★ Decision trees

Backward induction

Example of a complex decision tree

★ Beyond money lotteries

Expected Utility Theory

Decision to buy a house

- **NEW** (built 2015), costs \$350,000
- **OLD** (built 1980), costs \$300,000

You worry about the **total cost over the next 5 years**.

- **New** houses have a 25% probability of requiring a repair within 5 years and, on average, the repair would cost \$20,000.
- **Old** houses have a 60% probability of requiring a repair within 5 years and, on average, the repair would cost \$100,000.

Your options are: (1) buy house **N**, (2) buy house **O** or (3) pay \$1,000 to an **inspector** to inspect both houses. The inspector will be able to tell you if each house is good or bad.

- A **good new** house has probability 20% of requiring a repair (that costs \$20,000) and probability 80% of requiring no repair.
- A **bad new** house has probability 30% of requiring a repair (that costs \$20,000) and probability 70% of requiring no repair.
- A **good old** house has probability 50% of requiring a repair (that costs \$100,000) and probability 50% of requiring no repair.
- A **bad old** house has probability 70% of requiring a repair (that costs \$100,000) and probability 30% of requiring no repair.

Based on past data, the probabilities that the inspector will come up with the various verdicts are:

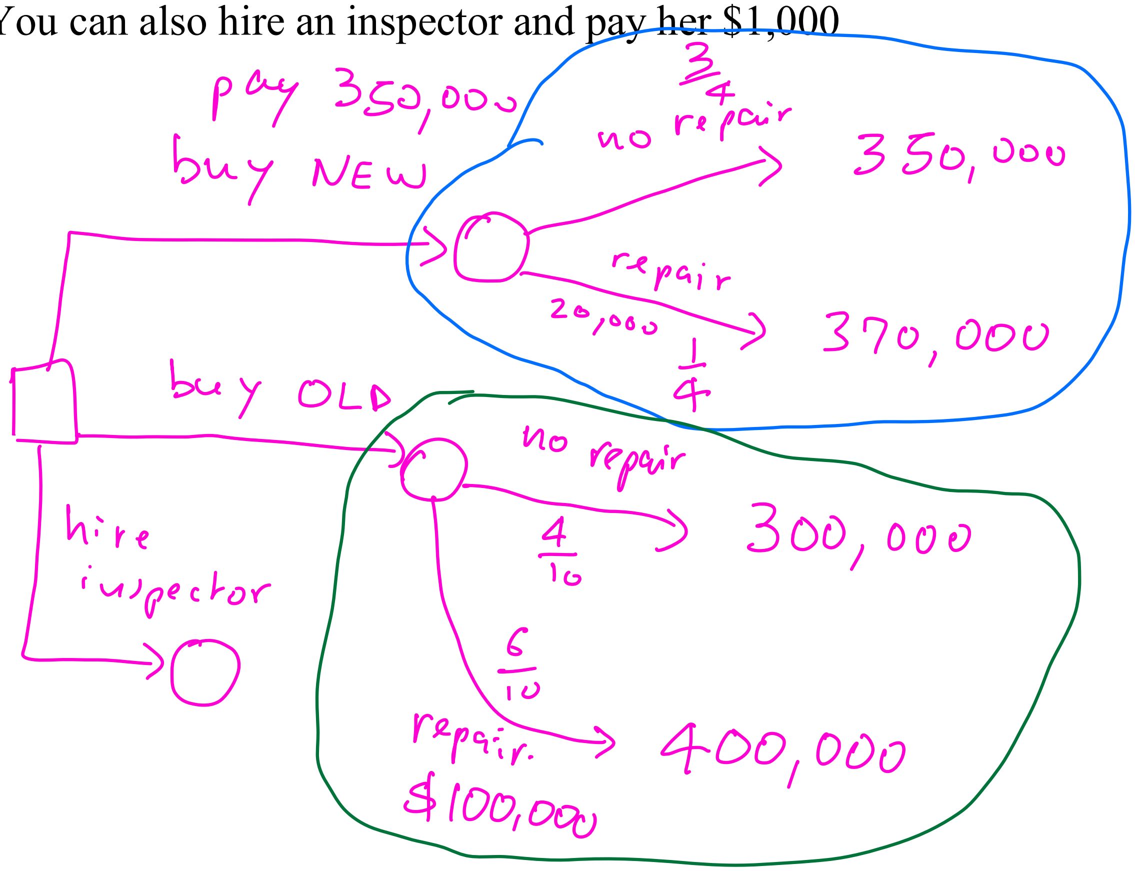
- Both good: 20%
- Both bad: 30%
- Old house good, new house bad: 20%
- Old house bad, new house good: 30%.

THIS IS A LOT OF INFORMATION!

- **NEW** costs $350,000. **New** houses have a 25% probability of requiring a repair within 5 years and, on average, the repair would cost $20,000.

- **OLD** costs $300,000. **Old** houses have a 60% probability of requiring a repair within 5 years and, on average, the repair would cost $100,000.

- You can also hire an inspector and pay her $1,000



Assuming risk neutrality

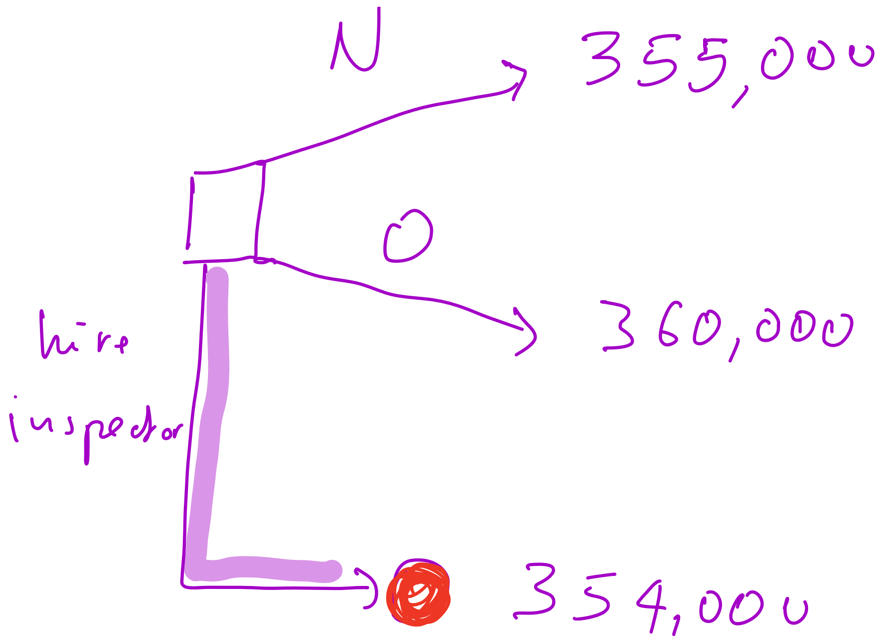
BY RISK NEUTRALITY:

$$N = \begin{pmatrix} 350{,}000 & 370{,}000 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

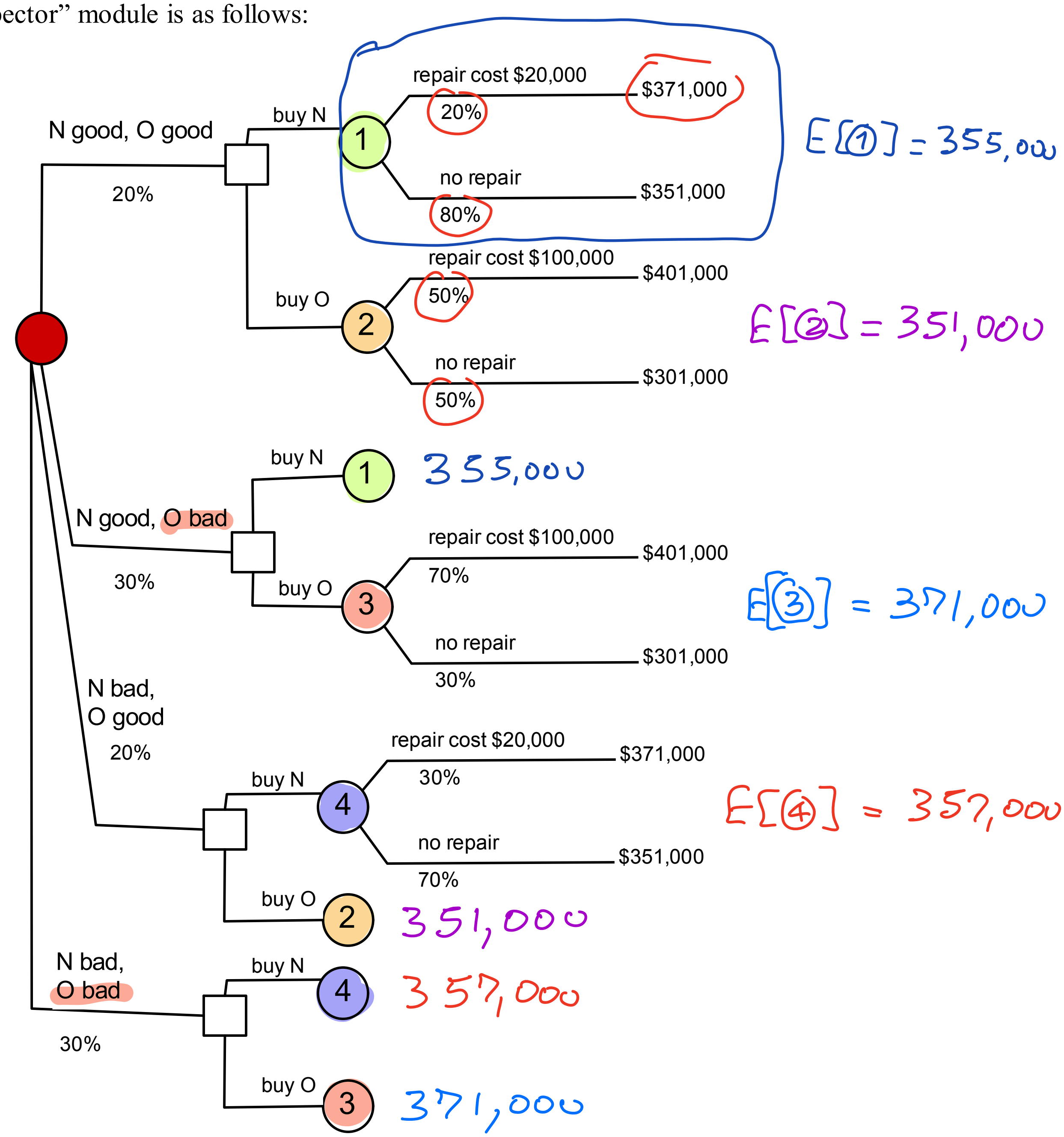
$$E[N] = \frac{3}{4}\, 350000 + \frac{1}{4}\, 370{,}000 = 355{,}000$$

$$O = \begin{pmatrix} 300{,}000 & 400{,}000 \\ \frac{4}{10} & \frac{6}{10} \end{pmatrix}$$

$$E[O] = \frac{4}{10}\, 300{,}000 + \frac{6}{10}\, 400{,}000 = 360{,}000$$


N
355,000
O
360,000
hire inspector
354,000

The “hire inspector” module is as follows:

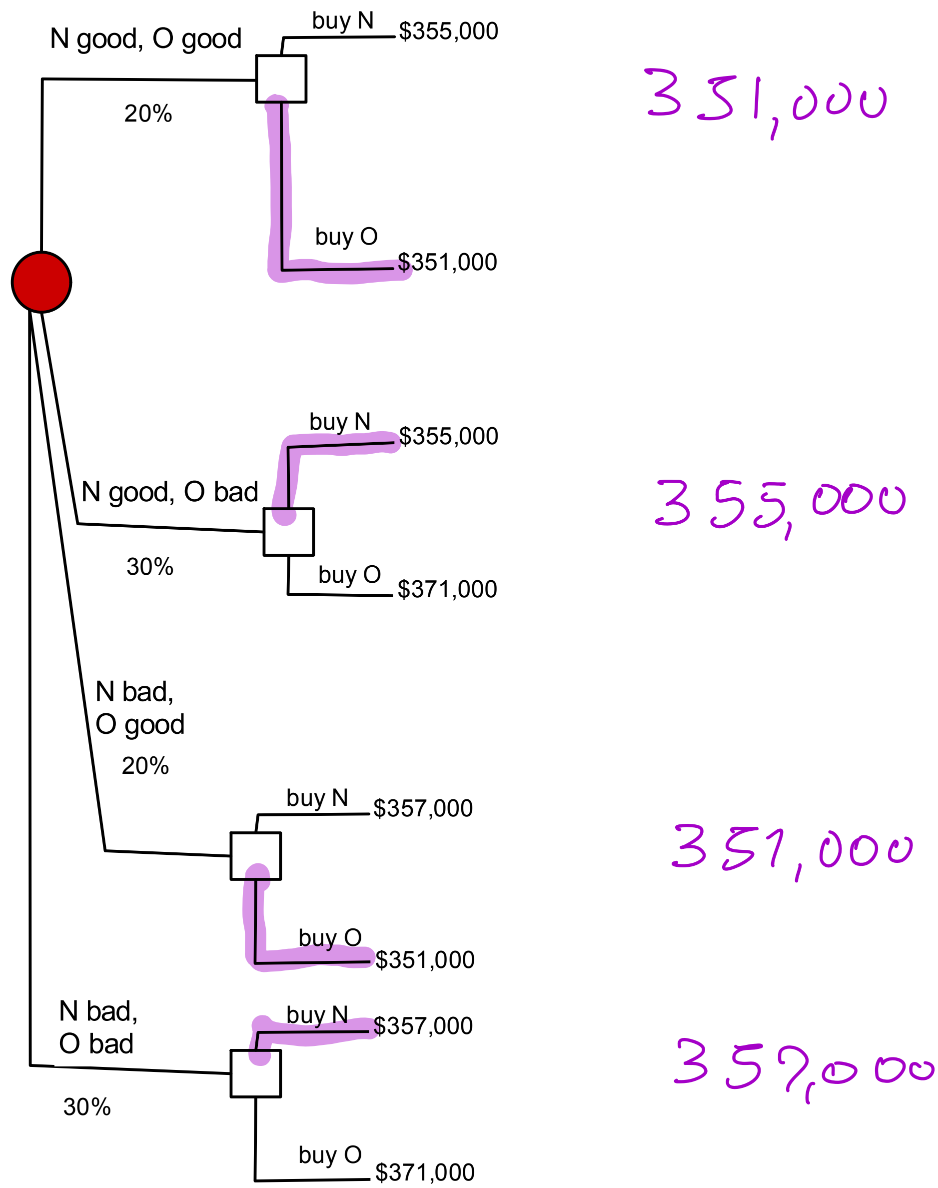


The expected values of the lotteries are:

- For ①:
- For ②:
- For ③:
- For ④:

Thus we can reduce this part of the tree to:

OBJECTIVE: pay the LOWEST 5-year cost



Thus we can reduce the option of hiring the inspector to the following lottery:

$$\begin{pmatrix} 351{,}000 & 355{,}000 & 351{,}000 & 357{,}000 \\ 20\% & 30\% & 20\% & 30\% \end{pmatrix}$$

Whose expected value is

$$E = \frac{2}{10}\,351 + \frac{3}{10}\,355 + \frac{2}{10}\,351 + \frac{3}{10}\,357 = 354{,}000$$

The optimal decision is:

1. **hire the inspector and then**
2. **(a) if both good, buy**

   (b) if N good and O bad, buy

   (c) if N bad and O good, buy

   (d) if both bad, buy

$$\text{operation } \; O = \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{pmatrix} \text{cured} & \text{permanent disability} \\ 90\% & 10\% \end{pmatrix} \end{array}$$

best z_1
z_3
z_4
worst z_2

$$\text{drug treatment } \; D = \begin{array}{c} \begin{array}{ccc} z_1 & z_3 & z_4 \end{array} \\ \begin{pmatrix} \text{cured} & \text{no benefit} & \text{adverse reaction} \\ 75\% & 10\% & 15\% \end{pmatrix} \end{array}$$

What is the expected value of lottery O?

What is the expected value of lottery D?

meaningless

Which of the two lotteries is better?

$$Z = \{ z_1, z_2, z_3, z_4 \}$$

$$L_1 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$L_2 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} z_2 & z_4 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$L_3 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} z_3 \\ 1 \end{pmatrix} = z_3$$

EXPECTED UTILITY THEORY

$Z = \{z_1, z_2, ..., z_m\}$ set of basic outcomes.

For every $i = 1, \dots, m$

A lottery is a probability distribution over Z: $L = \begin{pmatrix} z_1 & z_2 & \dots & z_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$ $0 \le p_i \le 1$, $\sum_{i=1}^{m} p_i = 1$

Let $\mathcal{L}$ be the set of lotteries. Suppose that the agent has a ranking $\succsim$ of the elements of $\mathcal{L}$:

if $L = \begin{pmatrix} z_1 & z_2 & \dots & z_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & \dots & z_m \\ q_1 & q_2 & \dots & q_m \end{pmatrix}$ then

$L \succ M$ means that L is better than M (in the eyes of the DM)

$L \sim M$ means that L is just as good as M (")

$L \succsim M$ L is at least as good as M

Rationality constraints on $\succsim$ (von Neumann-Morgenstern axioms):

…

$L = \begin{pmatrix} z_{best} & z_{worst} \\ p & 1-p \end{pmatrix}$ $M = \begin{pmatrix} z_{best} & z_{worst} \\ q & 1-q \end{pmatrix}$

$L \succ M$ if and only if $p > q$

Theorem 1 Let $Z = \{z_1, z_2, \ldots, z_m\}$ be a set of basic outcomes and L the set of lotteries over Z. If $\succsim$ satisfies the von Neumann-Morgenstern axionm then there exists a function $U : Z \to \mathbb{R}$, called a *von Neumann-Morgenstern utility function*, that assigns a number to every basic outcome and is such that, for any two lotteries $L = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ p_1 & p_2 & \ldots & p_m \end{pmatrix}$ and $M = \begin{pmatrix} z_1 & z_2 & \ldots & z_m \\ q_1 & q_2 & \ldots & q_m \end{pmatrix}$,

$p_1 z_1$ meaningless because z_1 might not be a number

$$L \succ M \quad \text{if and only if} \quad \underbrace{p_1 U(z_1) + p_2 U(z_2) + \ldots + p_m U(z_m)}_{\text{expected utility of lottery } L} > \underbrace{q_1 U(z_1) + q_2 U(z_2) + \ldots + q_m U(z_m)}_{\text{expected utility of lottery } M}$$

and

$$L \sim M \quad \text{if and only if} \quad \underbrace{p_1 U(z_1) + p_2 U(z_2) + \ldots + p_m U(z_m)}_{\text{expected utility of lottery } L} = \underbrace{q_1 U(z_1) + q_2 U(z_2) + \ldots + q_m U(z_m)}_{\text{expected utility of lottery } M}$$

$$L = \begin{pmatrix} z_1 & z_2 & z_4 \\ \frac{1}{8} & \frac{5}{8} & \frac{2}{8} \end{pmatrix}$$

2.25 3.33

EXAMPLE 1. $Z = \{z_1, z_2, z_3, z_4\}$ $\quad L = \underbrace{\begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{8} & \frac{5}{8} & 0 & \frac{2}{8} \end{pmatrix}}$ $\quad M = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ \frac{1}{6} & \frac{2}{6} & \frac{1}{6} & \frac{2}{6} \end{pmatrix}$

Suppose we know that $U = \begin{cases} z_1 & z_2 & z_3 & z_4 \\ 6 & 2 & 8 & 1 \end{cases}$

Then

$$\mathbb{E}[U(L)] = \frac{1}{8} U(z_1) + \frac{5}{8} U(z_2) + \frac{2}{8} U(z_4) = \frac{1}{8} 6 + \frac{5}{8} 2 + \frac{2}{8} 1 = \boxed{2.25}$$

$$\mathbb{E}[U(M)] = \frac{1}{6} U(z_1) + \frac{2}{6} U(z_2) + \frac{1}{6} U(z_3) + \frac{2}{6} U(z_4) =$$

$$= \frac{1}{6} 6 + \frac{2}{6} 2 + \frac{1}{6} 8 + \frac{2}{6} 1 = \boxed{3.33}$$

$$M \succ L$$

EXAMPLE 2.

z_1 z_3 z_2

$$A = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 50\% & 50\% \end{pmatrix} \qquad B = \begin{pmatrix} \text{paid 1-week vacation} \\ 100\% \end{pmatrix}$$

Suppose Ann says $\boxed{B \succ A}$ How would she rank

$$C = \begin{pmatrix} \text{paid 3-week vacation} & \text{no vacation} \\ 5\% & 95\% \end{pmatrix} \text{ and } D = \begin{pmatrix} \text{paid 1-week vacation} & \text{no vacation} \\ 10\% & 90\% \end{pmatrix} ?$$

		U
best	z_1	a
	z_2	b
worst	z_3	c

a, b, c are numbers

$a > b > c$

$$B = \begin{pmatrix} z_2 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} z_1 & z_3 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Since $B \succ A$ $\quad E[U(B)] > E[U(A)]$

$$1 \cdot U(z_2) > \frac{1}{2} U(z_1) + \frac{1}{2} U(z_3)$$

$$b > \frac{1}{2} a + \frac{1}{2} c$$

$$\boxed{2b > a + c}$$

$$\boxed{2b > a + c}$$

$$C = \begin{pmatrix} z_1 & z_3 \\ \frac{5}{100} & \frac{95}{100} \end{pmatrix} \qquad D = \begin{pmatrix} z_2 & z_3 \\ \frac{10}{100} & \frac{95}{100} \end{pmatrix}$$

$$E[U(C)] = \frac{5}{100} \cdot U(z_1) + \frac{95}{100} U(z_3) \qquad E[U(D)] = \frac{10}{100} U(z_2) + \frac{95}{100} U(z_3)$$

$$= \frac{5}{100} a + \frac{95}{100} c \quad < \quad = \frac{10}{100} b + \frac{90}{100} c$$

$$5a + 95c \quad \gtrless \quad 10b + 90c$$

subtract $90c$ from both

$$5a + 5c \qquad 10b$$

divide both by 5

$$a + c < 2b$$

So $D \succ C$