- Dominance
  - MaxiMin
  - LexiMin
- Money lotteries and attitudes to risk
- Decision trees
  - Backward induction
  - Example of a complex decision tree
- Beyond money lotteries
  - Expected Utility Theory

state → act ↓	s_1	s_2	s_3
a_1	4	3	1
a_2	6	2	2
a_3	5	3	2
a_4	6	1	0
a_5	3	2	5

Dominance:

So we can simplify

state → act ↓	s_1	s_2	s_3
a_2	6	2	2
a_3	5	3	2
a_5	3	2	5

What then?

First a different example:

state → act ↓	s_1	s_2	s_3
a_1	4	3	1
a_2	3	2	2
a_3	5	3	2
a_4	6	1	0
a_5	3	3	4

One criterion that can be used is the **MaxiMin** criterion.

Now back to the previous problem:

state → act ↓	s_1	s_2	s_3
a_2	6	2	2
a_3	5	3	2
a_5	3	2	5

MaxiMin =

A refinement is the **LexiMin**

state →	s_1	s_2	s_3
act ↓			
a_2	6	2	2
a_3	5	3	2
a_5	3	2	5

Here the LexiMin picks

One more example:

state →	s_1	s_2	s_3	s_4
act ↓				
a_1	2	3	1	5
a_2	6	2	2	3
a_3	5	3	2	4
a_4	6	1	0	7
a_5	3	2	5	1

MaxiMin =

LexiMin =

Special case: outcomes are sums of money

state →	s_1	s_2	s_3	s_4
act ↓				
a_1	\$12	\$30	\$0	\$18
a_2	\$36	\$6	\$24	\$12
a_3	\$6	\$42	\$12	\$0

Suppose that we are able to assign probabilities to the states:

state →	s_1	s_2	s_3	s_4
	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{1}{12}$

a_1 is the lottery

a_2 is the lottery

a_3 is the lottery

The expected values are:

Definition of attitude to risk ….

Given a money lottery *L*, imagine giving the individual a choice between *L* and the expected value of *L* for sure, that is, the choice

between $\begin{pmatrix} \mathbb{E}[L] \\ 1 \end{pmatrix}$ and L or, written more simply, between $\mathbb{E}[L]$ and L

If she says that

- $\mathbb{E}[L] \succ L$ we say that she is **risk** **relative to *L***
- $\mathbb{E}[L] \sim L$ we say that she is **risk** **relative to *L***
- $L \succ \mathbb{E}[L]$ we say that she is **risk** **relative to *L***

So in the above example, if we assume that the agent is risk neutral relative to every lottery and her preferences are transitive, then, since

$\mathbb{E}[a_1] = 10.5$

$\mathbb{E}[a_2] = 24$

$\mathbb{E}[a_3] = 14$

Can we infer risk attitudes from choices?

Let $L = \begin{pmatrix} \$40 & \$60 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ Then $\mathbb{E}[L] =$

Suppose Ann's preferences are transitive, she prefers more money to less and she says that she prefers \$49 to *L*.

Suppose Bob's preferences are transitive, he prefers more money to less and he says that he prefers \$51 to *L*.

DECISION TREES

Decision to buy a house

- **NEW** (built 2015), costs $350,000
- **OLD** (built 1980), costs $300,000

You worry about the **total cost over the next 5 years**.

- **New** houses have a 25% probability of requiring a repair within 5 years and, on average, the repair would cost $20,000.
- **Old** houses have a 60% probability of requiring a repair within 5 years and, on average, the repair would cost $100,000.

Your options are: (1) buy house **N**, (2) buy house **O** or (3) pay $1,000 to an **inspector** to inspect both houses. The inspector will be able to tell you if each house is good or bad.

- A **good new** house has probability 20% of requiring a repair (that costs $20,000) and probability 80% of requiring no repair.
- A **bad new** house has probability 30% of requiring a repair (that costs $20,000) and probability 70% of requiring no repair.
- A **good old** house has probability 50% of requiring a repair (that costs $100,000) and probability 50% of requiring no repair.
- A **bad old** house has probability 70% of requiring a repair (that costs $100,000) and probability 30% of requiring no repair.

Based on past data, the probabilities that the inspector will come up with the various verdicts are:

- Both good: 20%
- Both bad: 30%
- Old house good, new house bad: 20%
- Old house bad, new house good: 30%.

THIS IS A LOT OF INFORMATION!

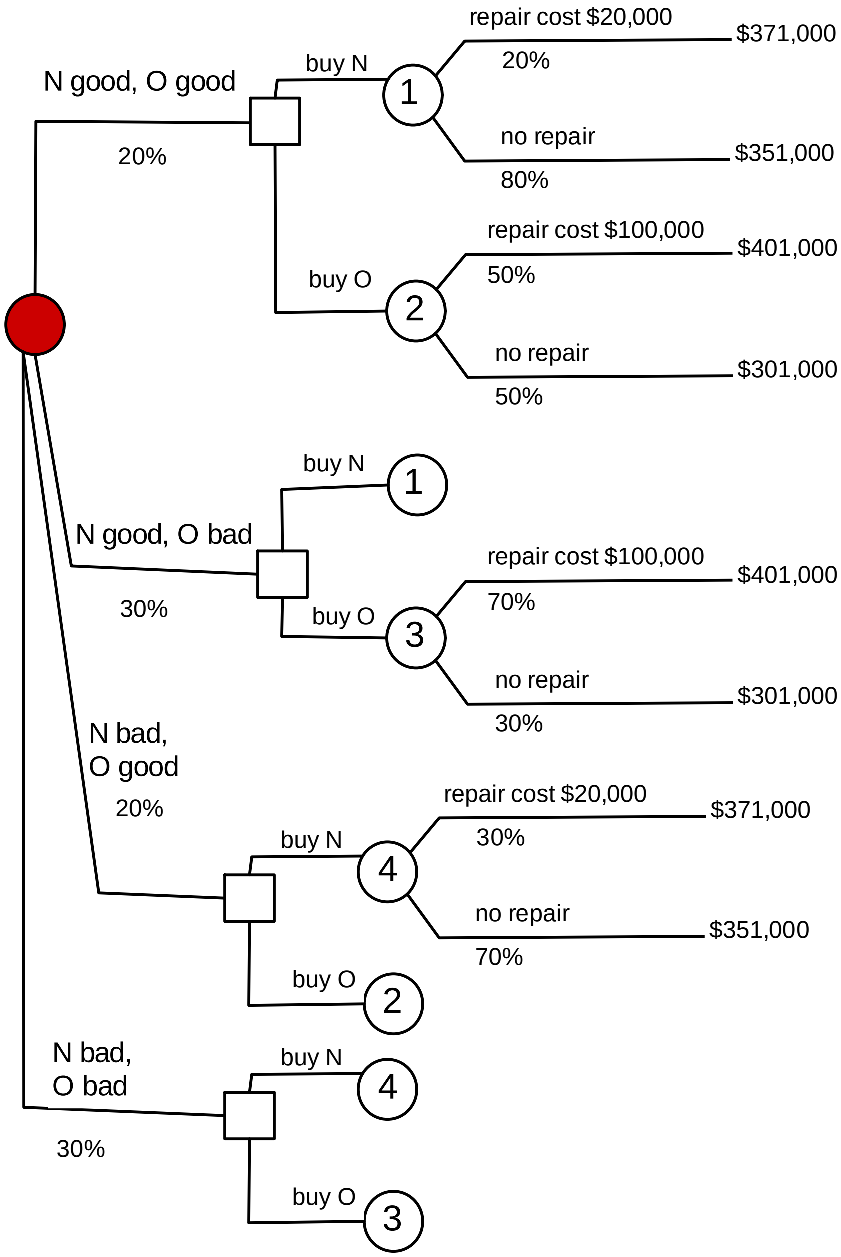
- **NEW** costs $350,000. **New** houses have a 25% probability of requiring a repair within 5 years and, on average, the repair would cost $20,000.

- **OLD** costs $300,000. **Old** houses have a 60% probability of requiring a repair within 5 years and, on average, the repair would cost $100,000.

- You can also hire an inspector and pay her $1,000

Assuming risk neutrality

The “hire inspector” module is as follows:

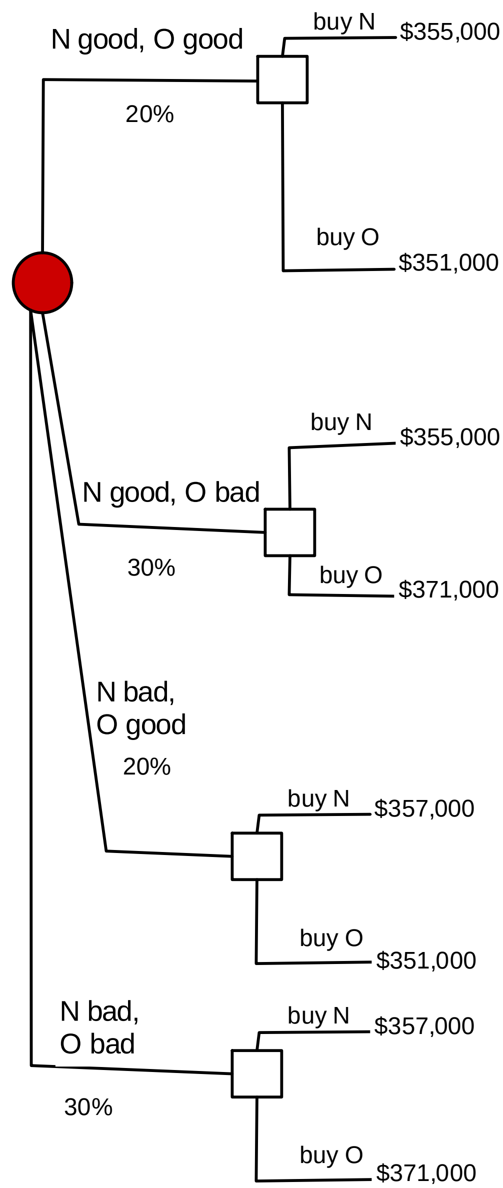


The expected values of the lotteries are:

- For ①:
- For ②:
- For ③:
- For ④:

Thus we can reduce this part of the tree to:

OBJECTIVE: pay the LOWEST 5-year cost



Thus we can reduce the option of hiring the inspector to the following lottery:

Whose expected value is

The optimal decision is:

1. **hire the inspector and then**
2. **(a) if both good, buy**

   (b) if N good and O bad, buy

   (c) if N bad and O good, buy

   (d) if both bad, buy

$$\text{operation } O = \begin{pmatrix} \text{cured} & \begin{matrix}\text{permanent} \\ \text{disability}\end{matrix} \\ \\ 90\% & 10\% \end{pmatrix}$$

$$\text{drug treatment } D = \begin{pmatrix} \text{cured} & \begin{matrix}\text{no} \\ \text{benefit}\end{matrix} & \begin{matrix}\text{adverse} \\ \text{reaction}\end{matrix} \\ \\ 75\% & 10\% & 15\% \end{pmatrix}$$

What is the expected value of lottery O?

What is the expected value of lottery D?

Which of the two lotteries is better?

EXPECTED UTILITY THEORY

$Z = \{z_1, z_2, ..., z_m\}$ set of basic outcomes.

A lottery is a probability distribution over *Z*: $L = \begin{pmatrix} z_1 & z_2 & ... & z_m \\ p_1 & p_2 & ... & p_m \end{pmatrix}$

Let $\mathcal{L}$ be the set of lotteries. Suppose that the agent has a ranking $\succsim$ of the elements of $\mathcal{L}$:

if $L = \begin{pmatrix} z_1 & z_2 & ... & z_m \\ p_1 & p_2 & ... & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & ... & z_m \\ q_1 & q_2 & ... & q_m \end{pmatrix}$ then

$L \succ M$ means that

$L \sim M$ means that

Rationality constraints on $\succsim$ (von Neumann-Morgenstern axioms):

…

Theorem 1 Let $Z = \{z_1, z_2, ..., z_m\}$ be a set of basic outcomes and L the set of lotteries over *Z*. If $\succsim$ satisfies the von Neumann-Morgenstern axionm then there exists a function $U : Z \to \mathbb{R}$, called a *von Neumann-Morgenstern utility function*, that assigns a number to every basic outcome and is such that, for any two lotteries $L = \begin{pmatrix} z_1 & z_2 & ... & z_m \\ p_1 & p_2 & ... & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & ... & z_m \\ q_1 & q_2 & ... & q_m \end{pmatrix}$,

$$L \succ M \quad \text{if and only if} \quad \underbrace{p_1U(z_1) + p_2U(z_2) + ... + p_mU(z_m)}_{\text{expected utility of lottery } L} > \underbrace{q_1U(z_1) + q_2U(z_2) + ... + q_mU(z_m)}_{\text{expected utility of lottery } M}$$

and

$$L \sim M \quad \text{if and only if} \quad \underbrace{p_1U(z_1) + p_2U(z_2) + ... + p_mU(z_m)}_{\text{expected utility of lottery } L} = \underbrace{q_1U(z_1) + q_2U(z_2) + ... + q_mU(z_m)}_{\text{expected utility of lottery } M}$$

2.25 3.33

EXAMPLE 1. $Z = \{z_1, z_2, z_3, z_4\}$ $L = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{8} & \frac{5}{8} & 0 & \frac{2}{8} \end{pmatrix}$ $M = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}$

Suppose we know that $U = \begin{cases} z_1 & z_2 & z_3 & z_4 \\ 6 & 2 & 8 & 1 \end{cases}$

Then

$\mathbb{E}[U(L)] =$

$\mathbb{E}[U(M)] =$

EXAMPLE 2.

$$A = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 50\% & 50\% \end{pmatrix} \qquad B = \begin{pmatrix} \text{paid 1-week vacation} \\ 100\% \end{pmatrix}$$

Suppose Ann says $\boxed{B \succ A}$ How would she rank

$$C = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 5\% & 95\% \end{pmatrix} \text{ and } D = \begin{pmatrix} \text{paid 1-week vacation} & \text{no vacation} \\ 10\% & 90\% \end{pmatrix} ?$$

Money lotteries

$$L = \begin{pmatrix} \$17 \\ 1 \end{pmatrix}$$

$$M = \begin{pmatrix} \$9 & \$25 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$\mathbb{E}[L] =$

$\mathbb{E}[M] =$

Suppose Bob's vNM utility function is: $U(\$x) = \sqrt{x}$

$\mathbb{E}[U(L)] =$

$\mathbb{E}[U(M)] =$

$$A = \begin{pmatrix} \$0 & \$100 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B = \begin{pmatrix} \$40 & \$60 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\mathbb{E}[A] = \qquad \mathbb{E}[B] =$$

Suppose Bob's vNM utility function is: $U(\$x) = \sqrt{x}$

$$\mathbb{E}[U(A)] =$$

$$\mathbb{E}[U(B)] =$$

$$A = \begin{pmatrix} \$4 & \$6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad B = \begin{pmatrix} \$5 \\ 1 \end{pmatrix}$$

$$\mathbf{U(\$x) = x^2}$$

Re-define attitudes to risk in terms of utility:

Risk-averse if

Risk-neutral if

Risk-loving if