Ranking lotteries

Given two money lotteries L and M when would any two individuals agree that L is better than M, no matter their attitude to risk? Assume throughout that every individual prefers more money to less, that is, that each individual's utility function is strictly increasing.

Everybody will agree that   is better than   .

What about   and   ?

$\mathbb{E}[L]$ and $\mathbb{E}[M] =$

For a risk-neutral person:

For a risk-averse person with utility function $U(x) = \sqrt{x}$

$\mathbb{E}[U(L)] =$ $\mathbb{E}[U(M)] =$

However, there are lotteries that can be unambiguously ranked in the sense that everybody ranks them the same way.

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \qquad M = \begin{pmatrix} \$x_1 & \$x_2 & \dots & \$x_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}.$$

Note that the **basic outcomes are the same** in both lotteries and for this part assume that the **prizes are listed in increasing order**: $0 \le x_1 < x_2 < \ldots < x_n$.

Define the **cumulative distribution function** (cdf) for lottery L as follows:

$P_i = p_1 + \ldots + p_i$ for every $i = 1, \ldots, n$:

$$L = \begin{pmatrix} \$x_1 & \$x_2 & \$x_3 & \dots & \$x_n \\ p_1 & p_2 & p_3 & \dots & p_n \\ P: & & & & \end{pmatrix}$$

P_i is the probability that $x \le x_i$.

define the cumulative probability distribution for lottery M as follows: $Q_i = q_1 + \ldots + q_i$ for every $i = 1, \ldots, n$:

$$M = \begin{pmatrix} \$x_1 & \$x_2 & \$x_3 & \ldots & \$x_n \\ q_1 & q_2 & q_3 & \ldots & q_n \end{pmatrix} \quad Q:$$

Definition. We say that L **first-order stochastically dominates** M and write $L >_{FSD} M$ if $P_i \le Q_i$ for ever $i = 1, 2, \ldots, n,$ with at least one strict inequality.

Example 1.

$$L = \begin{pmatrix} \$40 \\ 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} \$20 & \$60 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Example 2.

$$L = \begin{pmatrix} \$20 & \$40 & \$50 & \$60 \\ \frac{1}{12} & \frac{3}{12} & \frac{6}{12} & \frac{2}{12} \end{pmatrix} \text{ and } \mathrm{M} = \begin{pmatrix} \$20 & \$40 & \$50 & \$60 \\ \frac{1}{12} & \frac{4}{12} & \frac{5}{12} & \frac{2}{12} \end{pmatrix}.$$

Theorem. $L >_{FSD} M$ if and only if $\mathbb{E}[U(L)] > \mathbb{E}[U(M)]$ for every strictly increasing utility function U.

Thus if lottery L first-order stochastically dominates lottery M then it is unambiguously better than M, in the sense that everybody, no matter what their attitude to risk, prefers L to M.

Now **focus on risk-averse individuals** and ask when any two risk-averse individuals would agree that a lottery *M* is worse than another lottery *L*, in which case we can interpret this as ***M* being more risky than *L***.

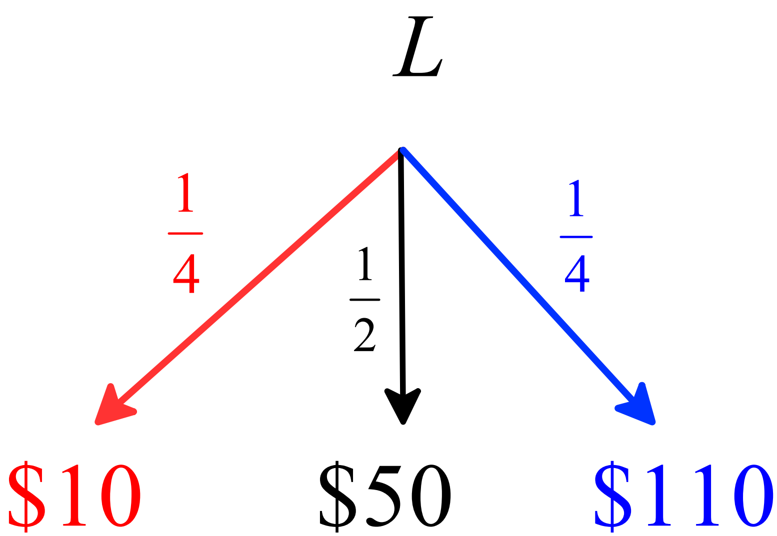
To begin with the two lotteries ought to be similar: $\mathbb{E}[L] = \mathbb{E}[M]$, in which case a risk-neutral individual would be indifferent between the two. Hence if a risk-averse person is not indifferent it must be because one is "more risky" than the other.

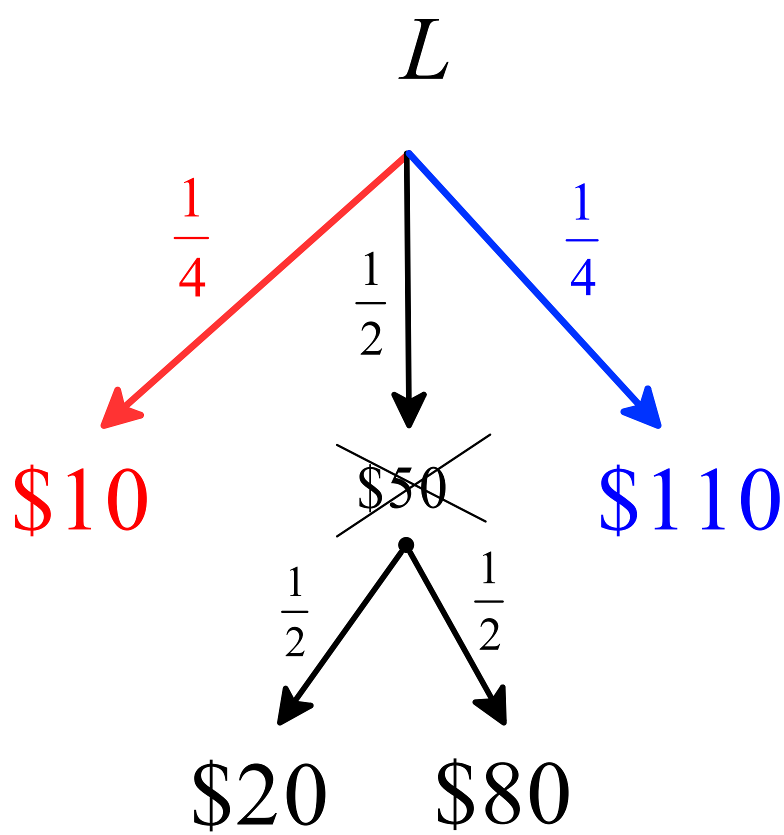
$$L = \begin{pmatrix} \$50 \\ 1 \end{pmatrix}$$

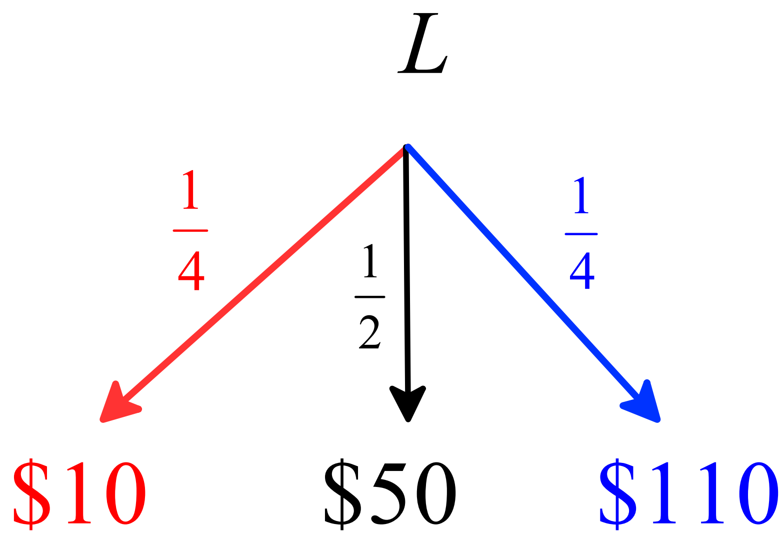
$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \text{ with } \quad \mathbb{E}[L] = 55$$

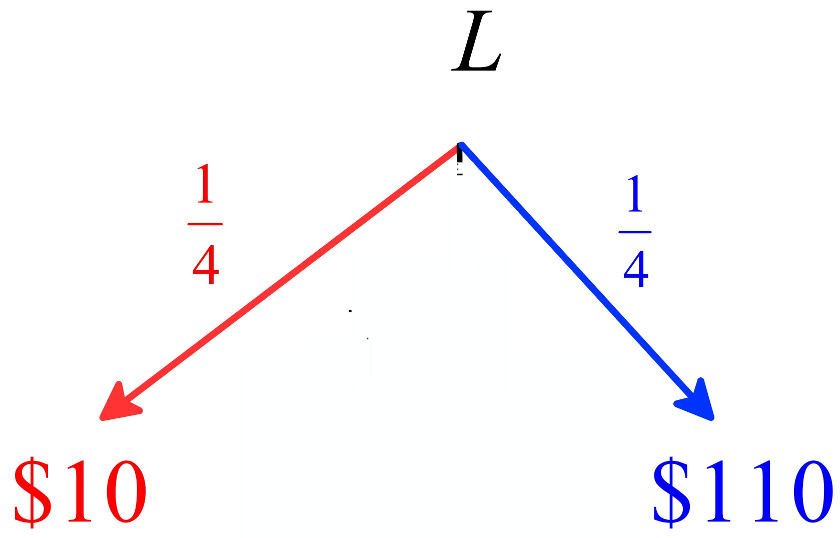
$$\begin{matrix} \$50 \\ \frac{1}{2} \end{matrix}$$

$$L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \text{ with } \quad \mathbb{E}[L] = 55$$






L
1/4
1/2
1/4
$10
$50
$110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Definition: M is a <u>mean-preserving spread</u> (<u>MPS</u>) of L if an outcome of L, say $\begin{pmatrix} \$x \\ p \end{pmatrix}$, is replaced by $\begin{pmatrix} \$y & \$x & \$z \\ r & p-q & s \end{pmatrix}$ with $0 < q \leq p$,

$y < x \quad z > x$

$r + s = q$

and $ry + sz = qx$

$$\begin{matrix} x \\ p \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} y & x & z \\ r & q & 1-r \end{matrix}$$

Example on the next page

We don't have to reduce the probability to zero: $L = \begin{pmatrix} \$10 & \$50 & \$110 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

Take away some of the probability of \$50, say $\frac{3}{10}$ and spread it between a lower amount, say \$15, and a higher amount, say \$90:

$$M = \begin{pmatrix} \$10 & \$15 & \$50 & \$90 & \$110 \\ & & & & \end{pmatrix}$$

For this to be a mean preserving spread we need

$$M = \begin{pmatrix} \$10 & \$15 & \$50 & \$90 & \$110 \\ & & & & \end{pmatrix}$$

Write $L >_{SSD} M$ to mean that ***L* dominates *M*** in the sense of **second-order stochastic dominance.**

Definition. $L >_{SSD} M$ if M can be obtained from L by a finite sequence of mean - preserving spreads, that is, if there is a sequence of money lotteries $\langle L_1, L_2, \ldots, L_m \rangle$ (with $m \geq 2$) such that:

(1) $L_1 = L$,
(2) $L_m = M$
(3) for every $i = 1, \ldots, m-1$, $L_i \rightarrow_{MPS} L_{i+1}$

Theorem. $L >_{SSD} M$ if and only if $\mathbb{E}[U(L)] > \mathbb{E}[U(M)]$ for every strictly increasing and strictly concave utility function *U*.