**PRACTICE FINAL: ANSWERS**

# 1.

**(a)** The possible outcomes are:

| | |
|---|---|
| $z_1$ | Stays home, gets A |
| $z_2$ | Stays home, gets C |
| $z_3$ | At party, good time with Kate, gets C |
| $z_4$ | At party, good time with Kate, gets F |
| $z_5$ | At party, rejected, gets C |
| $z_6$ | At party, rejected, gets F |
| $z_7$ | At party, not approached by Kate, gets C |
| $z_8$ | At party, not approached by Kate, gets F |

**(b)** Each state specifies whether the exam is easy or difficult, whether Kate is attracted to him or not and whether Kate is shy or not. Thus there are 8 states:

| | |
|---|---|
| $s_{e,a,s}$ | easy, attracted, shy |
| $s_{e,a,ns}$ | easy, attracted, not shy |
| $s_{e,na,s}$ | easy, not attracted, shy |
| $s_{e,na,ns}$ | easy, not attracted, not shy |
| $s_{ne,a,s}$ | not easy, attracted, shy |
| $s_{ne,a,ns}$ | not easy, attracted, not shy |
| $s_{ne,na,s}$ | not easy, not attracted, shy |
| $s_{ne,na,ns}$ | not easy, not attracted, not shy |

The decision problem can the be written as follows:

| | $s_{e,a,s}$ | $s_{e,a,ns}$ | $s_{e,na,s}$ | $s_{e,na,ns}$ | $s_{ne,a,s}$ | $s_{ne,a,ns}$ | $s_{ne,na,s}$ | $s_{ne,na,ns}$ |
|---|---|---|---|---|---|---|---|---|
| Stay home | $z_1$ | $z_1$ | $z_1$ | $z_1$ | $z_2$ | $z_2$ | $z_2$ | $z_2$ |
| Go to party, approach | $z_3$ | $z_3$ | $z_5$ | $z_5$ | $z_4$ | $z_4$ | $z_6$ | $z_6$ |
| Go to party, be cool | $z_7$ | $z_3$ | $z_7$ | $z_7$ | $z_8$ | $z_4$ | $z_8$ | $z_8$ |

**(c)** We can take values from 0 to 5 as follows:

| *Outcome* | $z_1$ | $z_2$ | $z_3$ | $z_4$ | $z_5$ | $z_6$ | $z_7$ | $z_8$ |
|---|---|---|---|---|---|---|---|---|
| *Utility* | 4 | 2 | 5 | 3 | 0 | 0 | 2 | 1 |

**(d)**

| | $s_{e,a,s}$ | $s_{e,a,ns}$ | $s_{e,na,s}$ | $s_{e,na,ns}$ | $s_{ne,a,s}$ | $s_{ne,a,ns}$ | $s_{ne,na,s}$ | $s_{ne,na,ns}$ |
|---|---|---|---|---|---|---|---|---|
| Stay home | 4 | 4 | 4 | 4 | 2 | 2 | 2 | 2 |
| Go to party, approach | 5 | 5 | 0 | 0 | 3 | 3 | 0 | 0 |
| Go to party, be cool | 2 | 5 | 2 | 2 | 1 | 3 | 1 | 1 |

No: for every two acts $x$ and $y$, there is a state where $x$ is better than $y$ and there is another state where $y$ is better than $x$.

**(e)** $E=\{s_{e,a,s}, s_{e,a,ns}, s_{e,a,ns}, s_{e,na,ns}\}$, $\neg E=\{s_{ne,a,s}, s_{ne,a,ns}, s_{ne,a,ns}, s_{ne,na,ns}\}$
$S=\{s_{e,a,s}, s_{e,na,s}, s_{ne,a,s}, s_{ne,na,s}\}$, $\neg S=\{s_{e,a,ns}, s_{e,na,ns}, s_{ne,a,ns}, s_{ne,na,ns}\}$
$A=\{s_{e,a,s}, s_{e,a,ns}, s_{ne,a,s}, s_{ne,a,ns}\}$, $\neg A=\{s_{e,na,s}, s_{e,na,ns}, s_{ne,na,s}, s_{ne,na,ns}\}$.

**(f)** $P(E\mid S)=P(E)$, $P(E\mid\neg S)=P(E)$, $P(E\mid A)=P(E)$, $P(E\mid\neg A)=P(E)$,
$P(S\mid E)=P(S)$, $P(S\mid\neg E)=P(S)$, etc.

**(g)** $P(E)=P(s_{e,a,s})+P(s_{e,a,ns})+P(s_{e,na,s})+P(s_{e,na,ns})=0.4$
$P(S)=P(s_{e,a,s})+P(s_{e,na,s})+P(s_{ne,a,s})+P(s_{ne,na,s})=0.8$
$P(A)=P(s_{e,a,s})+P(s_{e,a,ns})+P(s_{ne,a,s})+P(s_{ne,a,ns})=0.5$

**(h)** Here we are assuming strong independence as follows:

$$P(E\cap S)=P(E\mid S)\,P(S)\underset{\text{by independence}}{=}P(E)\,P(S)=0.4(0.8)=0.32$$
$$P(E\cap A)=P(E\mid A)\,P(A)\underset{\text{by independence}}{=}P(E)\,P(A)=0.4(0.5)=0.2$$
$$P(A\cap S)=P(A\mid S)\,P(S)\underset{\text{by independence}}{=}P(A)\,P(S)=0.5(0.8)=0.4$$

Then the values are as follows:

$$P(s_{e,a,s})=P(E\cap A\cap S)=P(E\cap A\mid S)\,P(S)\underset{\text{by independence}}{=}P(E\cap A)\,P(S)=0.2(0.8)=0.16$$
$$P(s_{e,a,ns})=P(E\cap A)-P(s_{e,a,s})=0.2-0.16=0.04$$
$$P(s_{e,na,s})=P(E\cap\neg A\cap S)=P(E\cap\neg A\mid S)\,P(S)\underset{\text{by independence}}{=}P(E\cap\neg A)\,P(S)$$
$$\underset{\text{by independence}}{=}P(E)\,P(\neg A)\,P(S)=0.4(0.5)(0.8)=0.16$$
$$P(s_{e,na,ns})=P(E\cap\neg A)-P(s_{e,na,s})=P(E)\,P(\neg A)-P(s_{e,na,s})=0.4(0.5)-0.16=0.04$$
$$P(s_{ne,a,s})=P(\neg E\cap A\cap S)=P(\neg E\cap A\mid S)\,P(S)\underset{\text{by independence}}{=}P(\neg E\cap A)\,P(S)$$
$$\underset{\text{by independence}}{=}P(\neg E)\,P(A)\,P(S)=0.6(0.5)(0.8)=0.24$$
$$P(s_{ne,a,ns})=P(\neg E\cap A)-P(s_{ne,a,s})=P(\neg E)\,P(A)-P(s_{ne,a,s})=0.6(0.5)-0.24=0.06$$
$$P(s_{ne,na,s})=P(\neg E\cap\neg A\cap S)=P(\neg E\cap\neg A\mid S)\,P(S)\underset{\text{by independence}}{=}P(\neg E\cap\neg A)\,P(S)$$
$$\underset{\text{by independence}}{=}P(\neg E)\,P(\neg A)\,P(S)=0.6(0.5)(0.8)=0.24$$
$$P(s_{ne,na,ns})=P(\neg E\cap\neg A)-P(s_{ne,na,s})=P(\neg E)\,P(\neg A)-P(s_{ne,na,s})=0.6(0.5)-0.24=0.06$$

Thus the probability distribution is as follows:

| $s_{e,a,s}$ | $s_{e,a,ns}$ | $s_{e,na,s}$ | $s_{e,na,ns}$ | $s_{ne,a,s}$ | $s_{ne,a,ns}$ | $s_{ne,na,s}$ | $s_{ne,na,ns}$ |
|---|---|---|---|---|---|---|---|
| 0.16 | 0.04 | 0.16 | 0.04 | 0.24 | 0.06 | 0.24 | 0.06 |

**(i)** Stay home = $\begin{pmatrix} z_1 & z_2 \\ 0.4 & 0.6 \end{pmatrix}$, To party/approach = $\begin{pmatrix} z_3 & z_4 & z_5 & z_6 \\ 0.2 & 0.3 & 0.2 & 0.3 \end{pmatrix}$

To party/cool = $\begin{pmatrix} z_3 & z_4 & z_7 & z_8 \\ 0.04 & 0.06 & 0.36 & 0.54 \end{pmatrix}$.

**(j)** We can normalize the utility function $U$ so that $U(z_3)=1$ and $U(z_5)=U(z_6)=0$. Since Jonathan is indifferent between $\begin{pmatrix} z_4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} z_3 & z_5 \\ 0.6 & 0.4 \end{pmatrix}$, it must be that $U(z_4)=0.6U(z_3)+0.4U(z_5)=(0.6)1+(0.4)0=0.6$. Thus the expected utility of party/approach is $0.2U(z_3)+0.3U(z_4)+0.2U(z_5)+0.3U(z_6)=0.2(1)+0.3(0.6)+0.2(0)+0.3(0)=0.38$. Hence, since he is indifferent between party/approach and staying home, it must be that the expected utility of staying home is equal to 0.38, that is, $0.4U(z_1)+0.6U(z_2)=0.38$. Thus all we know about the utility function is the following, with $1>x>0.6>y>z>0$ and $0.4x+0.6y=0.38$

| *Outcome* | *Utility* |
|---|---|
| $z_3$ | 1 |
| $z_1$ | $x$ |
| $z_4$ | 0.6 |
| $z_2$ | $y$ |
| $z_7$ | $y$ |
| $z_8$ | $z$ |
| $z_5$ | 0 |
| $z_6$ | 0 |

**(k)** Two questions: (1) what value of $p$ would make you indifferent between $z_1$ for sure and the lottery $\begin{pmatrix} z_3 & z_5 \\ p & 1-p \end{pmatrix}$? (2) what value of $q$ would make you indifferent between $z_8$ for sure and the lottery $\begin{pmatrix} z_3 & z_5 \\ q & 1-q \end{pmatrix}$? The answer to the first question gives the value of $U(z_1)$ and this, together with the equation $0.4U(z_1)+0.6U(z_2)=0.38$ enables you to figure out the value of $U(z_2)$. The answer to the second question gives the value of $U(z_8)$.

**(l)** Then Jonathan's utility function is

| *Outcome* | *Utility* |
|---|---|
| $z_3$ | 1 |
| $z_1$ | 0.8 |
| $z_4$ | 0.6 |
| $z_2$ | 0.1 |
| $z_7$ | 0.1 |
| $z_8$ | 0.05 |
| $z_5$ | 0 |
| $z_6$ | 0 |

Thus *EU*(stay home) = $0.4U(z_1)+0.6U(z_2)=0.4(0.8)+0.6(0.1)=0.38$

*EU*(party/approach) = $0.2U(z_3)+0.3U(z_4)+0.2U(z_5)+0.3U(z_6)$

$= 0.2(1) + 0.3(0.6) + 0.2(0) + 0.3(0) = 0.38$

$EU(\text{party/cool}) = 0.04U(z_3) + 0.06U(z_4) + 0.36U(z_7) + 0.54U(z_8)$

$= 0.04(1) + 0.06(0.6) + 0.36(0.1) + 0.54(0.05) = 0.139$

Thus Jonathan will either stay home or go to the party and approach Kate.

**2. (A)** Since the discount rate is $\rho = \frac{1}{9}$, the discount factor is $\delta = \dfrac{1}{1+\rho} = \frac{9}{10} = 0.9$. Thus

**(a)** $U_0(\$100 \text{ in 6 years}) = (0.9)^6(100) = 53.14$ and

$U_0(\$200 \text{ in 8 years}) = (0.9)^8(200) = 86.09$. Thus she chooses to get \$200 in 8 years.

**(b)** $U_6(\$100 \text{ now}) = 100$ and $U_6(\$200 \text{ in 2 years}) = (0.9)^2(200) = 162$. Thus she will choose to get \$200 two years later.

**(c)** Yes, her preferences are time consistent: she ranks the alternatives the same way at date 0 and at date 6.

**(B) (d)** $U_0(\$100 \text{ in 6 years}) = (0.6)(0.9)^6(100) = 31.89$ and

$U_0(\$200 \text{ in 8 years}) = (0.6)(0.9)^8(200) = 51.66$. Thus she chooses to get \$200 in 8 years.

**(e)** $U_6(\$100 \text{ now}) = 100$ and $U_6(\$200 \text{ in 2 years}) = (0.6)(0.9)^2(200) = 97.2$. Thus she will change her mind and choose \$100 right away.

**(f)** No, because she changes her initial plan after 6 years.

**3.** **(a)** With the Borda count and sincere voting *x* gets 22 points, *a* gets 17, *b* gets 16 and *c* gets 15. Thus the social ranking is

*x*
*a*
*b*
*c*

If, after the election, *x* drops out then the next best candidate will be chosen, that is candidate *a*.

**(b)** Eliminating *x* from the above profile we have:

| **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|
| *c* | *a* | *b* | *c* | *a* | *b* | *c* |
| *b* | *c* | *a* | *b* | *c* | *a* | *b* |
| *a* | *b* | *c* | *a* | *b* | *c* | *a* |

and using the Borda count with this profile we have that *a* gets 13 points, *b* gets 14 and *c* gets 15. Thus the social ranking becomes

*c*
*b*
*a*

that is, a complete reversal of the previous one! The winner is now *c*, who was the lowest ranked candidate before!

**4.** **(a)** When the range of the SCF has only two alternatives, plurality voting satisfies Unanimity, Non-dictatorship and Non-manipulability.

**(b)**

| 1’s ↓ \ 2’s → | *abc* | *acb* | *bac* | *bca* | *cab* | *cba* |
|---|---|---|---|---|---|---|
| *abc* | *a* | *a* | *a* | *a* | *a* | *a* |
| *acb* | *a* | *a* | *a* | *a* | *a* | *a* |
| *bac* | *a* | *a* | *b* | *b* | *b* | *b* |
| *bca* | *a* | *a* | *b* | *b* | *b* | *b* |
| *cab* | *a* | *a* | *c* | *c* | *c* | *c* |
| *cba* | *a* | *a* | *c* | *c* | *c* | *c* |

*3 reports abc*

| 1’s ↓ \ 2’s → | *abc* | *acb* | *bac* | *bca* | *cab* | *cba* |
|---|---|---|---|---|---|---|
| *abc* | *a* | *a* | *a* | *a* | *a* | *a* |
| *acb* | *a* | *a* | *a* | *a* | *a* | *a* |
| *bac* | *a* | *a* | *b* | *b* | *b* | *b* |
| *bca* | *a* | *a* | *b* | *b* | *b* | *b* |
| *cab* | *a* | *a* | *c* | *c* | *c* | *c* |
| *cba* | *a* | *a* | *c* | *c* | *c* | *c* |

*3 reports acb*

| 1’s ↓ \ 2’s → | *abc* | *acb* | *bac* | *bca* | *cab* | *cba* |
|---|---|---|---|---|---|---|
| *abc* | *a* | *a* | *b* | *b* | *a* | *a* |
| *acb* | *a* | *a* | *b* | *b* | *a* | *a* |
| *bac* | *b* | *b* | *b* | *b* | *b* | *b* |
| *bca* | *b* | *b* | *b* | *b* | *b* | *b* |
| *cab* | *c* | *c* | *b* | *b* | *c* | *c* |
| *cba* | *c* | *c* | *b* | *b* | *c* | *c* |

*3 reports bac*

| 1’s ↓ \ 2’s → | *abc* | *acb* | *bac* | *bca* | *cab* | *cba* |
|---|---|---|---|---|---|---|
| *abc* | *a* | *a* | *b* | *b* | *a* | *a* |
| *acb* | *a* | *a* | *b* | *b* | *a* | *a* |
| *bac* | *b* | *b* | *b* | *b* | *b* | *b* |
| *bca* | *b* | *b* | *b* | *b* | *b* | *b* |
| *cab* | *c* | *c* | *b* | *b* | *c* | *c* |
| *cba* | *c* | *c* | *b* | *b* | *c* | *c* |

*3 reports bca*

| 1’s ↓ \ ` | *abc* | *acb* | *bac* | *bca* | *cab* | *cba* |
|---|---|---|---|---|---|---|
| *abc* | *a* | *a* | *a* | *a* | *c* | *c* |
| *acb* | *a* | *a* | *a* | *a* | *c* | *c* |
| *bac* | *b* | *b* | *b* | *b* | *c* | *c* |
| *bca* | *b* | *b* | *b* | *b* | *c* | *c* |
| *cab* | *c* | *c* | *c* | *c* | *c* | *c* |
| *cba* | *c* | *c* | *c* | *c* | *c* | *c* |

*3 reports cab*

| 1’s ↓ \ ` | *abc* | *acb* | *bac* | *bca* | *cab* | *cba* |
|---|---|---|---|---|---|---|
| *abc* | *a* | *a* | *a* | *a* | *c* | *c* |
| *acb* | *a* | *a* | *a* | *a* | *c* | *c* |
| *bac* | *b* | *b* | *b* | *b* | *c* | *c* |
| *bca* | *b* | *b* | *b* | *b* | *c* | *c* |
| *cab* | *c* | *c* | *c* | *c* | *c* | *c* |
| *cba* | *c* | *c* | *c* | *c* | *c* | *c* |

*3 reports cba*

This SCF satisfies Freedom of Expression, Unanimity and Non-dictatorship but violates Non-manipulability.