PREDICTION IN BRANCHING TIME LOGIC

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Abstract

When we make a prediction we select, among the conceivable future descriptions of the world, those that appear to us to be most plausible. We capture this by means of two binary relations, $\prec_c$ and $\prec_p$: if t_1 and t_2 are points in time, we interpret $t_1 \prec_c t_2$ as saying that t_2 is in the *conceivable future* of t_1, while $t_1 \prec_p t_2$ is interpreted to mean that t_2 is in the *predicted future* of t_1. Within a branching-time framework we propose the following notion of "consistency of prediction". Suppose that at t_1 some future moment t_2 is predicted to occur, then (a) every moment t on the *unique* path from t_1 to t_2 should also be predicted at t_1 and (b) the prediction of t_2 should continue to hold at every such t. A sound and complete axiomatization is provided.

1. Introduction

When we make a prediction about the future we select, among the conceivable future descriptions of the world, those that appear to us to be most plausible. Thus the concept of prediction involves three notions: (1) time (predictions are about the future), (2) conceivable future states of the world and (3) a selection from the set of conceivable states of those that are considered most plausible. We propose a system of modal logic that incorporates these three elements. The notion of a multiplicity of possible future states is captured by what is known in temporal logic as *branching time* (where multiple paths between two instants are ruled out). To distinguish between conceivable and plausible future possibilities we introduce two binary relations, $\prec_c$ and $\prec_p$. If t_1 and t_2 are different points in time, we interpret $t_1 \prec_c t_2$ as saying that t_2 is in the *conceivable future* of t_1, while $t_1 \prec_p t_2$ is interpreted to mean that t_2 is in the *predicted future* of t_1. We propose the following requirement of *time consistency* of

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prediction. Suppose that at time t_1 a conceivable future development is represented by the path $t_1 t_2 t_3$ (that is, $t_1 \prec_c t_2$ and $t_2 \prec_c t_3$). This is shown in Figure 1, where a continuous arrow labelled 'c' from t to t' denotes that $t \prec_c t'$. Suppose also that t_3 lies in the predicted future of t_1 (that is, $t_1 \prec_p t_3$: this is shown in Figure 1 by a dotted arrow labelled 'p' from t_1 to t_3). Then we impose the following requirements:

(a) since reaching t_3 *requires* going through t_2, t_2 should lie in the predicted future of t_1 (that is, $t_1 \prec_p t_2$), and

(b) since reaching t_2 is a partial realization of (is consistent with) the prediction that t_3 will be reached, the prediction should continue to hold at t_2, that is, t_3 should be in the predicted future of t_2 ($t_2 \prec_p t_3$).

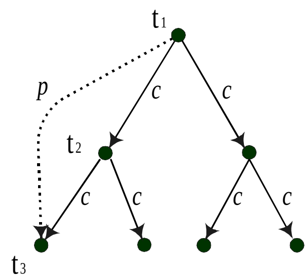

Figure 1

It is worth stressing that these two requirements are imposed within a framework where there is a *unique* path from t_1 to t_3, and t_2 belongs to that path. Indeed, the characteristic feature of branching time logic is that each instant has a unique past history (while the future is open). We provide a sound and complete axiomatization of this notion of consistency of prediction.

2. A minimal logic of prediction

Definition 2.1. *A branching-time frame is a triple* $\mathcal{F} = \langle T, \prec_c, \prec_p \rangle$ *where* T *is a (possibly infinite) set of instants and* $\prec_c$ *and* $\prec_p$ *are binary relations on* T *satisfying the following properties,* $\forall t_1, t_2, t_3 \in T$,

(R.0) antisymmetry of $\prec_c$: if $t_1 \prec_c t_2$ then $t_2 \nprec_c t_1$
(R.1) transitivity of $\prec_c$: if $t_1 \prec_c t_2$ and $t_2 \prec_c t_3$ then $t_1 \prec_c t_3$
(R.2) backward linearity of $\prec_c$: if $t_1 \prec_c t_3$ and $t_2 \prec_c t_3$ then either $t_1 = t_2$ or $t_1 \prec_c t_2$ or $t_2 \prec_c t_1$
(R.3) $\prec_p$ subrelation of $\prec_c$: if $t_1 \prec_p t_2$ then $t_1 \prec_c t_2$.

The interpretation of $t_1 \prec_c t_2$ is that t_2 is in the *conceivable future* of t_1, while if $t_1 \prec_p t_2$ we say that t_2 is in the *predicted future* of t_1. (R.0)-(R.2) constitute the definition of *branching time* in temporal logic.[1] In particular, (R.2) expresses the

[1]See, for example, van Benthem (1991), Burgess (1984), Goldblatt (1992), Øhrstrøm and Hasle (1995).

notion that, while a given moment may have different possible futures, its past is unique. In other words, there is at most one path between any two instants. (R.3) captures the notion that predicting the future consists in selecting a subset of the conceivable future states: those that are believed to be most plausible. Note that it is not assumed that the predicted future of a given moment be a unique history following that moment.[2] Furthermore, there is no requirement that the predicted future of a given moment be a *proper* subset of its conceivable future, that is, vague or trivial predictions are allowed.[3] As argued in the Introduction, the following seems a natural "consistency" requirement: if t_3 is in the predicted future of t_1, and t_2 is on the unique $\prec_c$-path from t_1 to t_3 then (i) t_2 should be in the predicted future of t_1 and (ii) t_3 should be in the predicted future of t_2. Formally ('CP' stands for 'Consistency of Prediction'), $\forall t_1, t_2, t_3 \in T$,

$$(CP) \qquad \text{if } \ t_1 \prec_p t_3 \text{ and } t_1 \prec_c t_2 \text{ and } t_2 \prec_c t_3 \text{ then } t_1 \prec_p t_2 \ \text{ and } t_2 \prec_p t_3.$$

The proof of the following lemma is straightforward and is omitted.[4]

Lemma 2.2. *In a branching-time frame, (CP) is equivalent to the conjunction of the following two properties:*
(R.4) backward linearity of $\prec_p$*: if* $t_1 \prec_p t_3$ *and* $t_2 \prec_p t_3$ *then either* $t_1 = t_2$ *or* $t_1 \prec_p t_2$ *or* $t_2 \prec_p t_1$
(R.5) if $t_1 \prec_p t_3$ *and* $t_2 \prec_c t_3$ *then either (a)* $t_1 = t_2$ *or (b)* $t_2 \prec_c t_1$ *or (c)* $t_1 \prec_c t_2$ *and* $t_2 \prec_p t_3$.

Definition 2.3. *A P-frame (P stands for 'Prediction') is a branching-time frame that satisfies (R.4) and (R.5).*

We now turn to the syntax. We consider a propositional language with four modal operators: G_c, H_c, G_p and H_p. The intended interpretation is as follows:
$G_c\phi$: "it is going to be the case in every *conceivable* future that ϕ"
$G_p\phi$: "it is going to be the case in every *predicted* future that ϕ"
$H_c\phi$: "it has always been the case that ϕ"
$H_p\phi$: "at every past date at which today was predicted it was the case that ϕ".

The formal language is built in the usual way from a countable set S of sentence letters, the connectives $\neg$ and $\vee$ (from which the other connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as usual) and the modal operators. Let $F_c\phi \stackrel{def}{=} \neg G_c \neg \phi$, $P_c\phi \stackrel{def}{=} \neg H_c \neg \phi$, $F_p\phi \stackrel{def}{=} \neg G_p \neg \phi$ and $P_p\phi \stackrel{def}{=} \neg H_p \neg \phi$. Thus, for example, the intended interpretation of $F_c\phi$ (resp. $F_p\phi$) is "at some *conceivable* (resp. *predicted*) future date it will be the case that ϕ ".

Given a frame $\langle T, \prec_c, \prec_p \rangle$ one obtains a *model based on it* by adding a function $V : S \rightarrow 2^T$ (where 2^T denotes the set of subsets of T) that associates with every

[2] That is, we do *not* require that if $t \prec_p t'$ and $t \prec_p t''$ then either $t' = t''$ or $t' \prec_c t''$ or $t'' \prec_c t'$.

[3] For example, suppose that $T = \{t_1, t_2, t_3, t_4\}$ and $\prec_c = \{(t_1,t_2),(t_1,t_3),(t_1,t_4)\}$. Suppose also that t_2 is a state where it is sunny, t_3 is a state where it rains and t_4 is a state where it snows. Then $\prec_p = \prec_c$ corresponds to the trivial prediction "tomorrow either it will be sunny or it will rain or it will snow", while $\prec_p = \{(t_1,t_2),(t_1,t_3)\}$ corresponds to the somewhat vague prediction "tomorrow either it will be sunny or it will rain (but it will not snow)" and $\prec_p = \{(t_1,t_2)\}$ corresponds to the sharp prediction "tomorrow it will be sunny".

[4] Detailed proofs of all the results given in this paper can be found in Bonanno (1998).

sentence letter q the set of dates at which q is true. Given a model and a formula ϕ, the truth set of ϕ, denoted by $\|\phi\|$, is defined as usual. In particular, $\|G_c\phi\| = \{t \in T : \forall t' \in T \text{ if } t \prec_c t' \text{ then } t' \in \|\phi\|\}$, $\|G_p\phi\| = \{t \in T : \forall t' \in T \text{ if } t \prec_p t' \text{ then } t' \in \|\phi\|\}$, etc. Thus $G_c\phi$ (resp. $G_p\phi$) is true at time t if ϕ is true at *every* conceivable (resp. predicted) future of t, while $F_c\phi$ (resp. $F_p\phi$) is true at time t if ϕ is true at *some* conceivable (resp. predicted) future of t. Similarly for $H_c\phi$, $H_p\phi$, $P_c\phi$ and $P_p\phi$. A formula ϕ is *valid in a model* if $\|\phi\| = T$, that is, if ϕ is true at every date $t \in T$. A formula ϕ is *valid in a frame* if it is valid in every model based on it. Consider the following axiom schemata:

(A.1) $G_c\phi \to G_cG_c\phi$
(A.2) $P_c\phi \wedge P_c\psi \to P_c(\phi \wedge \psi) \vee P_c(\phi \wedge P_c\psi) \vee P_c(P_c\phi \wedge \psi)$
(A.3) $G_c\phi \to G_p\phi$
(A.4) $P_p\phi \wedge P_p\psi \to P_p(\phi \wedge \psi) \vee P_p(\phi \wedge P_p\psi) \vee P_p(P_p\phi \wedge \psi)$
(A.5) $P_p\phi \wedge P_c\psi \to P_p(\phi \wedge \psi) \vee P_p(\phi \wedge P_c\psi) \vee P_p(P_c\phi \wedge \psi)$

The following characterization is straightforward and its proof is omitted.

Lemma 2.4. *Let $\mathcal{F} = \langle T, \prec_c, \prec_p \rangle$ be an arbitrary (not necessarily branching-time) frame. Then, for $i = 1, ..., 5$, $\mathcal{F}$ satisfies (R.i) if and only if (A.i) is valid in $\mathcal{F}$.*

We denote by $\mathbb{L}_0$ the basic system of temporal logic (see Burgess, 1984) and by $\mathbb{L}$ be the extension of $\mathbb{L}_0$ obtained by adding (A.1)-(A.5).

Theorem 2.5. *$\mathbb{L}$ is sound and complete with respect to the class of P-frames, that is, a formula is a theorem of $\mathbb{L}$ if and only if it valid in every P-frame.*

Soundness follows from Lemma 2.4. For the completeness proof we follow the constructive approach put forward by Burgess (1984). We shall only give the main steps. Let $Max\mathbb{L}$ be the set of maximal $\mathbb{L}$-consistent sets of formulae. Define the following relations on $Max\mathbb{L}$: $A \twoheadrightarrow_c B$ if and only if, for every formula ϕ, if $G_c\phi \in A$ then $\phi \in B$, and $A \twoheadrightarrow_p B$ if and only if, for every formula ϕ, if $G_p\phi \in A$ then $\phi \in B$.

The next two lemmas are well-known (cf. Burgess 1984, Lemmas 1.6 and 1.7).

Lemma 2.6. *Let $A, B \in Max\mathbb{L}$. Then the following are equivalent (as well as the corresponding versions with the 'p' subscript replaced by the 'c' subscript): (i) $A \twoheadrightarrow_p B$, (ii) for every formula ϕ, if $\phi \in A$ then $P_p\phi \in B$, (iii) for every formula ϕ, if $\phi \in B$ then $F_p\phi \in A$, (iv) for every formula ϕ, if $H_p\phi \in B$ then $\phi \in A$.*

Lemma 2.7. *Let $B \in Max\mathbb{L}$ and ϕ be any formula. Then (a) if $F_c\phi \in B$, then there exists a $D \in Max\mathbb{L}$ with $B \twoheadrightarrow_c D$ and $\phi \in D$; (b) if $P_c\phi \in B$, then there exists an $A \in Max\mathbb{L}$ with $A \twoheadrightarrow_c B$ and $\phi \in A$; (c) if $F_p\phi \in B$, then there exists a $D \in Max\mathbb{L}$ with $B \twoheadrightarrow_p D$ and $\phi \in D$; (d) if $P_p\phi \in B$, then there exists an $A \in Max\mathbb{L}$ with $A \twoheadrightarrow_p B$ and $\phi \in A$.*

Lemma 2.8. *The relation $\twoheadrightarrow_c$ on $Max\mathbb{L}$ is transitive and backward linear, $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$ and is backward linear, $\twoheadrightarrow_c$ and $\twoheadrightarrow_p$ satisfy the following property:*
if $A \twoheadrightarrow_p D$ and $B \twoheadrightarrow_c D$ then either $A = B$ or $B \twoheadrightarrow_c A$ or ($A \twoheadrightarrow_c B$ and $B \twoheadrightarrow_p D$).

Proof. We shall only prove the last property. Suppose that $A \twoheadrightarrow_p D$, $B \twoheadrightarrow_c D$, $A \neq B$ and $B \not\twoheadrightarrow_c A$ ($\not\twoheadrightarrow_c$ denotes the negation of $\twoheadrightarrow_c$). We need to show that $A \twoheadrightarrow_c B$ and $B \twoheadrightarrow_p D$. Since $A \neq B$, there exists a $\phi \in A$ such that $\neg\phi \in B$. Since $B \not\twoheadrightarrow_c A$, by Lemma 2.6 there exists a $\psi \in B$ such that $\neg P_c\psi \in A$. Thus

$$\phi \wedge \neg P_c\psi \in A \quad \textit{and} \quad \psi \wedge \neg\phi \in B. \tag{2.1}$$

Suppose that $A \not\twoheadrightarrow_c B$. Then by Lemma 2.6 there exists $\theta \in A$ such that $\neg P_c\theta \in B$. Then, using (2.1),

$$\theta \wedge \phi \wedge \neg P_c\psi \in A \quad \textit{and} \quad \psi \wedge \neg\phi \wedge \neg P_c\theta \in B. \tag{2.2}$$

Since $A \twoheadrightarrow_p D$, it follows from the first part of (2.2) and Lemma 2.6 that,

$$P_p\left(\theta \wedge \phi \wedge \neg P_c\psi\right) \in D. \tag{2.3}$$

Since $B \twoheadrightarrow_c D$, by the second part of (2.2) and Lemma 2.6,

$$P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right) \in D. \tag{2.4}$$

Thus, by (2.3) and (2.4) and axiom (A.5), either (i) $P_p(\theta\wedge\phi\wedge\neg P_c\psi\wedge\psi\wedge\neg\phi\wedge\neg P_c\theta) \in D$, or (ii) $P_p\left(\theta \wedge \phi \wedge \neg P_c\psi \wedge P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right)\right) \in D$, or (iii) $P_p\left(P_c\left(\theta \wedge \phi \wedge \neg P_c\psi\right) \wedge \psi \wedge \neg\phi \wedge \neg P_c\theta\right) \in D$. Case (i) is impossible because $\phi \wedge \neg\phi$ is a contradiction. Case (ii) is impossible because $P_c\left(\psi \wedge \neg\phi \wedge \neg P_c\theta\right)$ implies $P_c\psi$, contradicting $\neg P_c\psi$. Case (iii) is impossible, since $P_c\left(\theta \wedge \phi \wedge \neg P_c\psi\right)$ implies $P_c\theta$, contradicting $\neg P_c\theta$. Hence it must be that $A \twoheadrightarrow_c B$. Suppose now that $B \not\twoheadrightarrow_p D$. Then, by Lemma 2.6, $\exists\gamma \in B$ such that

$$\neg P_p\gamma \in D \tag{2.5}$$

Using the second part of (2.1),

$$\gamma \wedge \psi \wedge \neg\phi \in B. \tag{2.6}$$

Since $A \twoheadrightarrow_p D$, by the first part of (2.1) and Lemma 2.6,

$$P_p\left(\phi \wedge \neg P_c\psi\right) \in D. \tag{2.7}$$

Since $B \twoheadrightarrow_c D$, by (2.6) and Lemma 2.6,

$$P_c(\gamma \wedge \psi \wedge \neg\phi) \in D. \tag{2.8}$$

Thus by (2.7) and (2.8) and axiom (A.5) either (i) $P_p(\phi\wedge\neg P_c\psi\wedge\gamma\wedge\psi\wedge\neg\phi) \in D$, or (ii) $P_p\left(\phi \wedge \neg P_c\psi \wedge P_c\left(\gamma \wedge \psi \wedge \neg\phi\right)\right) \in D$, or (iii) $P_p\left(P_c\left(\phi \wedge \neg P_c\psi\right) \wedge \gamma \wedge \psi \wedge \neg\phi\right) \in D$. Case (i) is impossible because $\phi\wedge\neg\phi$ is a contradiction. Case (ii) is impossible because $P_c\left(\gamma \wedge \psi \wedge \neg\phi\right)$ implies $P_c\psi$, contradicting $\neg P_c\psi$. Case (iii) is impossible because it implies $P_p\gamma \in D$, contradicting (2.5). Thus it must be $B \twoheadrightarrow_p D$. ■

Definition 2.9. *A partial canonical P-frame is a quadruple* $\langle X, \prec_c, \prec_p, f\rangle$ *such that: (1)* X *is a finite set, (2)* $\langle X, \prec_c, \prec_p\rangle$ *is a P-frame and (3)* $f : X \to Max\mathbb{L}$*. A partial canonical P-frame is coherent if,* $\forall x, y \in X$*, (1) if* $x \prec_c y$ *then* $f(x) \twoheadrightarrow_c f(y)$*, and (2) if* $x \prec_p y$ *then* $f(x) \twoheadrightarrow_p f(y)$*.*

Let ϕ *be a formula and* $\langle X, \prec_c, \prec_p, f\rangle$ *a partial canonical P-frame. We say that* $F_c\phi$ *(resp.* $F_p\phi$*) is not satisfied at* $x \in X$*, if* $F_c\phi \in f(x)$ *(resp.* $F_p\phi \in f(x)$*) and there is no* $y \in X$ *such that* $x \prec_c y$ *(resp.* $x \prec_p y$*) and* $\phi \in f(y)$*. Similarly for* $P_c\phi$ *and* $P_p\phi$*.*

Given partial canonical P-frames $\langle X, \prec_c, \prec_p, f\rangle$ *and* $\langle X', \prec'_c, \prec'_p, f'\rangle$ *we say that the latter is an extension of the former if : (1)* $X \subseteq X'$*, (2)* $\prec_c \subseteq \prec'_c$*, (3)* $\prec_p \subseteq \prec'_p$*, and (4)* $f \subseteq f'$*.*

The following lemma gives the main step in the completeness proof.

Lemma 2.10. ***Extension Lemma****. Let* $\langle X, \prec_c, \prec_p, f\rangle$ *be a coherent partial canonical P-frame and let* ϕ *be a formula.*
(a) *Suppose that* $F_c\phi$ *is not satisfied at* $x \in X$*. Then there exists a coherent extension* $\langle X', \prec'_c, \prec'_p, f'\rangle$ *of* $\langle X, \prec_c, \prec_p, f\rangle$ *and a* $y \in X'$ *such that* $x \prec'_c y$ *and* $\phi \in f(y)$*.*
(b) *Suppose that* $F_p\phi$ *is not satisfied at* $x \in X$*. Then there exists a coherent extension* $\langle X', \prec'_c, \prec'_p, f'\rangle$ *and a* $y \in X'$ *such that* $x \prec'_p y$ *and* $\phi \in f(y)$*.*
(c) *Suppose that* $P_c\phi$ *is not satisfied at* $x \in X$*. Then there exists a coherent extension* $\langle X', \prec'_c, \prec'_p, f'\rangle$ *and a* $y \in X'$ *such that* $y \prec'_c x$ *and* $\phi \in f(y)$*.*
(d) *Suppose that* $P_p\phi$ *is not satisfied at* $x \in X$*. Then there exists a coherent extension* $\langle X', \prec'_c, \prec'_p, f'\rangle$ *and a* $y \in X'$ *such that* $y \prec'_p x$ *and* $\phi \in f(y)$*.*

Proof. **(a)** Let $x \in X$ and $F_c\phi \in f(x)$. By Lemma 2.7, $\exists B \in Max\mathbb{L}$ such that $f(x) \twoheadrightarrow_c B$ and $\phi \in B$. Construct the following extension of $\langle X, \prec_c, \prec_p, f\rangle$ obtained by (i) adding a new point y, (ii) assigning the set B to y, and (iii) adding the pair (x, y) to $\prec_c$ (and any new pair needed to preserve transitivity), while no pairs are added to $\prec_p$. Let $y \notin X$ and $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(x, y)\} \cup \{(v, y) : v \prec_c x\}$, $\prec'_p = \prec_p$ and $f' = f \cup \{(y, B)\}$. That the new frame is coherent follows from transitivity of $\twoheadrightarrow_c$ (cf. Lemma 2.8). It is also clear that the new frame is a P-frame, since the original frame was a P-frame, transitivity of $\prec_c$ has been preserved and no $\prec_p$-pairs have been added.

(b) Let $x \in X$ and $F_p\phi \in f(x)$. By Lemma 2.7 $\exists B \in Max\mathbb{L}$ such that $f(x) \twoheadrightarrow_p B$ and $\phi \in B$. Construct the following extension: let $y \notin X$ and $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(x, y)\} \cup \{(v, y) : v \prec_c x\}$, $\prec'_p = \prec_p \cup \{(x, y)\}$ and $f' = f \cup \{(y, B)\}$. That the new frame is coherent follows from Lemma 2.8 ($\twoheadrightarrow_p$ subrelation of $\twoheadrightarrow_c$ and transitivity of $\twoheadrightarrow_c$). Furthermore, given that the original frame was a P-frame, the new frame is also a P-frame: transitivity of $\prec_c$ has been preserved, and violation of (CP) in the new frame could only occur if $v \prec'_p y$ for some $v \in X$ such that $v \prec_c x$, but no such pair (v, y) has been added to $\prec_p$.

(c) Let $P_c\phi \in f(x)$ and suppose there is no $y \in X$ such that $y \prec_c x$ and $\phi \in f(y)$. We proceed by induction on the number n of $\prec_c$-predecessors of x in X. Suppose $n = 0$. Since $P_c\phi \in f(x)$, by Lemma 2.7 there exists a $B \in Max\mathbb{L}$ such that $B \twoheadrightarrow_c f(x)$ and $\phi \in B$. Construct the following extension obtained by (i) adding a new point y, (ii) assigning the set B to y, and (iii) adding the pair (y, x) to $\prec_c$

(and any new pair needed to preserve transitivity), while no pairs are added to $\prec_p$. Let $y \notin X$ and $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(y,x)\} \cup \{(y,v) : x \prec_c v\}$, $\prec'_p = \prec_p$ and $f' = f \cup \{(y,B)\}$. Coherence follows from transitivity of $\twoheadrightarrow_c$. Furthermore the new frame is a P-frame, since the original frame was a P-frame, transitivity of $\prec_c$ has been preserved and no $\prec_p$-pairs have been added. Suppose now that $n \geq 1$. Let x' be the *immediate* $\prec_c$-predecessor of x in X (recall that X is finite). By our supposition, $\phi \notin f(x')$. If $P_c\phi \in f(x')$ then we can reduce (by appealing to transitivity of $\prec_c$ and $\twoheadrightarrow_c$) to the case $n-1$ by replacing x with x'. Assume therefore that $P_c\phi \notin f(x')$. Then, by definition of maximal consistent set, $(\neg\phi \wedge \neg P_c\phi) \in f(x')$. We need to distinguish two cases. **CASE 1**: $x' \not\prec_p x$. By Lemma 2.7, since $P_c\phi \in f(x)$ there exists a $B \in Max\mathbb{L}$ such that $B \twoheadrightarrow_c f(x)$ and $\phi \in B$. Construct the following extension, obtained by (i) inserting a new point y between x' and x and assigning the set B to it, (ii) adding the pairs (x', y) and (y, x) to $\prec_c$ (and any new pair needed to preserve transitivity), while no pairs are added to $\prec_p$. Let $y \notin X$ and $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(x',y),(y,x)\} \cup \{(v,y) : v \prec_c x'\} \cup \{(y,w) : x \prec_c w\}$, $\prec'_p = \prec_p$ and $f' = f \cup \{(y,B)\}$. To verify coherence, besides appealing to transitivity of $\twoheadrightarrow_c$, we need to show that $f(x') \twoheadrightarrow_c B$. By coherence of $\langle X, \prec_c, \prec_p, f\rangle$, $f(x') \twoheadrightarrow_c f(x)$. Thus, since $B \twoheadrightarrow_c f(x)$, by backward linearity of $\twoheadrightarrow_c$ (cf. Lemma 2.8) either (i) $f(x') = B$ or (ii) $B \twoheadrightarrow_c f(x')$ or (iii) $f(x') \twoheadrightarrow_c B$. Case (i) is ruled out by $\phi \in B$ and $\phi \notin f(x')$. Case (ii) is ruled out by $\phi \in B$ and $P_c\phi \notin f(x')$ (cf. Lemma 2.6). Thus $f(x') \twoheadrightarrow_c B$. Furthermore, the new frame is a P-frame, since the original frame was a P-frame, transitivity of $\prec_c$ has been preserved and inserting a point between x' and x without adding any $\prec_p$-pairs would have violated property (CP) only if it had been the case that $x' \prec_p x$, contrary to our supposition. **CASE 2**: $x' \prec_p x$. By coherence of $\langle X, \prec_c, \prec_p, f\rangle$, since $(\neg\phi \wedge \neg P_c\phi) \in f(x')$, $P_p(\neg\phi \wedge \neg P_c\phi) \in f(x)$ (cf. Lemma 2.6). Thus $P_p(\neg\phi \wedge \neg P_c\phi) \wedge P_c\phi \in f(x)$. By definition of maximal consistent set, axiom (A.5) belongs to $f(x)$. Thus

$$P_p(\neg\phi \wedge \neg P_c\phi \wedge \phi) \vee P_p(\neg\phi \wedge \neg P_c\phi \wedge P_c\phi) \vee P_p(P_c(\neg\phi \wedge \neg P_c\phi) \wedge \phi) \in f(x)$$

But $P_p(\neg\phi \wedge \neg P_c\phi \wedge \phi) \notin f(x)$ because $(\neg\phi \wedge \phi)$ is a contradiction. For the same reason, $P_p(\neg\phi \wedge \neg P_c\phi \wedge P_c\phi) \notin f(x)$. Thus $P_p(P_c(\neg\phi \wedge \neg P_c\phi) \wedge \phi) \in f(x)$. Then by Lemma 2.7, $\exists D \in Max\mathbb{L}$ such that $D \twoheadrightarrow_p f(x)$ and $P_c(\neg\phi \wedge \neg P_c\phi) \wedge \phi \in D$. Construct the following extension: let $y \notin X$ and $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(x',y),(y,x)\} \cup \{(v,y) : v \prec_c x'\} \cup \{(y,w) : x \prec_c w\}$, $\prec'_p = \prec_p \cup \{(x',y),(y,x)\}$ and $f' = f \cup \{(y,D)\}$. To verify coherence, besides appealing to transitivity of $\twoheadrightarrow_c$ and the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$, we need to show that $f(x') \twoheadrightarrow_p D$. Since $f(x') \twoheadrightarrow_p f(x)$ (by coherence of $\langle X, \prec_c, \prec_p, f\rangle$) and $D \twoheadrightarrow_p f(x)$, by backward linearity of $\twoheadrightarrow_p$ (cf. Lemma 2.8) either (i) $f(x') = D$ or (ii) $D \twoheadrightarrow_p f(x')$ or (iii) $f(x') \twoheadrightarrow_p D$. Case (i) is ruled out by $\phi \in D$ and $\phi \notin f(x')$. Suppose (ii) were the case. Then, since $\phi \in D$, $P_p\phi \in f(x')$. But by (A.3) $P_p\phi \rightarrow P_c\phi \in f(x')$. Thus we would get $P_c\phi \in f(x')$, contradicting the fact that $\neg P_c\phi \in f(x')$. Hence it must be $f(x') \twoheadrightarrow_p D$. Furthermore, given that the original frame was a P-frame, the new frame is also P-frame: transitivity of $\prec_c$ has been preserved and property (CP) is preserved since the new path from x' to x is both a $\prec_p$-path and a $\prec_c$-path and no pairs of the form (v,y) with $v \prec_c x'$ or (y,w) with $x \prec_c w$ have been added to $\prec_p$.

(d) Let $P_p\phi \in f(x)$ and suppose there is no $y \in X$ such that $y \prec_p x$ and $\phi \in f(y)$. Since $P_p\phi \in f(x)$, by Lemma 2.7 there exists a $B \in Max\mathbb{L}$ such that

$$B \twoheadrightarrow_p f(x) \quad \text{and} \quad \phi \in B. \tag{2.9}$$

We need to distinguish several cases. **CASE 1***:* x has no $\prec_c$-predecessors (hence no $\prec_p$-predecessors) in X. Construct the following extension: let $y \notin X$ and $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(y,x)\} \cup \{(y,v) : x \prec_c v\}$, $\prec'_p = \prec_p \cup \{(y,x)\}$, $f' = f \cup \{(y,B)\}$. In this case coherence follows from (2.9), transitivity of $\twoheadrightarrow_c$ and the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$ (cf. Lemma 2.8). Furthermore, given that the original frame was a P-frame, the new frame is also a P-frame: transitivity of $\prec_c$ has been preserved and property (CP) is preserved since no pair of the form (y,w) with $x \prec_c w$ has been added to $\prec_p$. **CASE 2***:* x has at least one $\prec_c$-predecessor but no $\prec_p$-predecessors in X. Let x' be the *immediate* $\prec_c$-predecessor of x in X (recall that X is finite). If either $B = f(x')$ or $f(x') \twoheadrightarrow_p f(x)$ and $\phi \in f(x')$ then it is sufficient to add (x',x) to $\prec_p$. Suppose therefore that $B \neq f(x')$ and either $f(x') \not\twoheadrightarrow_p f(x)$ or $f(x') \twoheadrightarrow_p f(x)$ and $\phi \notin f(x')$. The case $B \neq f(x')$ and $f(x') \not\twoheadrightarrow_p f(x)$, in conjunction with $B \twoheadrightarrow_p f(x)$ and $f(x') \twoheadrightarrow_c f(x)$ (the latter following from coherence of the given frame), yields, by Lemma 2.8, $f(x') \twoheadrightarrow_c B$. Construct the following extension: let $y \notin X$, $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(x',y),(y,x)\} \cup \{(w,y) : w \prec_c x'\} \cup \{(y,z) : x \prec_c z\}$, $\prec'_p = \prec_p \cup \{(y,x)\}$, $f' = f \cup \{(y,B)\}$. Coherence follows from (2.9), the fact that $f(x') \twoheadrightarrow_c B$, transitivity of $\twoheadrightarrow_c$ and the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$ (cf. Lemma 2.8). Furthermore, the new frame is a P-frame, since transitivity of $\prec_c$ has been preserved and violation of property (CP) in the new frame could only arise if in the original frame $w \prec_p x$ for some $w \in X$ with either $w = x'$ or $w \prec_c x'$. However, since the original frame was a P-frame, in either case this would require $x' \prec_p x$, which, by coherence of the original frame, would imply that $f(x') \twoheadrightarrow_p f(x)$, contradicting our supposition. The only case that remains to be considered is the case where $f(x') \twoheadrightarrow_p f(x)$ and $\phi \notin f(x')$. In this case, by adding (x',x) to $\prec_p$ (coherence as well as the property of being a P-frame are clearly preserved) we can reduce to Case 3 below where x has at least one $\prec_p$-predecessor. **CASE 3***:* x has at least one $\prec_p$-predecessor (hence also at least one $\prec_c$-predecessor). Denote x by x_0 and let $x_1,...,x_n$ $(n \geq 1)$ be the $\prec_c$-predecessors of x_0 numbered so that, $\forall k = 1,...,n$, x_k is the immediate $\prec_c$-predecessor of x_{k-1} (recall that X is finite). By property (CP), there is an $m \leq n$ such that the $\prec_p$-predecessors of x_0 are precisely $x_1,...,x_m$ (and $\forall k = 1,...,m$, x_k is the immediate $\prec_p$-predecessor of x_{k-1}).[5] Since $P_p\phi$ is not satisfied at x_0, by coherence it must be that $\phi \notin f(x_k)$ for all $k = 1,...,m$. Thus, since $\phi \in B$, $B \neq f(x_k)$ for all $k = 1,...,m$. Furthermore, we can also assume that $B \neq f(x_0)$ (otherwise it would be sufficient to (i) add a copy of x_0, call it y, between x_0 and x_1, (ii) add (x_1,y) and (y,x_0) (and any pairs needed to preserve transitivity) to both $\prec_c$ and $\prec_p$). Thus

$$B \neq f(x_k), \;\; \forall k = 0,...,m \tag{2.10}$$

By coherence of the given frame, $f(x_k) \twoheadrightarrow_p f(x_0)$ $\forall k = 1,...,m$. This, together with (2.9) and (2.10) yields, by backward linearity of $\twoheadrightarrow_p$ (cf. Lemma 2.8),

[5] By (CP), if x_k is a $\prec_p$-predecessor of x_0 then the entire path from x_k to x_0 belongs to $\prec_p$.

$$\forall k = 1, ..., m, \text{ either } \quad f(x_k) \twoheadrightarrow_p B \text{ or } \quad B \twoheadrightarrow_p f(x_k). \tag{2.11}$$

Suppose first that $f(x_k) \twoheadrightarrow_p B$ for some $k \in \{1, ..., m\}$. Let k_0 be the smallest such integer. Then, for all $j \in \{1, ..., k_0 - 1\}$, $f(x_j) \not\twoheadrightarrow_p B$. It follows from this and (2.11) that, for all $j \in \{1, ..., k_0 - 1\}$, $B \twoheadrightarrow_p f(x_j)$. Thus, using also (2.9)

$$f(x_{k_0}) \twoheadrightarrow_p B \text{ and } \forall j \in \{0, ..., k_0 - 1\}, \ B \twoheadrightarrow_p f(x_j). \tag{2.12}$$

Construct the following extension: let $y \notin X$, $X' = X \cup \{y\}$, $\prec'_c = \prec_c \cup \{(x_{k_0}, y), (y, x_{k_0 - 1})\} \cup \{(w, y) : w \prec_c x_{k_0}\} \cup \{(y, z) : x_{k_0 - 1} \prec_c z\}$, $\prec'_p = \prec_p \cup \{(x_{k_0}, y)\} \cup \{(y, x_j) : j = 0, ..., k_0 - 1\}$, $f' = f \cup \{(y, B)\}$. Coherence follows from transitivity of $\twoheadrightarrow_c$, the fact that $\twoheadrightarrow_p$ is a subrelation of $\twoheadrightarrow_c$ (cf. Lemma 2.8) and (2.12). Furthermore, that the new frame is a P-frame is clear from the construction (in particular, both (x_{k_0}, y) and $(y, x_{k_0 - 1})$ have been added to $\prec_p$) and the fact that the original frame was a P-frame. The only case that remains to be considered is the case where $f(x_k) \not\twoheadrightarrow_p B$ for all $k = 1, ..., m$. Then, by (2.11), $B \twoheadrightarrow_p f(x_k)$, for all $k = 0, ..., m$. It follows from this and Lemma 2.6 that $P_p\phi \in f(x_k)$ for all $k = 1, ..., m$. In particular, $P_p\phi \in f(x_m)$. Since x_m has no $\prec_p$-predecessor in X, we can now construct a coherent extension of the initial frame where $P_p\phi$ is satisfied at x_m as explained in CASE 1 (if $n = m$) or in CASE 2 (if $n > m$)[6] using the set B of (2.9). The only modification of that construction consists in adding also the pair (y, x_0) to $\prec_p$ (and to $\prec_c$, as well as any pairs needed to preserve transitivity), thereby guaranteeing satisfaction of $P_p\phi$ at x_0. Coherence is guaranteed by the fact that $B \twoheadrightarrow_p f(x_0)$ (cf. (2.9)). Furthermore, property (CP) is preserved since the entire $\prec_c$-path from y to x_0 belongs to $\prec_p$. ■

The final step in the completeness proof (construction of a perfect chronicle) is entirely standard (cf. Burgess, 1984) and will be omitted.

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[6] This final step involves the possibility of going back to Case 2. Under Case 2 the possibility of a reduction to Case 3 arose. Note, however, that this does not involve a circular reasoning, because X is finite and hence the number of $\prec_c$-predecessors of x is finite. Each reduction (from Case 3 to Case 2 or *vice versa)* does not alter the number of $\prec_c$-predecessors of x while it shifts the argument up one step in the chain of $\prec_c$-predecessors of x (it turns an immediate $\prec_c$-predecessor into a $\prec_p$-predecessor also). Thus eventually one must fall within Case 1.