**Published in Synthese, 139 (2), March 2004, pp. 237-256 (and Knowledge, Rationality and Action, 2004, pp. 117-131)**

# A characterization of von Neumann games in terms of memory

Giacomo Bonanno*
Department of Economics,
University of California,
Davis, CA 95616-8578, USA
e-mail: gfbonanno@ucdavis.edu

First draft: May 2001. Final version: November 2003

**Abstract**

An information completion of an extensive game is obtained by extending the information partition of every player from the set of her decision nodes to the set of all nodes. The extended partition satisfies *Memory of Past Knowledge* (MPK) if at any node a player remembers what she knew at earlier nodes. It is shown that MPK can be satisfied in a game if and only if the game is von Neumann (vN) and satisfies memory at decision nodes (the restriction of MPK to a player's own decision nodes). A game is vN if any two decision nodes that belong to the same information set of a player have the same number of predecessors. By providing an axiom for MPK we also obtain a syntactic characterization of the said class of vN games.

## 1 Introduction

The standard definition of extensive game (see, for example, Selten, 1975) specifies a player's information only when it is her turn to move (that is, only at her decision nodes), thus providing only a partial description of what the player learns during any play of the game. For both conceptual and practical reasons (see, for example, Battigalli and Bonanno, 1999, and van Benthem, 2001), it may be desirable to express what a player knows also at nodes where she does not have to move, that is, at nodes that belong to another player. For example, one might want to model what information is (or can be) given to player $i$ after some other player has made a move, even if it is not player $i$'s turn to move. In order to be able to do so one needs to add, for every player, a partition of the set

*I am grateful to two anonymous referees for helpful comments and suggestions. A first draft of this paper was presented at the XIV Meeting on Game Theory and Applications, Ischia, July 2001.

of *all* nodes, which – when restricted to that player's decision nodes – coincides with her initial information partition (thus preserving the original information sets).[1] In this paper we study one aspect of memory within the context of such extended partitions.

In the philosophy literature the concept of memory has been identified with the retention of past knowledge (see, for example, Malcolm, 1963 and Munsat, 1966). In accordance with this, we define *Memory of Past Knowledge* ($MPK$) as the property that at any node the player remembers what she knew at earlier nodes. This is a natural property to consider and, indeed, the restriction of it to a player's own decision nodes is implied by the notion of perfect recall, which is routinely assumed in game theory. We show that $MPK$ can be satisfied only within the class of games that Kuhn (1953) calls von Neumann (vN) games. An extensive game is vN if any two decision nodes of player $i$ that belong to the same information set of player $i$ have the same number of predecessors. We prove that a game satisfies $MPK$ if and only if it is a vN game and, for each player, the restriction of $MPK$ to that player's decision nodes is satisfied. We call the latter property "Memory at Decision Nodes" ($MDN$).

We also show that an implication of $MPK$ is that, at every stage of the game, it is common knowledge among all the players that the play of the game has reached that stage (if node $x$ has $k$ predecessors, that is, if the path from the root to $x$ has length $k$, then we say that $x$ belongs to stage $k$). One can think of the stage of the game as the number of units of time that have elapsed since the beginning of the game. Thus $MPK$ implies that the time is always common knowledge among the players. In this respect vN games that satisfy $MPK$ are closely related to the synchronous systems studies in the computer science literature, where the agents have access to an external clock (see, for example, Halpern and Vardi, 1986).

In Section 3 we show that the proposed notion of memory on extended partitions does indeed capture the interpretation of memory as retention of past knowledge: we show that it is characterized by either of the following axioms:

1. If in the past the player knew $\phi$ then she knows now that in the past she knew $\phi$,

2. If the player knows $\phi$ now, then at every future time she will know that in the past she knew $\phi$.

Thus either axiom provides a syntactic characterization of the class of von Neumann games that satisfy Memory at Decision Nodes.

[1] To avoid confusion, throughout the paper we use the expression "player $i$'s information partition" to refer to the standard partition of $i$'s *decision* nodes. The elements of this partition will always be referred to as "information sets". On the other hand, player $i$'s partition of the set of *all* nodes will be called "$i$'s extended partition" and its elements will be called "cells".

## 2 Extended partitions and Memory

We use the tree-based definition of extensive game, which is due to Kuhn (1953). Since our analysis deals with the structure of moves and information, and is independent of payoffs, we shall focus on *extensive forms* and follow closely the definition given by Selten (1975). The first component of an extensive form is a finite or infinite rooted tree $\langle T, \to, t_0 \rangle$ where $t_0$ denotes the root and, for any two nodes $t, x \in T$, $t \to x$ denotes that $t$ is the *immediate predecessor* of $x$ (or $x$ is an *immediate successor* of $t$). For every node $t$ it is assumed that the number of immediate successors of $t$ is finite (possibly zero). We denote by $\prec$ the transitive closure of $\to$. Thus $t \prec x$ denotes that $t$ is a *predecessor* of $x$ or $x$ is a *successor* of $t$ (that is, there is a path from $t$ to $x$) and we use $t \precsim x$ to mean that either $t = x$ or $t \prec x$. For example, in the extensive form of Figure 1 we have that $t \to x$ and $t \prec z_3$. Let $Z$ be the set of *terminal nodes*, that is, nodes that have no successors and $X = T \backslash Z$ the set of *decision nodes.* For example, in Figure 1, $Z = \{z_1, z_2, ..., z_7\}$ and $X = \{t_0, t, t', y, x, x'\}$.

The second component of an extensive form is a set of *players* $N = \{1, 2, ..., n\}$ and a partition $\{X_i\}_{i \in N}$ of the set of decision nodes $X$. For every player $i \in N$, $X_i$ is the set of decision nodes of player $i$. In the extensive form of Figure 1, $N = \{1, 2\}$, the set of player 1's decision nodes is $X_1 = \{t_0, y\}$ and the set of player 2's decision nodes is $X_2 = \{t, t', x, x'\}$.

The third component is, for every player $i \in N$, an equivalence relation $\sim_i \subseteq X_i \times X_i$ (that is, a binary relation that is reflexive, symmetric and transitive) satisfying the following constraint: if $t, t' \in X_i$ and $t \sim_i t'$ then the number of immediate successors of $t$ is equal to the number of immediate successors of $t'$. The interpretation of $t \sim_i t'$ is that player $i$ cannot distinguish between $t$ and $t'$, that is, as far as she knows, she could be making a decision either at node $t$ or at node $t'$. The equivalence classes of $\sim_i$ partition $X_i$ and are called the *information sets* of player $i$. We denote by $\mathbb{H}_i$ the set of information sets of player $i$. In the extensive form of Figure 1, $\sim_1 = \{(t_0, t_0), (y, y)\}$ and $\sim_2 = \{(t, t), (t, t'), (t', t), (t', t'), (x, x), (x, x'), (x', x), (x', x')\}$. Thus, for example, player 2's information sets are $\{t, t'\}$ and $\{x, x'\}$, that is, $\mathbb{H}_2 = \{\{t, t'\}, \{x, x'\}\}$. We use the graphic convention of representing an information set as a rounded rectangle enclosing the corresponding nodes, if there are at least two nodes, while if an information set is a singleton we do not draw anything around it. Furthermore, since all the nodes in an information set belong to the same player, we write the corresponding player only once inside the rectangle.

The fourth, and last, component of an extensive form is, for every player $i \in N$, a *choice partition*, which, for each of her information sets, partitions the edges out of nodes in that information set (that is, the set of ordered pairs $(t, x)$ such that $t \to x$) into player $i$'s *choices* at that information set. If $(t, x)$ belongs to choice $c$ we write $t \to_c x$. The choice partition satisfies the following constraints: (1) if $t \to_c x$ and $t \to_c x'$ then $x = x'$, and (2) if $t \to_c x$ and $t \sim_i t'$ then there exists an $x'$ such that $t' \to_c x'$. The first condition says that a choice at a node selects a unique immediate successor, while the second condition says that if a choice is available at one node of an information set then it is available at every node in that information set. For example, in Figure 1, $x \to_g z_2$ and $x' \to_g z_4$, so that player 2's choice $g$ is $\{(x, z_2), (x', z_4)\}$. Graphically we represent choices by labeling the corresponding edges in such a way that two edges belong to the same choice if and only if they are assigned the same label.

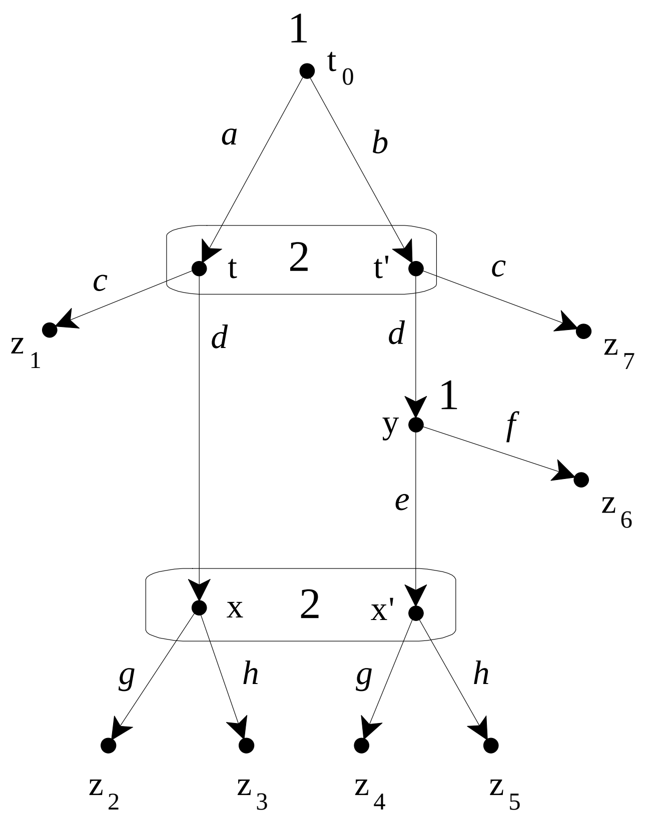


**Figure 1**

The main focus in game theory has been on games with perfect recall.[2] An extensive form is said to have *perfect recall* if "for every player $i$ and for any two information sets $g$ and $h$ of player $i$, if one vertex $y \in h$ comes after a choice $c$ at $g$ then every vertex $x \in h$ comes after this choice $c$" (Selten, 1975; p. 319 of Kuhn, 1997). For example, the extensive form of Figure 1 satisfies perfect recall (both $x$ and $x'$ come after the same choice at the earlier information set $\{t, t'\}$ of player 2, namely choice $d$ ). It is shown in Bonanno (2003) that perfect recall is equivalent to the conjunction of two independent properties, one expressing memory of past actions and the other memory of past knowledge. In

[2]The notion of perfect recall was introduced by Kuhn (1953), who interprets it as follows: "this condition is equivalent to the assertion that each player is allowed by the rules of the game to remember everything he knew at previous moves and all of his choices at those moves" (p. 65 of Kuhn, 1997). In the computer science literature the expression "perfect recall" has been used to denote a weaker property (see the next fotnote).

this paper we focus on the latter. We call "Memory at Decision Nodes" ($MDN$) the following property (which is a weakening of perfect recall): if one node in information set $h$ of player $i$ has a predecessor that belongs to information set $g$ of the same player $i$, then *every* node in $h$ has a predecessor in $g$.[3] Formally (recall that $\mathbb{H}_i$ denotes the set of information sets of player $i$):

$$\begin{array}{l}\text{if } x \prec y,\ x \in g \in \mathbb{H}_i,\ \ y \in h \in \mathbb{H}_i,\ \text{ and } y' \in h, \\ \text{then there exists an } x' \in g \ \text{ such that } x' \prec y'.\end{array} \qquad (MDN)$$

This means that, when it is her turn to move, a player always remembers what she knew at earlier decision nodes of hers. Note that this property is considerably weaker than perfect recall, since it is *independent of choices*. For example, if the extensive form of Figure 1 is modified in such a way that $(t, x)$ and $(t', y)$ belong to *different* choices of player 2,[4] then it will still satisfy $MDN$ but it will violate perfect recall. In this paper we *shall not assume perfect recall*, although we will restrict attention to extensive forms that satisfy the weaker property $MDN$. Our purpose is to study an extension of this property from the set of decision nodes of player $i$ to the set of all the nodes. This requires extending the notion of information set.

**Definition 1** *An* information completion *of an extensive form is an $n$-tuple $\langle \mathcal{K}_1, ..., \mathcal{K}_n \rangle$ where, for each player $i = 1, ..., n$, $\mathcal{K}_i$ is a partition of the set of nodes $T$ that agrees on player $i$'s information sets, in the sense that if node $t$ belongs to information set $h$ of player $i$ then the cell of $\mathcal{K}_i$ that contains $t$ − denoted by $\mathcal{K}_i(t)$ − coincides with $h$. Formally: if $t \in h \in \mathbb{H}_i$ then $\mathcal{K}_i(t) = h$.*

We call *Memory of Past Knowledge (MPK)* the extension of $MDN$ to the extended partition $\mathcal{K}_i$: $\forall x, y, y' \in T$, $\forall i \in N$,

$$\begin{array}{l}\text{if } \ x \prec y \ \text{ and } \ y' \in \mathcal{K}_i(y) \\ \text{then there exists an } \ x' \in \mathcal{K}_i(x) \ \text{ such that } \ x' \prec y'.\end{array} \qquad (MPK)$$

In Section 3 we show that $MPK$ does indeed correspond to the syntactic notion of remembering what one knew in the past. In this section we prove that $MPK$ can be only be satisfied in von Neumann extensive forms.

For every node $t \in T$, we denote by $\ell(t)$ the number of predecessors of $t$ (i.e. the length of the path from the root to $t$). The following definition is taken from Kuhn (1953; p. 52 of Kuhn, 1997).

**Definition 2** *An extensive form is* von Neumann *if, whenever $t$ and $x$ are decision nodes of player $i$ that belong to the same information set of player $i$, the number of predecessors of $t$ is equal to the number of predecessors of $x$. Formally: $\forall i \in N$, $\forall t, x \in T$, if $t, x \in h \in \mathbb{H}_i$ then $\ell(x) = \ell(t)$.*

[3] This property was first studied in the game theory literature by Okada (1987, p. 89). Ritzberger (1999, p. 77) calls it "strong ordering", while Kline (2002, p. 288) calls it "occurrence memory". An essentially identical property, called "no forgetting", was introduced in the computer science literature by Ladner and Reif (1986) and Halpern and Vardi (1986). It was later renamed as 'perfect recall' in Fagin *et al.* (1995). See also van der Meyden (1994).

[4] For example, if choice $c$ is $\{(t, z_1), (t', y)\}$ and choice $d$ is $\{(t, x), (t', z_7)\}$.

The extensive form shown in Figure 1 is *not* von Neumann (since $x$ and $x'$ belong to the same information set of player 2 and $\ell(x) = 2$ while $\ell(x') = 3$), while the one shown in Figure 2 *is* von Neumann.

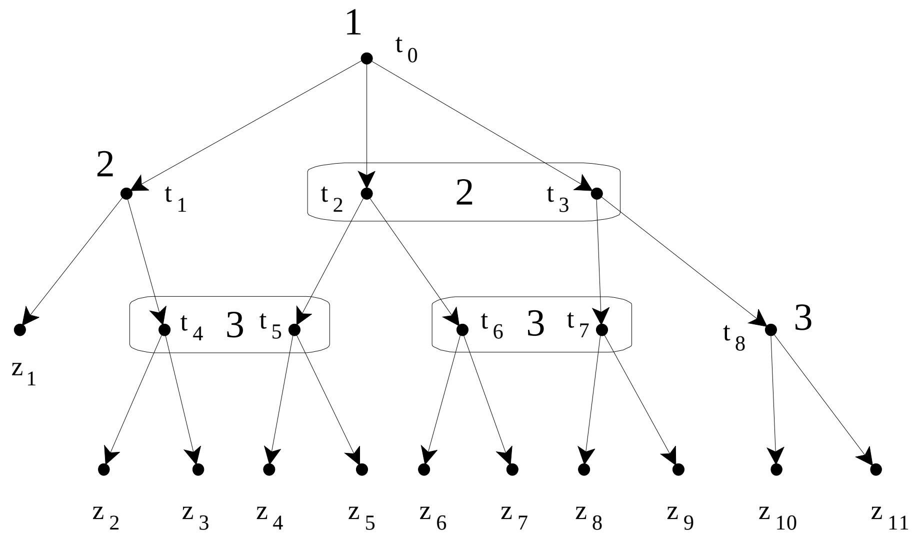


**Figure 2**

The proof of the following proposition requires several steps and is relegated to Appendix A. For every integer $k \geq 0$ we denote by $T^k$ the set of $k$-stage nodes: $T^k = \{t \in T : \ell(t) = k\}$.[5]

**Proposition 3** *Let $G$ be an arbitrary extensive form and $\langle \mathcal{K}_1, ..., \mathcal{K}_n \rangle$ an information completion of it that satisfies MPK. Then*

*(1) $G$ is von Neumann, and*

*(2) For every $t \in T$, $i \in N$ and $k \geq 0$, if $t \in T^k$ then $\mathcal{K}_i(t) \subseteq T^k$.*

Part (2) of Proposition 3 implies that at every node $t$ it is common knowledge among all the players that the play of the game has reached the stage $k = \ell(t)$. In fact, since $\mathcal{K}_i(t) \subseteq T^k$ for all $i$, the cell of the common knowledge partition containing $t$ is also a subset of $T^k$. Thus at every node the number of moves made up to that point is common knowledge among all the players (although some players may be uncertain as to what moves have been made).

The following result, due to Battigalli and Bonanno (1999), gives the converse to Proposition 3.

**Proposition 4** *Let $G$ be a von Neumann extensive form that satisfies property MDN. Then there exists an information completion $\langle \mathcal{K}_1, ..., \mathcal{K}_n \rangle$ of it that satisfies MPK.*

[5]For example, in the game of Figure 2, $T^1 = \{t_1, t_2, t_3\}$, $T^2 = \{z_1, t_4, t_5, t_6, t_7, t_8\}$, etc.

Typically, there will be several information completions that satisfy $MPK$. The finest of all such completions (capturing the maximum amount of information that can be conveyed to the players, without violating memory) is obtained as follows. First some notation. For every node $t$ and for every player $i$, let $H_i(t) \subseteq \mathbb{H}_i$ be the set of information sets of player $i$ that are crossed by paths starting at $t$: $H_i(t) = \{h \in \mathbb{H}_i : t \precsim y \text{ for some } y \in h\}$. For example, in the extensive form of Figure 2, $H_3(t_1) = \{\{t_4, t_5\}\}$, $H_3(t_2) = \{\{t_4, t_5\}, \{t_6, t_7\}\}$, $H_3(t_3) = \{\{t_6, t_7\}, \{t_8\}\}$, $H_3(t_4) = H_3(t_5) = \{t_4, t_5\}$, etc. Next we introduce, for every player $i$, a binary relation on $T$, denoted by $\approx_i$. Let $v, w \in T$; then $v \approx_i w$ if and only if, either (1) $v = w$, or (2) $\ell(v) = \ell(w)$ and $H_i(v) \cap H_i(w) \neq \varnothing$. For example, in the extensive form of Figure 2, $t_1 \approx_3 t_2$ and $t_2 \approx_3 t_3$ but *not* $t_1 \approx_3 t_3$ The relation $\approx_i$ is clearly reflexive and symmetric, but, in general, it is not transitive (as in the case of Figure 2). Let $\approx_i^*$ denote the transitive closure of $\approx_i$. Thus $v \approx_i^* w$ if and only if there exists a finite sequence of nodes $\{y_1, y_2, ..., y_m\}$ such that $y_1 = v$, $y_m = w$ and, for all $k = 1, ..., m-1$, $y_k \approx_i y_{k+1}$ . Then $\approx_i^*$ is an equivalence relation on $T$. Let $\mathcal{K}_i(t)$ be the equivalence class of $t$ generated by $\approx_i^*$ and $\mathcal{K}_i$ the set of equivalence classes, that is, $\mathcal{K}_i(t) = \{v \in T : t \approx_i^* v\}$ and $\mathcal{K}_i = \{S \subseteq T : S = \mathcal{K}_i(t) \text{ for some } t \in T\}$ . For example, in the extensive form of Figure 2, $\mathcal{K}_3(t_0) = \{t_0\}$, $\mathcal{K}_3(t_1) = \mathcal{K}_3(t_2) = \mathcal{K}_3(t_3) = \{t_1, t_2, t_3\}$, $\mathcal{K}_3(t_4) = \mathcal{K}_3(t_5) = \{t_4, t_5\}$, etc. It is shown in Battigalli and Bonanno (1999) that the information completion defined above is the finest completion that satisfies $MPK$.

By Propositions 3 and 4, the class of vN games that satisfy property $MDN$ is *precisely* the class of games where there exists an information completion that satisfies $MPK$.

By Proposition 3 an extensive form which is not von Neumann cannot have an information completion that satisfies $MPK$, even if it satisfies $MDN$. We illustrate this by means of the extensive form of Figure 1, which satisfies property $MDN$. Consider an information completion $\mathcal{K}_2$ for player 2. Since information completions preserve information sets, it must be that $\mathcal{K}_2(t) = \mathcal{K}_2(t') = \{t, t'\}$ and $\mathcal{K}_2(x) = \mathcal{K}_2(x') = \{x, x'\}$. By $MPK$, since $y \prec x'$ and $x \in \mathcal{K}_2(x')$ there must be a node $v \in \mathcal{K}_2(y)$ such that $v \prec x$. The only predecessors of $x$ are $t$ and $t_0$. We cannot have $t \in \mathcal{K}_2(y)$, since that would imply (by definition of partition) that $y \in \mathcal{K}_2(t)$, contradicting the fact that $\mathcal{K}_2(t) = \{t, t'\}$. On the other hand, if $t_0 \in \mathcal{K}_2(y)$ then, since $t' \prec y$ and $t_0 \in \mathcal{K}_2(y)$, $MPK$ would require the existence of a $v \in \mathcal{K}_2(t')$ such that $v \prec t_0$. But $\mathcal{K}_2(t') = \{t, t'\}$.

# 3 Syntactic Characterization of Memory

In this section we provide a syntactic characterization of $MPK$. We interpret the precedence relation $\prec$ as a temporal relation and associate with it the standard past and future operators from basic temporal logic (see, for example, Prior, 1956, Burgess, 1984, or Goldblatt, 1992). To the extended partition $\mathcal{K}_i$ of player $i$ we associate a knowledge operator for player $i$.

Given an extensive form and an information completion of it, by *frame* we mean the collection $\langle T, \prec, \{\mathcal{K}_i\}_{i\in N}\rangle$ where $T$ is the set of nodes, $\prec$ the precedence relation on $T$ and $\mathcal{K}_i$ is player $i$'s extended partition of $T$.

We consider a propositional language with the following modal operators: the temporal operators $G$ and $H$ and, for every player $i$, the knowledge operator $K_i$. The intended interpretation is as follows:

$G\phi$ : "it is **G**oing to be the case at every future time that $\phi$"
$H\phi$ : "it **H**as always been the case that $\phi$"
$K_i\phi$ : "player $i$ **K**nows that $\phi$".

The formal language is built in the usual way from a countable set $S$ of atomic propositions, the connectives $\neg$ (for "not") and $\vee$ (for "or") and the modal operators.[6] Let $P\phi \overset{def}{=} \neg H\neg\phi$. Thus the interpretation is:

$P\phi$ : "at *some* **P**ast time it was the case that $\phi$".

Given a frame $\langle T, \prec, \{\mathcal{K}_i\}_{i\in N}\rangle$ one obtains a *model based on it* by adding a function $V : S \to 2^T$ (where $2^T$ denotes the set of subsets of $T$) that associates with every atomic proposition $q \in S$ the set of nodes at which $q$ is true. Given a model and a formula $\phi$, the truth set of $\phi$ - denoted by $V(\phi)$ - is defined as usual. In particular,

$V(G\phi) = \{t \in T : \forall t' \in T \text{ if } t \prec t' \text{ then } t' \in V(\phi)\}$,
$V(H\phi) = \{t \in T : \forall t'' \in T \text{ if } t'' \prec t \text{ then } t'' \in V(\phi)\}$,
$V(K_i\phi) = \{t \in T : \mathcal{K}_i(t) \subseteq V(\phi)\}$.

An alternative notation for $t \in V(\phi)$ is $t \models \phi$. A formula $\phi$ is *valid in a model* if $t \models \phi$ for all $t \in T$, that is, if $\phi$ is true at every node. A formula $\phi$ is *valid in a frame* if it is valid in every model based on it.

Finally, we say that a property of frames is *characterized by* an axiom if (1) the axiom is valid in any frame that satisfies the property and, conversely, (2) whenever the axiom is valid in a frame, then the frame satisfies the property.

The following proposition, which is proved in Appendix B, provides a characterization of $MPK$.[7]

**Proposition 5** *$MPK$ is characterized by either of the following axioms:*

$$PK_i\phi \to K_iPK_i\phi \tag{M1}$$

[6]See, for example, Chellas (1984). The connectives $\wedge$ (for "and") and $\to$ (for "if ... then") are defined as usual: $\phi \wedge \psi \overset{def}{=} \neg(\neg\phi \vee \neg\psi)$ and $\phi \to \psi \overset{def}{=} \neg\phi \vee \psi$.

[7]An alternative axiom for the property that we call 'memory of past knowledge' was suggested by Ladner and Reif (1986): $K_iG\phi \to GK_i\phi$. Halpern and Vardi (1986) provided a sound and complete axiomatization of systems that satisfy 'memory of past knowledge' ( they called this property 'no forgetting') and are synchronous (i.e. the agents have access to an external clock). The key axiom is $K_i \bigcirc \phi \to \bigcirc K_i\phi$, where $\bigcirc$ is the 'next time' operator, that is, $t \models \bigcirc\phi$ if $\phi$ is true at every immediate successor of $t$. As pointed out in Section 1, synchronous systems are closely related to von Neumann games.

$$K_i\phi \to GK_iPK_i\phi. \tag{M2}$$

$M1$ says that if, at some time in the past, player $i$ knew $\phi$, then she knows now that in the past she knew $\phi$. While $M1$ is backward-looking, $M2$ is forward-looking: it says that if player $i$ knows $\phi$ now, then at every future time she will know that some time in the past she knew $\phi$.

By Propositions 3 and 5, if an extensive form has an information completion that validates axiom $M1$ then the extensive form is von Neumann and satisfies property $MDN$. Conversely, by Propositions 4 and 5 , a von Neumann extensive form that satisfies $MDN$ has an information completion that validates axiom $M1$. Thus axiom $M1$ provides a syntactic characterization of the class of von Neumann games that satisfy $MDN$. The same is true of axiom $M2$.

# 4 Conclusion

An information completion of an extensive form is obtained by extending the information partition of every player from the set of her decision nodes to the set of all nodes. One can then define, for the extended partition, the following notion of memory: at any node a player remembers what she knew at earlier nodes. We showed that this property can be satisfied in an extensive form if and only if the extensive form is von Neumann and satisfies the restriction of the property to a player's own decision nodes. We also provided two equivalent axioms for the proposed notion of memory thus obtaining a syntactic characterization of the said class of von Neumann games.

**APPENDICES**

# A Proofs for Section 2

In this appendix we prove Proposition 3 of Section 2. For the reader's convenience we repeat the definition of $MPK$:

if $t \prec x$ and $x' \in \mathcal{K}_i(x)$ then there exists a $t' \in \mathcal{K}_i(t)$ such that $t' \prec x'$.

We say that at node $x$ there is "time uncertainty" for player $i$ if the cell $\mathcal{K}_i(x)$ of her extended partition $\mathcal{K}_i$ contains a predecessor of $x$, that is, if there is a path in the tree that crosses the cell of player $i$'s extended partition that contains $x$ more than once.[8]

**Definition 6** *At* $x \in T$ *there is* time uncertainty for player i *if there exists a* $t \in \mathcal{K}_i(x)$ *such that* $t \prec x$.

[8] When restricted to a player's information sets, time uncertainty coincides with the notion of absent-mindedness (cf. Piccione and Rubinstein, 1997, p.10 and Kline, 2002, p. 289).

The following lemma states that in information completions that satisfy $MPK$, time uncertainty "propagates into the past".

**Lemma 7** *Fix an arbitrary extensive form and let $\langle \mathcal{K}_1, ..., \mathcal{K}_n \rangle$ be an information completion of it that satisfies $MPK$. Then the following is true for every node $x$ and every player $i$: if at $x$ there is time uncertainty for player $i$, then there exists a $t \in T$ such that (1) $t \prec x$ and (2) at $t$ there is time uncertainty for player $i$.*

**Proof.** Let $x$ and $i$ be such that there exists a $t \in \mathcal{K}_i(x)$ with $t \prec x$. By $MPK$ (letting $x' = t$) there exists a $t' \in \mathcal{K}_i(t)$ such that $t' \prec t$. Thus at $t$ there is time uncertainty for player $i$. ∎

The following proposition says that $MPK$ rules out time uncertainty.

**Proposition 8** *Fix an arbitrary extensive form and let $\langle \mathcal{K}_1, ..., \mathcal{K}_n \rangle$ be an information completion of it that satisfies $MPK$. Then for every node $x$ and every player $i$ there cannot be time uncertainty at $x$ for player $i$.*

**Proof.** Suppose that there is a node $t_1$ and a player $i$ at which there is time uncertainty for player $i$. By Lemma 7 there is an infinite sequence $\langle t_1, t_2, ... \rangle$ such that, for all $k \geq 1$, $t_{k+1} \prec t_k$ and at $t_{k+1}$ there is time uncertainty for player $i$. Since $\langle T, \prec \rangle$ is a rooted tree, it has no cycles. Thus, for all $j, k \geq 1$ with $j \neq k$, $t_j \neq t_k$, contradicting the fact that in a rooted tree every node has a finite number of predecessors. ∎

The following proposition states that a situation like the one illustrated in Figure 3 (where rounded rectangles represent cells of $\mathcal{K}_i$) is not compatible with $MPK$.

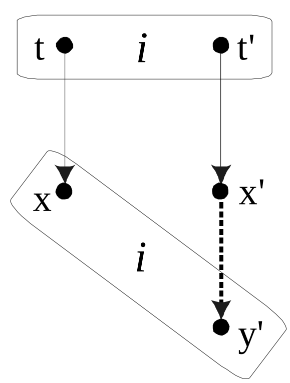


Figure 3

**Proposition 9** *Let $G$ be an arbitrary extensive form and $\langle \mathcal{K}_1, ..., \mathcal{K}_n \rangle$ an information completion of it that satisfies $MPK$. Then the following is true for all $t, t', x, x', y' \in T$ and $i \in N$:*

*if* $t \rightarrow x$, $t' \in \mathcal{K}_i(t)$, $t' \rightarrow x'$, $x' \precsim y'$, *and* $y' \in \mathcal{K}_i(x)$, *then* $y' = x'$.

**Proof.** Suppose not. Then there exist $t, t', x, x', y' \in T$ and $i \in N$ such that $t \to x$ (that is, $t$ is the immediate predecessor of $x$), $t' \in \mathcal{K}_i(t)$, $t' \to x'$ (that is, $t'$ is the immediate predecessor of $x'$), $y' \in \mathcal{K}_i(x)$ and $x' \prec y'$. Since $t' \to x'$, by Proposition 8 it must be that

$$t' \notin \mathcal{K}_i(x') \tag{1}$$

(otherwise there would be time uncertainty for player $i$ at $x'$). It follows that

$$t \notin \mathcal{K}_i(x'). \tag{2}$$

In fact, if it were the case that $t \in \mathcal{K}_i(x')$, then we would have (by definition of partition) that $\mathcal{K}_i(t) = \mathcal{K}_i(x')$ and, since $t' \in \mathcal{K}_i(t)$, $t' \in \mathcal{K}_i(x')$, contradicting (1). Since $y' \in \mathcal{K}_i(x)$, $\mathcal{K}_i(y') = \mathcal{K}_i(x)$. Thus, since $x \in \mathcal{K}_i(x)$,

$$x \in \mathcal{K}_i(y'). \tag{3}$$

By $MPK$ it follows from $x' \prec y'$ and (3) that there exists an $x''$ such that

$$x'' \in \mathcal{K}_i(x') \tag{4}$$

and

$$x'' \prec x. \tag{5}$$

Since $t$ is the immediate predecessor of $x$ ($t \to x$), it follows from (5) that either $x'' = t$, or $x'' \prec t$. The case $x'' = t$ yields a contradiction between (4) and (2). Suppose, therefore, that $x'' \prec t$. By $MPK$ it follows from $t' \to x'$ and (4) that there exists a $t'' \in \mathcal{K}_i(t')$ such that $t'' \prec x''$. From $t' \in \mathcal{K}_i(t)$ and $t'' \in \mathcal{K}_i(t')$ we get (by definition of partition) that

$$t'' \in \mathcal{K}_i(t). \tag{6}$$

From $t'' \prec x''$ and $x'' \prec t$ we get (by transitivity of $\prec$) that $t'' \prec t$. This, in conjunction with (6), yields time uncertainty at $t$ for player $i$, contradicting Proposition 8. ■

**Proof of Proposition 3.** Fix an arbitrary player $i$ and an arbitrary node $x$. Let $k = \ell(x)$. First we prove part (2), namely that $\mathcal{K}_i(x) \subseteq T^k$. We do this by induction. First of all, it must be that $\mathcal{K}_i(t_0) = \{t_0\}$ (where $t_0$ is the root of the tree). In fact, if there were a $t \neq t_0$ with $t \in \mathcal{K}_i(t_0)$, then we would have $\mathcal{K}_i(t) = \mathcal{K}_i(t_0)$ and, since $t_0 \in \mathcal{K}_i(t_0)$, $t_0 \in \mathcal{K}_i(t)$. Thus, since $t_0 \prec t$, there would be time uncertainty at $t$ for player $i$, contradicting Proposition 8. Thus the statement is true for $k = 0$. Next we show that if it is true for all $k \leq m$ then it is true for $k = m + 1$. Fix a node $x \in T^{m+1}$ and an arbitrary $y' \in \mathcal{K}_i(x)$. Then $\mathcal{K}_i(y') = \mathcal{K}_i(x)$. By the induction hypothesis, $\ell(y') \geq m + 1$.[9]

[9] Suppose, to the contrary, that $\ell(y') = j$ with $j < m+1$. Then, by the induction hypothesis, $\mathcal{K}_i(y') \subseteq T^j$. Since $x \in \mathcal{K}_i(x)$ and $\mathcal{K}_i(x) = \mathcal{K}_i(y')$, $x \in \mathcal{K}_i(y')$. Thus $x \in T^j$, contradicting the hypothesis that $x \in T^{m+1}$.

Suppose that $\ell(y') > m+1$. Let $t \in T^m$ be the immediate predecessor of $x$. Since $t \to x$ and $y' \in \mathcal{K}_i(x)$, by $MPK$ there exists a $t' \in \mathcal{K}_i(t)$ such that $t' \prec y'$. By the induction hypothesis, $\mathcal{K}_i(t) \subseteq T^m$ and therefore $t' \in T^m$. Let $x'$ be the immediate successor of $t'$ on the path from $t'$ to $y'$. Since $\ell(t') = m$, $\ell(x') = m+1$. Thus, since $\ell(y') > m+1$, $x' \neq y'$. Thus we have that all of the following are true, contradicting Proposition 9: $t \to x$, $t' \in \mathcal{K}_i(t)$, $t' \to x'$, $x' \precsim y'$, $y' \in \mathcal{K}_i(x)$ and $y' \neq x'$. Thus we have shown that for every player $i$ and node $x$, $\mathcal{K}_i(x) \subseteq T^{\ell(x)}$, completing the proof of part (2) of Proposition 3. To prove part (1) it is sufficient to recall that, by definition of information completion, if node $x$ belongs to information set $h$ of player $i$, then $\mathcal{K}_i(x) = h$. Thus the extensive form is von Neumann. ■

# B Proofs for Section 3

**Proof of Proposition 5.** Assume $MPK$. We show that both $(M1)$ and $(M2)$ are valid. For $(M1)$: suppose that $x \models PK_i\phi$. Then there exists a $t$ such that $t \prec x$ and $t \models K_i\phi$, that is, $\mathcal{K}_i(t) \subseteq V(\phi)$. Fix an arbitrary $x' \in \mathcal{K}_i(x)$. By $MPK$ there exists a $t' \in \mathcal{K}_i(t)$ such that $t' \prec x'$. Since $t' \in \mathcal{K}_i(t)$, $\mathcal{K}_i(t') = \mathcal{K}_i(t)$ and, therefore, since $\mathcal{K}_i(t) \subseteq V(\phi)$, $t' \models K_i\phi$. Thus $x' \models PK_i\phi$ and $x \models K_iPK_i\phi$. For $(M2)$: suppose that $t \models K_i\phi$. Fix arbitrary $x$ and $x'$ such that $t \prec x$ and $x' \in \mathcal{K}_i(x)$. By $MPK$ there exists a $t' \in \mathcal{K}_i(t)$ such that $t' \prec x'$. Since $t' \in \mathcal{K}_i(t)$, $\mathcal{K}_i(t') = \mathcal{K}_i(t)$ and, therefore, $t' \models K_i\phi$. Thus $x' \models PK_i\phi$ and $x \models K_iPK_i\phi$ and $t \models GK_iPK_i\phi$.

To prove the converse, assume that $MPK$ does not hold, that is, there exist $i \in N$ and $t, x, x' \in T$ such that all of the following hold:

$$t \prec x \tag{7}$$

$$x' \in \mathcal{K}_i(x) \tag{8}$$

$$\forall t' \in T,\ \ if\ \ t' \prec x'\ \ then\ \ t' \notin \mathcal{K}_i(t). \tag{9}$$

We want to show that both $(M1)$ and $(M2)$ can be falsified. Let $q$ be an atomic sentence and construct a model where $V(q) = \mathcal{K}_i(t)$. Then

$$t \models K_iq. \tag{10}$$

For every $t'$ such that $t' \prec x'$, by (9) $t' \notin \mathcal{K}_i(t) = V(q)$ and therefore

$$t' \nvDash q. \tag{11}$$

It follows from (11) that

$$t' \nvDash K_iq. \tag{12}$$

In fact, if it were the case that $\mathcal{K}_i(t') \subseteq V(q) = \mathcal{K}_i(t)$ then, since $t' \in \mathcal{K}_i(t')$ we would have $t' \models q$, contradicting (11). It follows from (12) that $x' \nvDash PK_iq$. Hence, by (8),

$$x \nvDash K_iPK_iq. \tag{13}$$

By (7) and (10), $x \models PK_iq$. This, together with (13), falsifies $(M1)$ at $x$. By (13) and (7), $t \nvDash GK_iPK_iq$. This, together with (10), falsifies $(M2)$ at $t$. ■

# References


[1] Battigalli, P. and G. Bonanno (1999), Synchronic information, knowledge and common knowledge in extensive games, *Research in Economics*, 53, 77-99.

[2] van Benthem, J. (2001), Games in dynamic epistemic logic, *Bulletin of Economic Research*, 53, 219-248.

[3] Bonanno, G. (2003), Memory and perfect recall in extensive games, *Games and Economic Behavior,* In Press.

[4] Burgess, J. (1984), Basic tense logic, in: D. Gabbay and F. Guenthner (eds.), *Handbook of philosophical logic*, Vol. II, D. Reidel Publishing Company, 89-133.

[5] Chellas, B. (1984), *Modal logic: an introduction*, Cambridge University Press.

[6] Fagin, R, J. Halpern, Y. Moses and M. Vardi (1995), *Reasoning about knowledge,* MIT Press.

[7] Goldblatt, R. (1992), *Logics of time and computation*, CSLI Lecture Notes No. 7.

[8] Halpern, J. and M. Vardi (1986), The complexity of reasoning about knowledge and time, *Proceedings 18th ACM Symposium on Theory of Computing,* 304-315.

[9] Kline, J. J. (2002), Minimum memory for equivalence between *ex ante* optimality and time consistency, *Games and Economic Behavior,* 38, 278-305.

[10] Kuhn, H. W. (1953), Extensive games and the problem of information, in: H. W. Kuhn and W. W. Tucker (eds.), *Contributions to the theory of games*, Vol. II, Princeton University Press, 193-216. Reprinted in Kuhn (1997), 46-68.

[11] Kuhn, H. W. (1997), *Classics in game theory*, Princeton University Press.

[12] Ladner, R. and J. Reif (1986), The logic of distributed protocols (preliminary report), in: J. Halpern, Ed., *Theoretical aspects of reasoning about knowledge: Proceedings of the 1986 conference,* Morgan Kaufmann, 207-222.

[13] Malcolm, N. (1963), *Knowledge and certainty*, Prentice-Hall.

[14] van der Meyden, R. (1994), Axioms for knowledge and time in distributed systems with perfect recall, *Proc. IEEE Symposium on Logic in Computer Science*, 448-457.

[15] Munsat, S. (1966), *The concept of memory*, Random House.

[16] Okada, A. (1987), Complete inflation and perfect recall in extensive games, *International Journal of Game Theory, 16, 85-91.*

[17] Piccione, M. and A. Rubinstein (1997), On the interpretation of decision problems with imperfect recall, *Games and Economic Behavior,* 20, 3-24.

[18] Prior, A.N. (1956), *Time and modality*, Oxford University Press.

[19] Ritzberger, K. (1999), Recall in extensive form games, *International Journal of Game Theory*, 28, 69-87.

[20] Selten, R. (1975), Re-examination of the perfectness concept for equilibrium points in extensive games, *International Journal of Game Theory,* 4, 25-55. Reprinted in Kuhn (1997), 317-354.