta, \gamma\}$ *and by (R.2)* $\beta \notin \mathcal{K}(\alpha)$ *and* $\gamma \notin \mathcal{K}(\alpha)$.[7] *In order to guarantee seriality of* $\mathcal{K}$ *one needs to require* $\mathcal{I}$ *to satisfy the following property (for which reflexivity is a sufficient condition):* [8]

$$\forall\omega \in \Omega,\ \exists\omega' \in \mathcal{I}(\omega) \text{ such that } \mathcal{I}(\omega') = \mathcal{I}(\omega). \tag{R.4}$$

By Proposition 6, knowledge satisfies the K45 logic (cf. Chellas, 1984) and if the information function satisfies (R.4) then knowledge obeys the KD45 logic. If information is always correct ($\mathcal{I}$ is reflexive) then knowledge satisfies the S5 logic, which is the logic usually identified with the notion of knowledge. Thus, when $\mathcal{I}$ is reflexive, $\mathcal{K}$ gives rise to a partition of Ω, although $\mathcal{I}$ itself need not be a partition, as the following example shows: $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{I}(\alpha) = \{\alpha, \beta, \gamma\}$, $\mathcal{I}(\beta) = \mathcal{I}(\gamma) = \{\beta, \gamma\}$. By (R.1)-(R.3) $\mathcal{K}(\alpha) = \{\alpha\}$ and $\mathcal{K}(\beta) = \mathcal{K}(\gamma) = \{\beta, \gamma\}$. Thus $\mathcal{K}$ is a partition but $\mathcal{I}$ is not (it is reflexive and transitive but not euclidean).

That (R.1)-(R.3) capture the notion of knowledge as belief based *exclusively* on information is confirmed by the fact that *when* $\mathcal{I}$ *is euclidean,* $\mathcal{K}$ *coincides with* $\mathcal{I}$.[9] Indeed, when $\mathcal{I}$ is euclidean there are no further inferences that can be drawn by reasoning about the available information and therefore knowledge and information coincide.

We conclude this section by interpreting knowledge as defined above in terms of signals. Intuitively, it seems natural to require the individual to

[7] A possible interpretation of this example is the following: α, β and γ are diseases, of which only β and γ are currently known. There are three symptoms: s_1, s_2 and s_3; α's only symptom is s_1, while β has symptoms s_1 and s_2 and γ has symptoms s_1 and s_3. If the true disease is the as yet unknown disease α, then a rational doctor would be left in a state of confusion, not having α in her database and having to rule out β because of the missing symptom s_2 and γ because of the missing symptom s_3.

[8] Because of transitivity of $\mathcal{I}$, $(R.4)$ is equivalent to the seemingly weaker condition: $\forall\omega \in \Omega$, $\exists\omega' \in \mathcal{I}(\omega)$ such that $\mathcal{I}(\omega) \subseteq \mathcal{I}(\omega')$. In terms of signals, a sufficient (but not necessary) condition for (R.4) to hold is the following: $\forall\omega \in \Omega$, $\exists\omega' \in \Omega_K$ such that $\sigma(\omega') = \sigma(\omega)$.

[9] Since $\mathcal{I}$ is transitive, euclideanness implies that if $\beta \in \mathcal{I}(\alpha)$ then $\mathcal{I}(\alpha) = \mathcal{I}(\beta)$. Thus, by (R.3), $\mathcal{I}(\alpha) \subseteq \mathcal{K}(\alpha)$. Together with (R.1) this yields $\mathcal{K}(\alpha) = \mathcal{I}(\alpha)$.

allow as possible states only those which would have produced *exactly* the signals that are actually observed.[10] The following proposition confirms this conjecture in all those situations where among the known states there is at least one with the same associated signals as the true state.

Proposition 9 *Let the information function $\mathcal{I}$ be given as in Definition 1 and let the knowledge function $\mathcal{K}$ be obtained from $\mathcal{I}$ according to (R.1)-(R.3). Then, for every $\omega \in \Omega$, if $\{\omega' \in \Omega_K : \sigma(\omega') = \sigma(\omega)\} \neq \emptyset$, then $\mathcal{K}(\omega) = \{\omega' \in \Omega_K : \sigma(\omega') = \sigma(\omega)\}$.*

Proof. Fix an arbitrary $\omega \in \Omega$ and assume that $\{\omega' \in \Omega_K : \sigma(\omega') = \sigma(\omega)\} \neq \emptyset$. First we show that if $\alpha \in \mathcal{K}(\omega)$ then $\alpha \in \Omega_K$ and $\sigma(\alpha) = \sigma(\omega)$. By (R.1), $\alpha \in \mathcal{I}(\omega)$. Thus, by Definition 1, $\alpha \in \Omega_K$ and $\sigma(\alpha) \supseteq \sigma(\omega)$. To show that $\sigma(\omega) \supseteq \sigma(\alpha)$, fix an arbitrary $\omega' \in \Omega_K$ such that $\sigma(\omega') = \sigma(\omega)$ (it exists because of our supposition). Then, by Definition 1, $\omega' \in \mathcal{I}(\omega)$. Since $\alpha \in \mathcal{K}(\omega)$, by (R.2) $\mathcal{I}(\alpha) = \mathcal{I}(\omega)$. Thus $\omega' \in \mathcal{I}(\alpha)$ and, therefore, by Definition 1, $\sigma(\omega') \supseteq \sigma(\alpha)$. Since $\sigma(\omega') = \sigma(\omega)$, it follows that $\sigma(\omega) \supseteq \sigma(\alpha)$. Next we show that if $\alpha \in \Omega_K$ and $\sigma(\alpha) = \sigma(\omega)$ then $\alpha \in \mathcal{K}(\omega)$. By Definition 1, $\alpha \in \mathcal{I}(\omega)$ and $\mathcal{I}(\alpha) = \mathcal{I}(\omega)$. Thus, by (R.3), $\alpha \in \mathcal{K}(\omega)$. ■

The condition postulated in Proposition 9, namely that there be a state in the "known database" that produces the same signals as the true state ($\{\omega' \in \Omega_K : \sigma(\omega') = \sigma(\omega)\} \neq \emptyset$) is trivially satisfied whenever $\Omega = \Omega_K$, that is, whenever all the states are in the database. It is also clear that $\Omega = \Omega_K$ if and only if both $\mathcal{I}$ and $\mathcal{K}$ are reflexive.

4 Remarks on the syntactic approach

In this section we investigate the axiomatization of some of the properties of information and knowledge introduced in the previous two sections. First some preliminaries. Consider a propositional language with two modal operators, I and K, whose intended interpretation is:

$I\phi$: "the individual is informed that ϕ".
$K\phi$: "the individual knows that ϕ".

[10] I am grateful to Pierpaolo Battigalli for making this point.

The formal language is built in the familiar way from the following components: a countable set S of sentence letters (representing atomic propositions), the connectives $\neg$ and $\vee$ (from which the other connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators I and K.[11]

A *frame* is a triple $\langle \Omega, \mathcal{I}, \mathcal{K} \rangle$, where $\mathcal{I} : \Omega \rightarrow 2^{\Omega}$ and $\mathcal{K} : \Omega \rightarrow 2^{\Omega}$. Given a frame, one obtains a *model based on it* by adding a function $V : S \rightarrow 2^{\Omega}$ that associates with every sentence letter p the set of states at which p is true. For non-modal formulae, truth at a state in a model is defined as usual.[12] Validation for modal formulae is as follows:

$\omega \models I\phi$ iff $\omega' \models \phi$ for all ω' such that $\omega' \in \mathcal{I}(\omega)$

$\omega \models K\phi$ iff $\omega' \models \phi$ for all ω' such that $\omega' \in \mathcal{K}(\omega)$.

Given a model, we denote by $||\phi||$ the *truth set* of formula ϕ, that is, $||\phi|| = \{\omega \in \Omega : \omega \models \phi\}$. A formula ϕ is *valid in a model* if $||\phi|| = \Omega$; it is *valid in a frame* if it is valid in every model based on it.

We say that an axiom *corresponds to* (or *characterizes*) a property of a frame if the property is necessary and sufficient for the axiom to be valid in the frame. For example, as is well known (cf. Chellas, 1984), the axiom $I(I\phi \rightarrow \phi)$ characterizes secondary reflexivity of $\mathcal{I}$ while $I\phi \rightarrow II\phi$ corresponds to transitivity of $\mathcal{I}$. Property (R.1) corresponds to the axiom $I\phi \rightarrow K\phi$ and, as the next proposition shows, (R.2) corresponds to the conjunction of the following axioms:

$$I\phi \rightarrow KI\phi \tag{A.2a}$$

and

$$\neg I\phi \rightarrow K\neg I\phi. \tag{A.2b}$$

[11] The set Φ of formulae is thus obtained from the sentence letters by closing with respect to negation, disjunction and the modal operators: (i) for every $p \in S$, $(p) \in \Phi$, (ii) if $\phi, \psi \in \Phi$ then all of the following belong to Φ: $(\neg\phi)$, $(\phi \vee \psi)$, $I\phi$ and $K\phi$.

[12] Given a model, we write $\omega \models \phi$ to denote that ϕ *is true at state* ω and $\omega \not\models \phi$ to denote that ϕ is false at ω. If p is a sentence letter, $\omega \models p$ iff $\omega \in V(p)$; furthermore, $\omega \models \neg\phi$ iff $\omega \not\models \phi$ and $\omega \models (\phi \vee \psi)$ iff either $\omega \models \phi$ or $\omega \models \psi$. It follows that $\omega \models (\phi \wedge \psi)$ iff $\omega \models \phi$ and $\omega \models \psi$, and $\omega \models (\phi \rightarrow \psi)$ iff $\omega \models \psi$ whenever $\omega \models \phi$.

Proposition 10 *The conjunction of (A.2a) and (A.2b) characterizes (R.2), that is, if a frame satisfies (R.2) then (A.2a) and (A.2b) are valid in it and, conversely, if (A.2a) and (A.2b) are valid in a frame then the frame satisfies (R.2).*

Proof. (R.2) is equivalent to the conjunction of (R.2a) and (R.2b) below:
(R.2a) if $\beta \in \mathcal{K}(\alpha)$ then $\mathcal{I}(\beta) \subseteq \mathcal{I}(\alpha)$
(R.2b) if $\beta \in \mathcal{K}(\alpha)$ then $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta)$.
First we show that (A.2a) characterizes (R.2a). Fix a frame that satisfies (R.2a), arbitrary $\alpha \in \Omega$ and formula ϕ and suppose that $\alpha \models I\phi$, that is, $\mathcal{I}(\alpha) \subseteq ||\phi||$ (recall that $||\phi||$ denotes the truth set of ϕ). Fix an arbitrary $\beta \in \mathcal{K}(\alpha)$. By (R.2a), $\mathcal{I}(\beta) \subseteq \mathcal{I}(\alpha)$ and, therefore, $\mathcal{I}(\beta) \subseteq ||\phi||$, that is, $\beta \models I\phi$. Hence $\alpha \models KI\phi$. Conversely, take a frame that does not satisfy (R.2a). Then there exist α, β, γ such that $\beta \in \mathcal{K}(\alpha)$, $\gamma \in \mathcal{I}(\beta)$ and $\gamma \notin \mathcal{I}(\alpha)$. Construct a model based on this frame where, for some atomic sentence p, $\|p\| = \mathcal{I}(\alpha)$. Then $\alpha \models Ip$. Furthermore, since $\gamma \in \mathcal{I}(\beta)$ and $\gamma \notin \|p\|$, $\beta \models \neg Ip$; thus, since $\beta \in \mathcal{K}(\alpha)$, $\alpha \models \neg KIp$.
Next we show that (A.2b) characterizes (R.2b). Fix a frame that satisfies (R.2b), arbitrary $\alpha \in \Omega$ and formula ϕ and suppose that $\alpha \models \neg I\phi$. Then there exists a $\gamma \in \mathcal{I}(\alpha)$ such that $\gamma \not\models \phi$. Fix an arbitrary $\beta \in \mathcal{K}(\alpha)$. By (R.2b), $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta)$ and therefore $\gamma \in \mathcal{I}(\beta)$. Thus $\beta \models \neg I\phi$ and, therefore, since $\beta \in \mathcal{K}(\alpha)$ was chosen arbitrarily, $\alpha \models K\neg I\phi$. Conversely, take a frame that does not satisfy (R.2b). Then there exist α, β, γ such that $\beta \in \mathcal{K}(\alpha)$, $\gamma \in \mathcal{I}(\alpha)$ and $\gamma \notin \mathcal{I}(\beta)$. Construct a model based on this frame where, for some atomic sentence p, $\|p\| = \mathcal{I}(\beta)$. Thus $\beta \models Ip$ and, since $\beta \in \mathcal{K}(\alpha)$, $\alpha \models \neg K\neg Ip$. Furthermore, since $\gamma \in \mathcal{I}(\alpha)$ and $\gamma \notin \|p\|$, $\alpha \models \neg Ip$. ■

On the other hand – unlike (R.1) and (R.2) (and secondary reflexivity and transitivity of $\mathcal{I}$) – property (R.3) is not characterizable in the above modal language.[13]

5 Belief and knowledge

We use the term "belief" for those beliefs that, while still based on information, are not *fully* justifiable on the basis of it. Such beliefs are derived from the available information but incorporate an element of arbitrariness, e.g. intuition, a hunch, a guess, etc. Let $\mathcal{B} : \Omega \to 2^{\Omega}$ be the *belief function.*

[13] I am grateful to Johan van Benthem for pointing this out.

$\mathcal{B}(\omega)$ is interpreted as the set of states that the individual considers possible when the true state is ω. The interaction between knowledge and belief has been studied extensively in the literature (see, for example, Battigalli and Bonanno, 1997, Halpern, 1991, van der Hoek and Meyer, 1995, Kraus and Lehmann, 1988, Lenzen, 1978). In this literature the relationship between belief and knowledge is captured by the following properties (in brackets, next to each property, is the characterizing modal axiom: B denotes the belief operator associated with $\mathcal{B}$).

$$\forall \omega \in \Omega, \;\; \mathcal{B}(\omega) \subseteq \mathcal{K}(\omega) \qquad (K\phi \rightarrow B\phi) \qquad \text{(R.5)}$$

$$if\ \omega' \in \mathcal{K}(\omega)\ then\ \mathcal{B}(\omega') \subseteq \mathcal{B}(\omega) \qquad (B\phi \rightarrow KB\phi). \qquad \text{(R.6)}$$

(R.5) states that beliefs are based on knowledge, while (R.6) requires that the individual know his own beliefs. Further rationality requirements on beliefs are given by the introspection properties (positive introspection, captured by transitivity, and negative introspection, captured by euclideanness):

$$if\ \omega' \in \mathcal{B}(\omega)\ \ then\ \ \mathcal{B}(\omega') \subseteq \mathcal{B}(\omega) \qquad (B\phi \rightarrow BB\phi) \qquad \text{(R.7)}$$

$$if\ \omega' \in \mathcal{B}(\omega)\ \ then\ \ \mathcal{B}(\omega) \subseteq \mathcal{B}(\omega') \qquad (\neg B\phi \rightarrow B\neg B\phi). \qquad \text{(R.8)}$$

It is also customary to require beliefs to be consistent ($\mathcal{B}$ to be serial: $\forall \omega \in \Omega$, $\mathcal{B}(\omega) \neq \emptyset$). In our framework, however, since seriality of knowledge cannot be guaranteed (cf. Remark 7) requiring seriality of $\mathcal{B}$ would be in conflict with (R.5). Thus we can impose the weaker requirement that $\mathcal{B}$ be serial when $\mathcal{K}$ is:

$$\forall \omega \in \Omega, \;\; if\ \mathcal{K}(\omega) \neq \emptyset \;\; then\ \ \mathcal{B}(\omega) \neq \emptyset. \qquad (1)$$

Since (R.4) guarantees seriality of $\mathcal{K}$, when (R.4) is satisfied (1) is equivalent to seriality of $\mathcal{B}$.[14]

A possible belief that satisfies (R.5)-(R.8) (as well as seriality) in the example of Figure 1 of Section 3 is the following: $\mathcal{B}(\alpha) = \mathcal{B}(\beta) = \{\beta\}$,

[14] For this reason, in the list of properties given at the beginning of next section we replace (1) with seriality of $\mathcal{B}$.

$\mathcal{B}(\gamma) = \{\gamma\}$ (thus − when α is the actual thief − although on the basis of the available information α cannot be ruled out as a suspect, the investigator somehow becomes mistakenly convinced that the thief was β).

Remark 11 *It follows from (R.6)-(R.8) that if $\beta \in \mathcal{K}(\alpha)$ and $\mathcal{B}(\beta) \neq \emptyset$ then $\mathcal{B}(\beta) = \mathcal{B}(\alpha)$.*[15]

6 Updating knowledge and belief

We now turn to the intertemporal framework where information changes over time and our objective is to investigate how knowledge and belief should be updated in the face of new evidence. Let $\mathbb{N}$ be the set of natural numbers. For every $t \in \mathbb{N}$ let $\mathcal{I}_t$, $\mathcal{K}_t$ and $\mathcal{B}_t$ denote the information, knowledge and belief functions at date t. We shall assume that, for every $t \in \mathbb{N}$, $\mathcal{I}_t$, $\mathcal{K}_t$ and $\mathcal{B}_t$ satisfy properties (R.1)-(R.8) as well as seriality of $\mathcal{B}$.[16] That is, for all $t \in \mathbb{N}$ and $\omega, \omega' \in \Omega$,

$$\mathcal{K}_t(\omega) \subseteq \mathcal{I}_t(\omega) \quad \text{(R.1)}$$
$$if\ \omega' \in \mathcal{K}_t(\omega)\ then\ \mathcal{I}_t(\omega') = \mathcal{I}_t(\omega) \quad \text{(R.2)}$$
$$if\ \omega' \in \mathcal{I}_t(\omega)\ and\ \mathcal{I}_t(\omega') = \mathcal{I}_t(\omega)\ then\ \omega' \in \mathcal{K}_t(\omega) \quad \text{(R.3)}$$
$$\exists \omega'' \in \mathcal{I}_t(\omega)\ such\ that\ \mathcal{I}_t(\omega) = \mathcal{I}_t(\omega'') \quad \text{(R.4)}$$
$$\mathcal{B}_t(\omega) \subseteq \mathcal{K}_t(\omega) \quad \text{(R.5)}$$
$$if\ \omega' \in \mathcal{K}_t(\omega)\ then\ \mathcal{B}_t(\omega') \subseteq \mathcal{B}_t(\omega) \quad \text{(R.6)}$$
$$if\ \omega' \in \mathcal{B}_t(\omega)\ then\ \mathcal{B}_t(\omega') \subseteq \mathcal{B}_t(\omega) \quad \text{(R.7)}$$
$$if\ \omega' \in \mathcal{B}_t(\omega)\ then\ \mathcal{B}_t(\omega) \subseteq \mathcal{B}_t(\omega') \quad \text{(R.8)}$$
$$\mathcal{B}_t(\omega) \neq \emptyset. \quad \text{(R.9)}$$

The topic of belief revision has received considerable attention in the literature. The notion of rational belief revision is normally identified with the "conservativity principle" which states that 'When changing beliefs in

[15] **Proof.** Let $\beta \in \mathcal{K}(\alpha)$. By (R.6) $\mathcal{B}(\beta) \subseteq \mathcal{B}(\alpha)$. To show that $\mathcal{B}(\alpha) \subseteq \mathcal{B}(\beta)$, fix an arbitrary $\gamma \in \mathcal{B}(\beta)$ (it exists by our supposition). By (R.7) $\mathcal{B}(\gamma) \subseteq \mathcal{B}(\beta)$. Since $\gamma \in \mathcal{B}(\beta)$ and $\mathcal{B}(\beta) \subseteq \mathcal{B}(\alpha)$, $\gamma \in \mathcal{B}(\alpha)$. Thus, by (R.8), $\mathcal{B}(\alpha) \subseteq \mathcal{B}(\gamma)$. Hence $\mathcal{B}(\alpha) \subseteq \mathcal{B}(\beta)$. ■

[16] Some of the properties listed below are redundant, that is, they can be derived from the others. For example, (R.7) can be derived from (R.5) and (R.6). We nevertheless list them separately in order to highlight the role that each of them plays in the proofs.

response to new evidence, you should continue to believe as many of the old beliefs as possible' (Harman, 1986, p. 46). This means that if an individual gets new information which is not inconsistent with her previous beliefs, then (1) she has to maintain all the beliefs she previously had and (2) the change in her beliefs should be minimal in the sense that every new proposition that she believes must be deducible from her old beliefs and the new information (see, for example, Gärdenfors, 1988, Stalnaker, 1984). In the case where information is identified with knowledge and the latter satisfies the S5 logic (that is, $\mathcal{K}$ is reflexive, transitive and euclidean, thus a partition) then the conservativity principle is fully captured by the following property, which is the qualitative version of Bayes' rule (cf. Battigalli and Bonanno, 1997): $\forall t \in \mathbb{N}$, $\forall \omega \in \Omega$,

$$if \;\; \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset \; then \; \mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega). \tag{C}$$

Condition *(C)* says that if what the individual knows now (at time $t+1$) is compatible with what he believed before (at time t), then what he believes now coincides with the intersection of what he believed before and what he cannot rule out based on his current knowledge.

Since reflexivity is *not* satisfied in general by the information and knowledge functions that we consider, and, furthermore, the information function is not necessarily euclidean, we will investigate the extent to which in our framework condition (C) applies to the relationship between information and knowledge and between knowledge and belief.

Remark 12 *Even when $\mathcal{I}$ (and thus $\mathcal{K}$) is reflexive, condition (C) restated in terms of $\mathcal{I}$ and $\mathcal{K}$ does not hold in general, that is, it is not the case that $\mathcal{K}_{t+1}(\omega) = \mathcal{K}_t(\omega) \cap \mathcal{I}_{t+1}(\omega)$, as the following example shows: $\Omega = \{\alpha, \beta, \gamma\}$, $\forall \omega \in \Omega$, $\mathcal{I}_t(\omega) = \{\alpha, \beta, \gamma\}$, $\mathcal{I}_{t+1}(\alpha) = \{\alpha, \beta, \gamma\}$ and $\mathcal{I}_{t+1}(\beta) = \mathcal{I}_{t+1}(\gamma) = \{\beta, \gamma\}$. Then, $\forall \omega \in \Omega$, $\mathcal{K}_t(\omega) = \{\alpha, \beta, \gamma\}$, while $\mathcal{K}_{t+1}(\alpha) = \{\alpha\}$ and $\mathcal{K}_{t+1}(\beta) = \mathcal{K}_{t+1}(\gamma) = \{\beta, \gamma\}$. Here we have that $\mathcal{I}_{t+1}(\alpha) \cap \mathcal{K}_t(\alpha) = \{\alpha, \beta, \gamma\}$, while $\mathcal{K}_{t+1}(\alpha) = \{\alpha\}$.*[17]

[17] A story consistent with this example is the following: a patient suffers from symptoms which are associated with either food poisoning (α) or a bacterial infection (β) or a viral infection (γ). Initially the doctor cannot tell which is the true illness. A day later the patient reports no changes in his symptoms. The doctor concludes that it must be a case of food poisoning (α), because if it had been an infection (either bacterial or viral) then the patient would have developed high fever.

We now introduce a further property of the information function, which is natural in our framework where information is defined as possibilities arising from signals. According to Definition 1 (adding a time subscript) the information function $\mathcal{I}_t : \Omega \to 2^\Omega$ is given by: $\mathcal{I}_t(\omega) = \{\omega' \in \Omega_K^t : \sigma_t(\omega') \supseteq \sigma_t(\omega)\}$, where $\Omega_K^t \subseteq \Omega$ is the database of known states at date t. It is very plausible to assume that the database can only expand over time, that is, that known states are never forgotten (and possibly new states are learned). If one makes this assumption, namely that $\Omega_K^t \subseteq \Omega_K^{t+1}$, then it follows from the definition of $\mathcal{I}_t(\omega)$ that if $\omega \in \mathcal{I}_t(\omega)$ then $\omega \in \mathcal{I}_{t+1}(\omega)$.[18] We introduce this assumption explicitly:

$$\forall \omega \in \Omega, \ if \ \omega \in \mathcal{I}_t(\omega) \ then \ \omega \in \mathcal{I}_{t+1}(\omega). \qquad \text{(R.10)}$$

Note that (R.10) is much weaker than reflexivity of $\mathcal{I}$: in fact it is consistent with (R.10) to have that, for some state ω and for every date t, $\omega \notin \mathcal{I}_t(\omega)$ (and therefore $\omega \notin \mathcal{K}_t(\omega)$)

The following proposition says that, when (R.10) is satisfied, the interaction of knowledge and belief expressed by (C) is fully captured by the axiom $B_t\phi \leftrightarrow B_tB_{t+1}\phi$. In one direction ($B_t\phi \to B_tB_{t+1}\phi$) the axiom says that the individual must anticipate continuing to believe everything that she currently believes, while in the other direction ($B_tB_{t+1}\phi \to B_t\phi$) it says that if the individual anticipates believing something in the future then she must believe it now.[19]

Proposition 13 *Let $\mathcal{I}$, $\mathcal{K}$ and $\mathcal{B}$ satisfy (R.1)-(R.10). Then the following are equivalent*

(1) *if* $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$ *then* $\mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$

(2) *the axiom* $B_t\phi \leftrightarrow B_tB_{t+1}\phi$ *is valid.*

Proposition 13 generalizes Proposition 1 in Battigalli and Bonanno (1997) to the case where $\mathcal{K}$ is not reflexive. Furthermore, in our framework (unlike

[18] If $\omega \in \mathcal{I}_t(\omega)$ then $\omega \in \Omega_K^t$. Since $\Omega_K^t \subseteq \Omega_K^{t+1}$, it follows that $\omega \in \Omega_K^{t+1}$. Hence, since $\sigma_{t+1}(\omega) \supseteq \sigma_{t+1}(\omega)$, $\omega \in \mathcal{I}_{t+1}(\omega)$.

[19] Thus in frames that satisfy (R.1)-(R.10) the axiom $B_t\phi \leftrightarrow B_tB_{t+1}\phi$ characterizes the qualitative rule of conditionalization (C). It follows that the plausibility of this axiom is equivalent to the plausibility of the semantic rule of conditionalization.

theirs) $\mathcal{K}$ is a derived concept rather than a primitive. The proof of Proposition 13 is split into the following four lemmas. It can be seen from these lemmas that (R.10) is required to prove the implication in one direction only.

Lemma 14 *Assume (R.5)-(R.9). If $B_t\phi \to B_tB_{t+1}\phi$ is valid then the following holds: $\forall\omega \in \Omega$, if $\mathcal{B}_t(\omega)\cap\mathcal{K}_{t+1}(\omega) \neq \emptyset$ then $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega)\cap\mathcal{K}_{t+1}(\omega)$.*

Proof. Assume that $B_t\phi \to B_tB_{t+1}\phi$ is valid in the frame. Fix an arbitrary $\omega \in \Omega$ and suppose that $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$. By (R.5), $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{K}_{t+1}(\omega)$. Thus we only need to show that $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega)$. Suppose not, that is, suppose there exists an $\alpha \in \mathcal{B}_{t+1}(\omega)$ such that $\alpha \notin \mathcal{B}_t(\omega)$. Construct a model where, for some atomic proposition p, $||p|| = \Omega\backslash\{\alpha\}$. Then $\omega \models B_tp$. Thus, by our hypothesis, $\omega \models B_tB_{t+1}p$. Let $\beta \in \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$ (it exists by our supposition). Since $\beta \in \mathcal{B}_t(\omega)$ and $\omega \models B_tB_{t+1}p$, $\beta \models B_{t+1}p$, that is,

$$\alpha \notin \mathcal{B}_{t+1}(\beta). \tag{2}$$

By (R.9) $\mathcal{B}_{t+1}(\beta) \neq \emptyset$. Thus, by Remark 11, since $\beta \in \mathcal{K}_{t+1}(\omega)$, $\mathcal{B}_{t+1}(\beta) = \mathcal{B}_{t+1}(\omega)$. Then, since $\alpha \in \mathcal{B}_{t+1}(\omega)$, $\alpha \in \mathcal{B}_{t+1}(\beta)$, contradicting (2). ■

Lemma 15 *Assume (R.5)-(R.9). If $B_tB_{t+1}\phi \to B_t\phi$ is valid then the following holds: $\forall\omega \in \Omega$, $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\omega)$.*

Proof. Assume that $B_tB_{t+1}\phi \to B_t\phi$ is valid in the frame. Fix an arbitrary $\omega \in \Omega$ and suppose it is not the case that $\mathcal{B}_t(\omega)\cap\mathcal{K}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\omega)$, that is, there exists an $\alpha \in \mathcal{B}_t(\omega)\cap\mathcal{K}_{t+1}(\omega)$ such that $\alpha \notin \mathcal{B}_{t+1}(\omega)$. Construct a model where, for some atomic proposition p, $||p|| = \Omega\backslash\{\alpha\}$. Then, since $\alpha \in \mathcal{B}_t(\omega)$,

$$\omega \not\models B_tp. \tag{3}$$

By (R.9) $\mathcal{B}_{t+1}(\alpha) \neq \emptyset$. Thus, by Remark 11, since $\alpha \in \mathcal{K}_{t+1}(\omega)$, $\mathcal{B}_{t+1}(\alpha) = \mathcal{B}_{t+1}(\omega)$. It follows that, since $\alpha \notin \mathcal{B}_{t+1}(\omega)$,

$$\alpha \notin \mathcal{B}_{t+1}(\alpha). \tag{4}$$

Next we show that

$$\forall\gamma \in \mathcal{B}_t(\omega),\ \alpha \notin \mathcal{B}_{t+1}(\gamma). \tag{5}$$

Suppose not, that is, suppose there exists a $\gamma \in \mathcal{B}_t(\omega)$ such that $\alpha \in \mathcal{B}_{t+1}(\gamma)$. Then by (R.5) $\alpha \in \mathcal{K}_{t+1}(\gamma)$ and, therefore, by Remark 11, $\mathcal{B}_{t+1}(\alpha) = \mathcal{B}_{t+1}(\gamma)$. Thus by (4) $\alpha \notin \mathcal{B}_{t+1}(\gamma)$, yielding a contradiction. It follows from (5) that $\omega \models B_t B_{t+1} p$. Thus, by our supposition, $\omega \models B_t p$, contradicting (3). ■

Remark 16 *In general, in frames that satisfy (R.5)-(R.9) the converse of Lemma 14 is not true, as the following example shows:* $\Omega = \{\alpha, \beta\}$, $\mathcal{I}_t(\alpha) = \mathcal{K}_t(\alpha) = \mathcal{B}_t(\alpha) = \{\alpha\}$, $\mathcal{I}_t(\beta) = \mathcal{K}_t(\beta) = \mathcal{B}_t(\beta) = \{\beta\}$ *and,* $\forall \omega \in \Omega$, $\mathcal{I}_{t+1}(\omega) = \mathcal{K}_{t+1}(\omega) = \mathcal{B}_{t+1}(\omega) = \{\beta\}$. *Here it is true at every state* ω *that if* $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$ *then* $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$ *(the antecedent is false at* α*). Yet it is not true that* $B_t\phi \to B_t B_{t+1}\phi$ *is valid. In fact, consider a model where, for some atomic sentence* p, $||p|| = \{\alpha\}$. *Then* $\alpha \models B_t p$ *but* $\alpha \nvDash B_t B_{t+1} p$ *since* $\alpha \in \mathcal{B}_t(\alpha)$ *and* $\beta \in \mathcal{B}_{t+1}(\alpha)$ *and* $\beta \nvDash p$.

The following lemma shows that in the presence of (R.10) the converse of Lemma 14 does hold.

Lemma 17 *Assume (R.1), (R.3), (R.5), (R.7), (R.8) and (R.10). Assume also that,* $\forall \omega \in \Omega$, *if* $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \neq \emptyset$ *then* $\mathcal{B}_{t+1}(\omega) \subseteq \mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega)$. *Then* $B_t\phi \to B_t B_{t+1}\phi$ *is valid.*

Proof. Suppose not, that is, suppose that in some model, for some ω and ϕ it is the case that $\omega \models B_t\phi$ and $\omega \nvDash B_t B_{t+1}\phi$. Then there exist $\alpha \in \mathcal{B}_t(\omega)$ and $\beta \in \mathcal{B}_{t+1}(\alpha)$ such that $\beta \nvDash \phi$. Thus

$$\beta \notin \mathcal{B}_t(\omega). \tag{6}$$

Since $\alpha \in \mathcal{B}_t(\omega)$, by (R.7) and (R.8), $\mathcal{B}_t(\alpha) = \mathcal{B}_t(\omega)$. Thus

$$\alpha \in \mathcal{B}_t(\alpha) \tag{7}$$

and by (6)

$$\beta \notin \mathcal{B}_t(\alpha). \tag{8}$$

By (7), (R.5) and (R.1), $\alpha \in \mathcal{I}_t(\alpha)$. Hence, by (R.10), $\alpha \in \mathcal{I}_{t+1}(\alpha)$. It follows from this and (R.3) that

$$\alpha \in \mathcal{K}_{t+1}(\alpha). \tag{9}$$

By (7) and (9), $\mathcal{B}_t(\alpha) \cap \mathcal{K}_{t+1}(\alpha) \neq \emptyset$. Hence by our hypothesis $\mathcal{B}_{t+1}(\alpha) \subseteq \mathcal{B}_t(\alpha) \cap \mathcal{K}_{t+1}(\alpha)$. Hence, since $\beta \in \mathcal{B}_{t+1}(\alpha)$, we get that $\beta \in \mathcal{B}_t(\alpha)$, contradicting (8). ■

Remark 18 *In general, in frames that satisfy (R.5)-(R.9) the converse of Lemma 15 is not true, as the following example shows:* $\Omega = \{\alpha, \beta\}$, $\forall \omega \in \Omega$, $\mathcal{I}_t(\omega) = \mathcal{K}_t(\omega) = \mathcal{B}_t(\omega) = \{\alpha, \beta\}$ *and* $\mathcal{I}_{t+1}(\omega) = \mathcal{K}_{t+1}(\omega) = \mathcal{B}_{t+1}(\omega) = \{\alpha\}$. *Here it is true at every state* ω *that* $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\omega)$. *Yet it is not true that* $B_t B_{t+1}\phi \to B_t\phi$ *is valid. In fact, consider a model where, for some atomic sentence* p, $||p|| = \{\alpha\}$. *Then* $\alpha \models B_t B_{t+1} p$ *but* $\alpha \nvDash B_t p$.

The following lemma shows that in the presence of (R.10) the converse of Lemma 15 does hold.

Lemma 19 *Assume (R.1), (R.3), (R.5), (R.8) and (R.10). Assume also that,* $\forall \omega \in \Omega$, $\mathcal{B}_t(\omega) \cap \mathcal{K}_{t+1}(\omega) \subseteq \mathcal{B}_{t+1}(\omega)$. *Then* $B_t B_{t+1}\phi \to B_t\phi$ *is valid.*

Proof. Suppose not, that is, suppose that in some model, for some ω and ϕ it is the case that $\omega \models B_t B_{t+1}\phi$ and $\omega \nvDash B_t\phi$, that is, $\forall \alpha \in \mathcal{B}_t(\omega)$, $\mathcal{B}_{t+1}(\alpha) \subseteq ||\phi||$ and there exists a $\beta \in \mathcal{B}_t(\omega)$ such that $\beta \nvDash \phi$. Then

$$\beta \notin \mathcal{B}_{t+1}(\beta). \tag{10}$$

Since $\beta \in \mathcal{B}_t(\omega)$, by (R.5) and (R.1), $\beta \in \mathcal{I}_t(\omega)$. Hence by secondary reflexivity of $\mathcal{I}_t$ (see Proposition 4), $\beta \in \mathcal{I}_t(\beta)$. Thus, by (R.10), $\beta \in \mathcal{I}_{t+1}(\beta)$ and hence, by (R.3), $\beta \in \mathcal{K}_{t+1}(\beta)$. Since $\beta \in \mathcal{B}_t(\omega)$, by (R.8) $\beta \in \mathcal{B}_t(\beta)$. Thus $\beta \in \mathcal{B}_t(\beta) \cap \mathcal{K}_{t+1}(\beta)$. By our hypothesis, $\mathcal{B}_t(\beta) \cap \mathcal{K}_{t+1}(\beta) \subseteq \mathcal{B}_{t+1}(\beta)$. Thus $\beta \in \mathcal{B}_{t+1}(\beta)$, contradicting (10). ■

It is easily seen that Proposition 13 is a corollary of the above four lemmas.

Remark 20 *In Proposition 13, one cannot replace* $\mathcal{K}$ *with* $\mathcal{I}$ *in* (C), *that is,* (C) *cannot be replaced by*

$$if \;\; \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega) \neq \emptyset \; then \; \mathcal{B}_{t+1}(\omega) = \mathcal{B}_t(\omega) \cap \mathcal{I}_{t+1}(\omega). \tag{C'}$$

In fact, while it is the case that (C') *implies validity of* $B_t\phi \leftrightarrow B_t B_{t+1}\phi$,[20]

[20]**Proof.** First we show validity of $B_t\phi \to B_t B_{t+1}\phi$. Suppose that $\alpha \models B_t\phi$, that is, $\mathcal{B}_t(\alpha) \subseteq ||\phi||$. Fix an arbitrary $\beta \in \mathcal{B}_t(\alpha)$. We want to show that $\beta \models B_{t+1}\phi$, that is, $\mathcal{B}_{t+1}(\beta) \subseteq ||\phi||$. By (R.7) and (R.8), $\mathcal{B}_t(\alpha) = \mathcal{B}_t(\beta)$. Thus $\beta \in \mathcal{B}_t(\beta)$ and by (R.5) and (R.1) $\beta \in \mathcal{I}_t(\beta)$. Thus, by (R.10), $\beta \in \mathcal{I}_{t+1}(\beta)$. Hence, $\mathcal{B}_t(\beta) \cap \mathcal{I}_{t+1}(\beta) \neq \emptyset$ and by (C'), $\mathcal{B}_{t+1}(\beta) \subseteq \mathcal{B}_t(\beta)$. Thus, since $\mathcal{B}_t(\beta) = \mathcal{B}_t(\alpha) \subseteq ||\phi||$, $\mathcal{B}_{t+1}(\beta) \subseteq ||\phi||$.

Next we show validity of $B_t B_{t+1}\phi \to B_t\phi$. Suppose that $\alpha \models B_t B_{t+1}\phi$, that is, $\forall \omega \in \mathcal{B}_t(\alpha)$, $\mathcal{B}_{t+1}(\omega) \subseteq ||\phi||$. Fix an arbitrary $\beta \in \mathcal{B}_t(\alpha)$. We want to show that $\beta \in ||\phi||$. Since $\beta \in \mathcal{B}_t(\alpha)$, it follows from (R.8) that $\beta \in \mathcal{B}_t(\beta)$ and by (R.5) and (R.1) $\beta \in \mathcal{I}_t(\beta)$. Thus, by (R.10), $\beta \in \mathcal{I}_{t+1}(\beta)$. It follows from (C') that $\beta \in \mathcal{B}_{t+1}(\beta)$. Hence, since $\beta \in \mathcal{B}_t(\alpha)$, $\beta \in ||\phi||$. ■

the converse is not true, as the following example shows: $\Omega = \{\alpha, \beta, \gamma\}$, $\forall\omega \in \Omega$, $\mathcal{I}_t(\omega) = \mathcal{K}_t(\omega) = \mathcal{B}_t(\omega) = \{\alpha, \beta, \gamma\}$, $\mathcal{I}_{t+1}(\alpha) = \{\alpha, \beta, \gamma\}$, $\mathcal{I}_{t+1}(\beta) = \mathcal{I}_{t+1}(\gamma) = \{\beta, \gamma\}$, $\mathcal{K}_{t+1}(\alpha) = \mathcal{B}_{t+1}(\alpha) = \{\alpha\}$, $\mathcal{K}_{t+1}(\beta) = \mathcal{B}_{t+1}(\beta) = \mathcal{K}_{t+1}(\gamma) = \mathcal{B}_{t+1}(\gamma) = \{\beta, \gamma\}$. *For every* $\omega \in \Omega$, $\omega \models B_t\phi$ *iff* $||\phi|| = \Omega$; *hence,* $\forall\omega \in \Omega$ *and for any such formula* ϕ, $\omega \models B_{t+1}\phi$ *and, therefore,* $\omega \models B_tB_{t+1}\phi$. *Thus* $B_t\phi \to B_tB_{t+1}\phi$ *is valid. However,* $\mathcal{B}_{t+1}(\alpha) = \{\alpha\} \neq \mathcal{B}_t(\alpha) \cap \mathcal{I}_{t+1}(\alpha) = \{\alpha, \beta, \gamma\}$.[21]

The axiom considered in Proposition 13 is *forward*-looking, that is, it involves beliefs about future beliefs. Consider now the similar, although *backward*-looking, axiom: $B_tB_{t-1}\phi \leftrightarrow B_{t-1}\phi$. In one direction ($B_tB_{t-1}\phi \to B_{t-1}\phi$) the axiom says that the individual's beliefs about her past beliefs are always correct,[22] while in the other direction ($B_{t-1}\phi \to B_tB_{t-1}\phi$) it says that the individual always remembers her past beliefs. The following proposition shows that, when information is correct ($\mathcal{I}$ is reflexive), the axiom characterizes the following property of *knowledge*, which says that the individual's knowledge becomes more refined as time progresses:[23]

$$\forall t \in \mathbb{N},\ \forall\omega \in \Omega,\ \ \mathcal{K}_t(\omega) \subseteq \mathcal{K}_{t-1}(\omega). \qquad (\mathrm{R}_K)$$

Proposition 21 *Assume (R.1)-(R.9). If* $\mathcal{I}$ *is reflexive, then the following are equivalent:*

(1) *the axiom* $B_tB_{t-1}\phi \leftrightarrow B_{t-1}\phi$ *is valid,*
(2) *property* (R_K) *is satisfied.*

By (R.3), reflexivity of $\mathcal{I}$ implies reflexivity of $\mathcal{K}$. Thus the above proposition is a consequence of the following two lemmas. Note that the first lemma does *not* require reflexivity of $\mathcal{K}$.

Lemma 22 *Assume (R.5)-(R.9). If* (R_K) *holds, then the axiom* $B_tB_{t-1}\phi \leftrightarrow B_{t-1}\phi$ *is valid.*

[21] The crucial property that is missing here is the counterpart of (R.6) for information: while the individual always knows his own beliefs, it is not the case that he is informed of his own beliefs. In this example, $\beta \in \mathcal{I}_{t+1}(\alpha)$ but $\mathcal{B}_{t+1}(\alpha) \cap \mathcal{B}_{t+1}(\beta) = \emptyset$.

[22] Recall that even if $\mathcal{K}$ is reflexive, $\mathcal{B}$ in general is not: mistaken beliefs are allowed (and, indeed, quite common). While allowing for mistaken beliefs about facts, this axiom rules out incorrect beliefs about one's own past beliefs.

[23] 'R_K' stands for "refinement of knowledge".

Proof. First we prove validity of $B_tB_{t-1}\phi \to B_{t-1}\phi$. Let $\alpha \models B_tB_{t-1}\phi$, that is, $\forall\omega \in \mathcal{B}_t(\alpha)$, $\mathcal{B}_{t-1}(\omega) \subseteq ||\phi||$. We want to show that $\alpha \models B_{t-1}\phi$, that is, $\mathcal{B}_{t-1}(\alpha) \subseteq ||\phi||$. By (R.9) there exists a $\beta \in \mathcal{B}_t(\alpha)$. Then $\mathcal{B}_{t-1}(\beta) \subseteq ||\phi||$. By (R.5) $\mathcal{B}_t(\alpha) \subseteq \mathcal{K}_t(\alpha)$ and by hypothesis $\mathcal{K}_t(\alpha) \subseteq \mathcal{K}_{t-1}(\alpha)$. Thus $\beta \in \mathcal{K}_{t-1}(\alpha)$; furthermore, by (R.9) $\mathcal{B}_{t-1}(\beta) \neq \emptyset$; thus, by Remark 11, $\mathcal{B}_{t-1}(\beta) = \mathcal{B}_{t-1}(\alpha)$. Hence $\mathcal{B}_{t-1}(\alpha) \subseteq ||\phi||$. Next we prove validity of $B_{t-1}\phi \to B_tB_{t-1}\phi$. Suppose $\alpha \models B_{t-1}\phi$, that is, $\mathcal{B}_{t-1}(\alpha) \subseteq ||\phi||$. Fix an arbitrary $\beta \in \mathcal{B}_t(\alpha)$. We need to show that $\mathcal{B}_{t-1}(\beta) \subseteq ||\phi||$. By (R.5) $\mathcal{B}_t(\alpha) \subseteq \mathcal{K}_t(\alpha)$ and by hypothesis $\mathcal{K}_t(\alpha) \subseteq \mathcal{K}_{t-1}(\alpha)$. Thus $\beta \in \mathcal{K}_{t-1}(\alpha)$; furthermore, by (R.9) $\mathcal{B}_{t-1}(\beta) \neq \emptyset$; thus, by Remark 11, $\mathcal{B}_{t-1}(\beta) = \mathcal{B}_{t-1}(\alpha)$. Hence $\mathcal{B}_{t-1}(\beta) \subseteq ||\phi||$. ■

Lemma 23 *Assume (R.1)-(R.3) and (R.5)-(R.9). If $\mathcal{I}$ (and thus $\mathcal{K}$) is reflexive and the axiom $B_tB_{t-1}\phi \leftrightarrow B_{t-1}\phi$ is valid, then (R_K) is satisfied.*

Proof. Suppose that $\mathcal{I}$ (and thus $\mathcal{K}$) is reflexive and (R_K) is violated. Then there exist α, β such that $\beta \in \mathcal{K}_t(\alpha)$ and $\beta \notin \mathcal{K}_{t-1}(\alpha)$. Then $\mathcal{K}_{t-1}(\alpha) \cap \mathcal{K}_{t-1}(\beta) = \emptyset$ (recall that $\mathcal{K}$ is transitive and euclidean: cf. Proposition 6; note: this is where reflexivity of $\mathcal{K}$ is plays a crucial role)[24]. By (R.5), $\mathcal{B}_{t-1}(\alpha) \subseteq \mathcal{K}_{t-1}(\alpha)$ and $\mathcal{B}_{t-1}(\beta) \subseteq \mathcal{K}_{t-1}(\beta)$. Thus $\mathcal{B}_{t-1}(\alpha) \cap \mathcal{B}_{t-1}(\beta) = \emptyset$. Construct a model where for some atomic sentence p, $||p|| = \mathcal{B}_{t-1}(\alpha)$. Then $\alpha \models B_{t-1}p$ and $\beta \not\models B_{t-1}p$. By (R.9), $\mathcal{B}_t(\beta) \neq \emptyset$; thus, since $\beta \in \mathcal{K}_t(\alpha)$, by Remark 11 $\mathcal{B}_t(\beta) = \mathcal{B}_t(\alpha)$. Thus, for every formula ϕ, $\alpha \models B_t\phi$ if and only if $\beta \models B_t\phi$. If $\alpha \models B_tB_{t-1}p$ then $\beta \models B_tB_{t-1}p$ and the formula $B_tB_{t-1}p \to B_{t-1}p$ is false at β. If $\alpha \not\models B_tB_{t-1}p$ then the formula $B_{t-1}p \to B_tB_{t-1}p$ is false at α. In either case we get a contradiction with our hypothesis that the axiom $B_tB_{t-1}\phi \leftrightarrow B_{t-1}\phi$ is valid. ■

Remark 24 *The following frame, which satisfies (R.5)-(R.9), shows that in Lemma 23 the hypothesis that $\mathcal{I}$ is reflexive is crucial: $\Omega = \{\alpha, \beta\}$, $\forall\omega \in \Omega$, $\mathcal{I}_{t-1}(\omega) = \mathcal{K}_{t-1}(\omega) = \mathcal{B}_{t-1}(\omega) = \{\alpha\}$, $\mathcal{I}_t(\omega) = \mathcal{K}_t(\omega) = \{\alpha, \beta\}$ and $\mathcal{B}_t(\omega) = \{\alpha\}$. In any model based on this frame, for every formula ϕ, $B_tB_{t-1}\phi \leftrightarrow B_{t-1}\phi$ is valid but (R_K) is violated.*

One case which is of some interest is the case where the information available at any date does not contradict the information available at earlier

[24]Suppose $\gamma \in \mathcal{K}_{t-1}(\alpha) \cap \mathcal{K}_{t-1}(\beta)$. By transitivity and euclideanness of $\mathcal{K}$ (cf. Proposition 6), $\mathcal{K}_{t-1}(\gamma) = \mathcal{K}_{t-1}(\alpha)$ and $\mathcal{K}_{t-1}(\gamma) = \mathcal{K}_{t-1}(\beta)$. Thus $\mathcal{K}_{t-1}(\alpha) = \mathcal{K}_{t-1}(\beta)$. By reflexivity of $\mathcal{K}$, $\beta \in \mathcal{K}_{t-1}(\beta)$. Thus $\beta \in \mathcal{K}_{t-1}(\alpha)$, a contradiction.

dates, that is, the individual becomes *more informed* as time progresses. This is expressed by the following property:[25]

$$\forall t \in \mathbb{N}, \ \ \forall \omega \in \Omega, \ \ \mathcal{I}_t(\omega) \subseteq \mathcal{I}_{t-1}(\omega). \qquad (\mathrm{R}_I)$$

Remark 25 *From the facts that knowledge is based on information (R.1) and that new information never contradicts past information (R_I) one might be led to conclude that what the individual knows at a date t should be a refinement of what he knew at date $t-1$, that is, that (R_K) should hold. The following example shows that this is not the case. Let $\Omega = \{\alpha, \beta, \gamma, \delta\}$, $\mathcal{I}_{t-1}(\alpha) = \mathcal{I}_{t-1}(\gamma) = \mathcal{I}_{t-1}(\delta) = \{\beta, \gamma, \delta\}$ and $\mathcal{I}_{t-1}(\beta) = \{\beta\}$. Then (R.1)-(R.3) imply that $\mathcal{K}_{t-1}(\alpha) = \mathcal{K}_{t-1}(\gamma) = \mathcal{K}_{t-1}(\delta) = \{\gamma, \delta\}$ and $\mathcal{K}_{t-1}(\beta) = \{\beta\}$. Suppose now that $\mathcal{I}_t(\alpha) = \mathcal{I}_t(\beta) = \{\beta\}$ and $\mathcal{I}_t(\gamma) = \mathcal{I}_t(\delta) = \{\gamma, \delta\}$. Then, by (R.1)-(R.3), $\mathcal{K}_t(\alpha) = \mathcal{K}_t(\beta) = \{\beta\}$ and $\mathcal{K}_t(\gamma) = \mathcal{K}_t(\delta) = \{\gamma, \delta\}$. In this example ($R_I$) is satisfied and yet $\mathcal{K}_t(\alpha) \cap \mathcal{K}_{t-1}(\alpha) = \emptyset$, that is, what the individual knows at date t is inconsistent with what she knew at date $t-1$.*

The following proposition shows that when information is correct ($\mathcal{I}$ is reflexive), property (R_I) *is* transferred to knowledge, that is, (R_K) holds. Thus if information is always correct and new information never contradicts past information, then it is never the case that the individual knew something in the past which she does not know now.

Proposition 26 *Let $\mathcal{I}$ and $\mathcal{K}$ satisfy (R.1)-(R.3) and (R_I). If, in addition, $\mathcal{I}$ is reflexive then (R_K) holds.*[26]

Proof. Fix arbitrary $t \in \mathbb{N}$ and $\alpha, \beta \in \Omega$ and suppose that $\beta \in \mathcal{K}_t(\alpha)$. We want to show that $\beta \in \mathcal{K}_{t-1}(\alpha)$. By (R.1), $\beta \in \mathcal{I}_t(\alpha)$. Thus, by ($\mathrm{R}_I$), $\beta \in \mathcal{I}_{t-1}(\alpha)$. Suppose that $\beta \notin \mathcal{K}_{t-1}(\alpha)$. Then, since $\beta \in \mathcal{I}_{t-1}(\alpha)$, it follows from (R.3) that $\mathcal{I}_{t-1}(\beta) \neq \mathcal{I}_{t-1}(\alpha)$. Hence, by transitivity of $\mathcal{I}$ (Proposition 4), $\alpha \notin \mathcal{I}_{t-1}(\beta)$.[27] Since $\beta \in \mathcal{K}_t(\alpha)$, by (R.2), $\mathcal{I}_t(\beta) = \mathcal{I}_t(\alpha)$. By reflexivity of $\mathcal{I}$, $\alpha \in \mathcal{I}_t(\alpha)$. Thus $\alpha \in \mathcal{I}_t(\beta)$. Hence, by ($\mathrm{R}_I$), $\alpha \in \mathcal{I}_{t-1}(\beta)$, yielding a contradiction. ■

[25] 'R_I' stands for "refinement of information".

[26] Note that, even if $\mathcal{I}$ is reflexive, the converse is not true, that is, (R_K) does not imply (R_I), as the following example shows: $\Omega = \{\alpha, \beta, \gamma\}$, $\mathcal{I}_{t-1}(\alpha) = \mathcal{I}_{t-1}(\beta) = \mathcal{K}_{t-1}(\alpha) = \mathcal{K}_{t-1}(\beta) = \{\alpha, \beta\}$, $\mathcal{I}_{t-1}(\gamma) = \mathcal{K}_{t-1}(\gamma) = \{\gamma\}$, $\mathcal{I}_t(\alpha) = \mathcal{I}_t(\beta) = \{\alpha, \beta, \gamma\}$, $\mathcal{K}_t(\alpha) = \mathcal{K}_t(\beta) = \{\alpha, \beta\}$, $\mathcal{I}_t(\gamma) = \mathcal{K}_t(\gamma) = \{\gamma\}$. Here ($\mathrm{R}_K$) is satisfied but not ($\mathrm{R}_I$).

[27] By transitivity of $\mathcal{I}$, since $\beta \in \mathcal{I}_{t-1}(\alpha)$, $\mathcal{I}_{t-1}(\beta) \subseteq \mathcal{I}_{t-1}(\alpha)$. If it were the case that $\alpha \in \mathcal{I}_{t-1}(\beta)$, then – again by transitivity of $\mathcal{I}$ – we would have that $\mathcal{I}_{t-1}(\alpha) \subseteq \mathcal{I}_{t-1}(\beta)$ and therefore $\mathcal{I}_{t-1}(\beta) = \mathcal{I}_{t-1}(\alpha)$.

7 Conclusion

We modeled information as possibilities consistent with signals received from the environment. Knowledge was then defined as belief obtained from the available information by reasoning about the signals received as well as those that could have been received, but were not. We required knowledge to be fully justifiable on the basis of the available information. The term 'belief', on the other hand, was used to refer to those beliefs that are based on information but not necessarily *only* on information. We investigated the relationship between information, knowledge and belief, as well as the issue of updating knowledge and belief in response to changes in information.

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