- Expected Utility Theory
- The Allais Paradox
- Normalization of the utility function
- Decision trees revisited
- Hurwicz index
- MinMax Regret

Theorem 2. Let $\succsim$ be a von Neumann-Morgenstern ranking of the set of basic lotteries $\mathcal{L}$. Then the following are true.

(A) If $U: Z \to \mathbb{R}$ is a von Neumann-Morgenstern utility function that represents $\succsim$, then, for any two real numbers a and b with $a > 0$, the function $V: Z \to \mathbb{R}$ defined by $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$) is also a von Neumann-Morgenstern utility function that represents $\succsim$.

(B) If $U: Z \to \mathbb{R}$ and $V: Z \to \mathbb{R}$ are two von Neumann-Morgenstern utility functions that represent $\succsim$, then there exist two real numbers a and b with $a > 0$ such that $V(z_i) = aU(z_i) + b$ ($i = 1,2,\ldots,m$).

$$U = \begin{cases} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \\ 10 & 6 & 16 & 8 & 6 & 14 \end{cases}$$

$$\text{operation } O = \begin{pmatrix} \text{cured} & \text{permanent disability} \\ 90\% & 10\% \end{pmatrix}$$

$$\text{drug treatment } D = \begin{pmatrix} \text{cured} & \text{no benefit} & \text{adverse reaction} \\ 75\% & 10\% & 15\% \end{pmatrix}$$

What should you do?

$$\text{operation } O = \begin{pmatrix} \text{cured} & \begin{matrix}\text{permanent}\\ \text{disability}\end{matrix} \\ \\ 90\% & 10\% \\ \end{pmatrix}$$

$$\text{drug treatment } D = \begin{pmatrix} \text{cured} & \begin{matrix}\text{no}\\ \text{benefit}\end{matrix} & \begin{matrix}\text{adverse}\\ \text{reaction}\end{matrix} \\ \\ 75\% & 10\% & 15\% \\ \end{pmatrix}$$

The Allais paradox

(Maurice Allais, 1952)

$$A = \begin{pmatrix} \$1M \\ 100\% \end{pmatrix} \quad \text{versus} \quad B = \begin{pmatrix} \$2.5M & \$1M & 0 \\ 10\% & 89\% & 1\% \end{pmatrix}$$

$$C = \begin{pmatrix} \$1M & 0 \\ 11\% & 89\% \end{pmatrix} \quad \text{versus} \quad D = \begin{pmatrix} \$2.5M & 0 \\ 10\% & 90\% \end{pmatrix}.$$

$$A \succ B \quad \text{if and only if} \quad \mathbb{E}[U(A)] > \mathbb{E}[U(B)]$$

First question to ask yourself:
what is my ranking of the basic outcomes?

state $\rightarrow$	s_1	s_2	s_3
act $\downarrow$			
a_1	z_1	z_2	z_3
a_2	z_4	z_5	z_6
a_3	z_7	z_8	z_9

state $\rightarrow$	s_1	s_2	s_3		
act $\downarrow$				best	z_8
					z_3
a_1	z_1	z_2	z_3		z_1, z_9
a_2	z_4	z_5	z_6		z_2, z_6
a_3	z_7	z_8	z_9		z_4, z_5
				worst	z_7

Note:

- a_1

Thus …

state $\rightarrow$	s_1	s_2	s_3			Utility
act $\downarrow$				best	z_8	1
a_1	z_1	z_2	z_3		z_3	
a_3	z_7	z_8	z_9		z_1, z_9	
					z_2	
				worst	z_7	0

Three questions to ask yourself:

Note that p is the probability of the worst outcome, not the best

(1) What p is such that $\begin{pmatrix} z_3 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_7 & z_8 \\ p & 1-p \end{pmatrix}$? Suppose the answer is

(2) What p is such that $\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_7 & z_8 \\ p & 1-p \end{pmatrix}$? Suppose the answer is

(3) What p is such that $\begin{pmatrix} z_2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_7 & z_8 \\ p & 1-p \end{pmatrix}$? Suppose the answer is

		Utility
best	z_8	1
	z_3	$\frac{3}{4}$
	z_1, z_9	$\frac{2}{3}$
	z_2	$\frac{2}{5}$
worst	z_7	0

In order not to deal with fractions, rescale the utility function by multiplying each number by 60:

		Utility
best	z_8	60
	z_3	45
	z_1, z_9	40
	z_2	24
worst	z_7	0

state $\rightarrow$	s_1	s_2	s_3
act $\downarrow$			
a_1	40	24	45
a_3	0	60	40

Next step: try to assign probabilities to the states (from objective data or some subjective assessment). Suppose you assess the following:

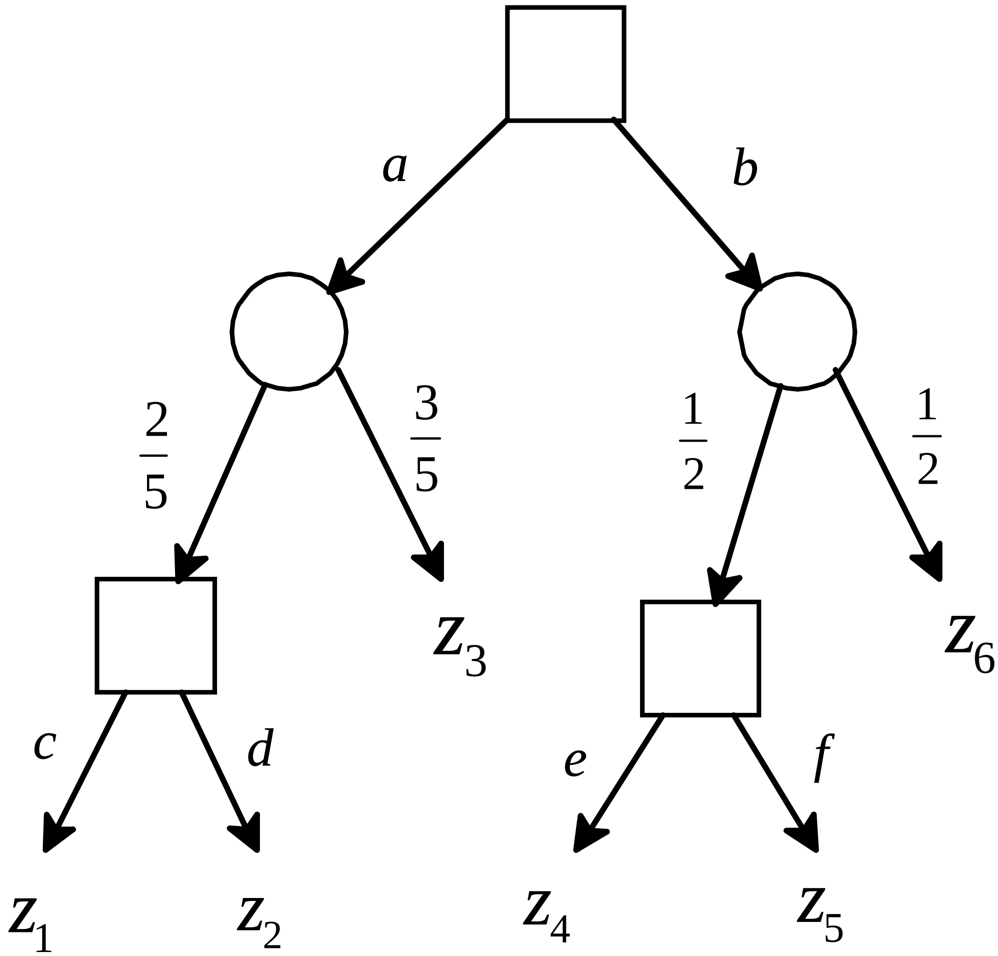
state:	s_1	s_2	s_3
probability:	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

Then: $\mathbb{E}[U(a_1)] =$

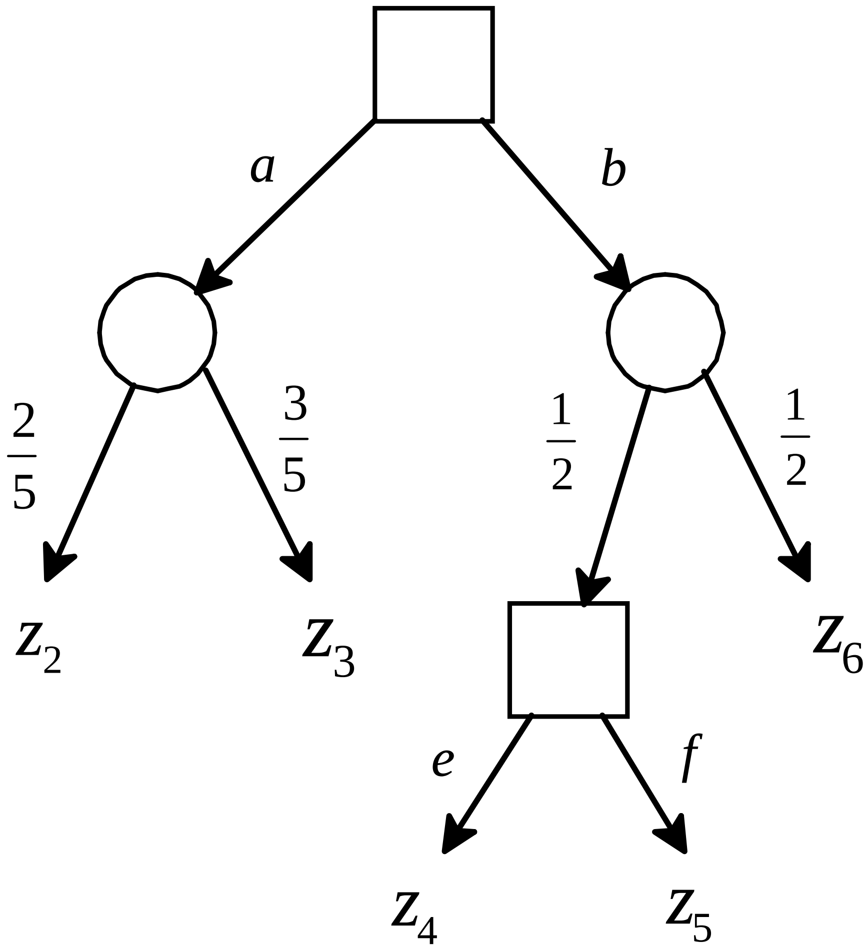
$\mathbb{E}[U(a_3)] =$

Hence you should take action

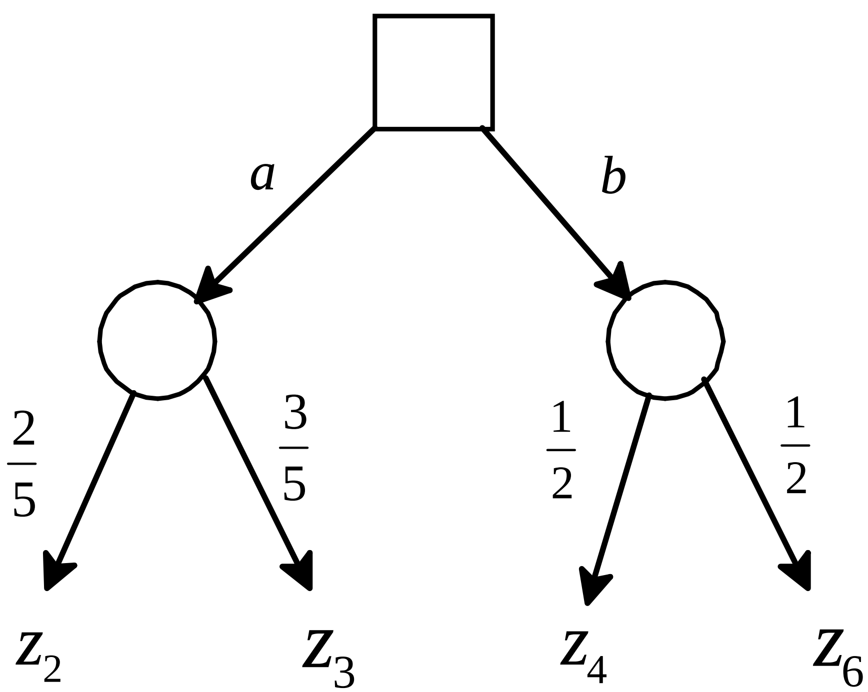
Decision tree

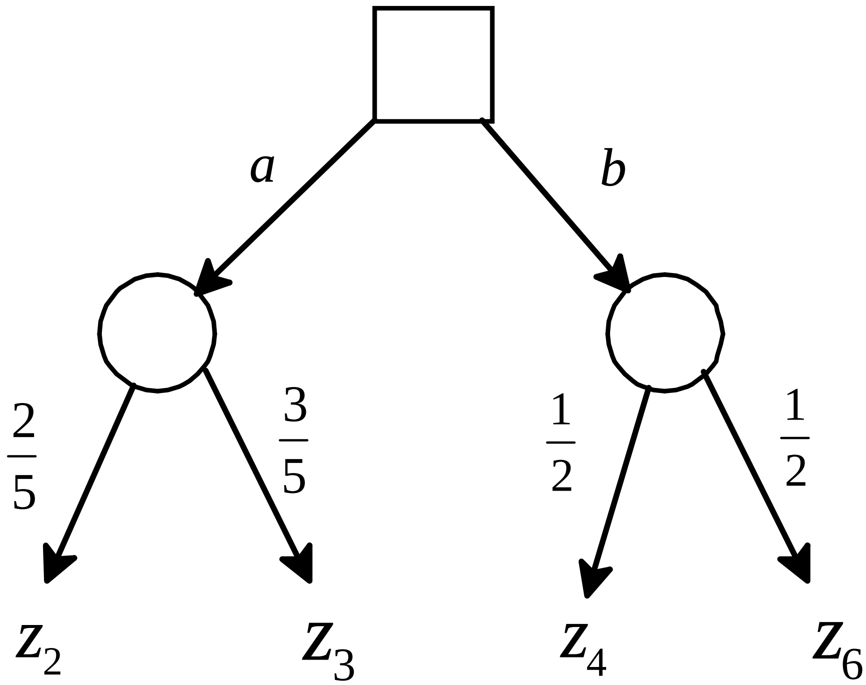


First question to ask yourself: how do I rank z_1 and z_2? Suppose that the answer is $z_2 \succ z_1$.



Second question to ask yourself: how do I rank z_4 and z_5? Suppose that the answer is $z_4 \succ z_5$.

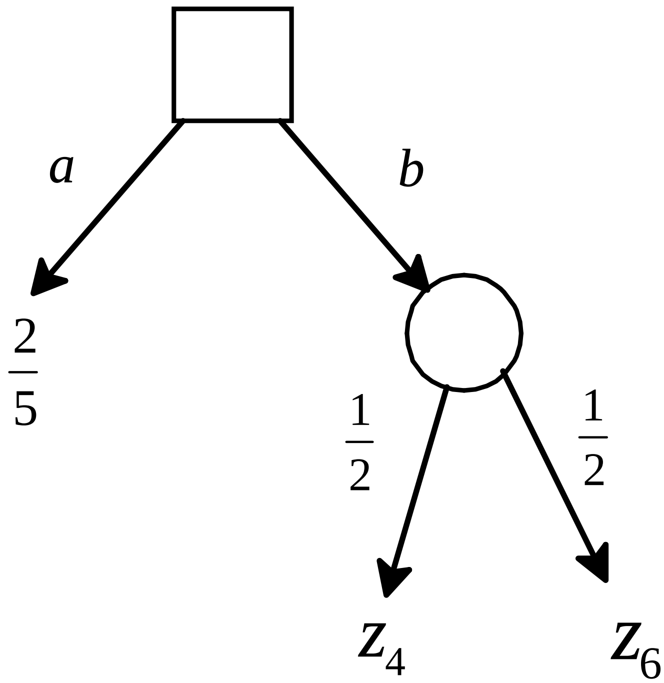




Next question: how do I rank the remaining four outcomes? Suppose:

		Utility
best	z_2	1
	z_6	
	z_4	
worst	z_3	0

This is sufficient to eliminate the random event on the left:

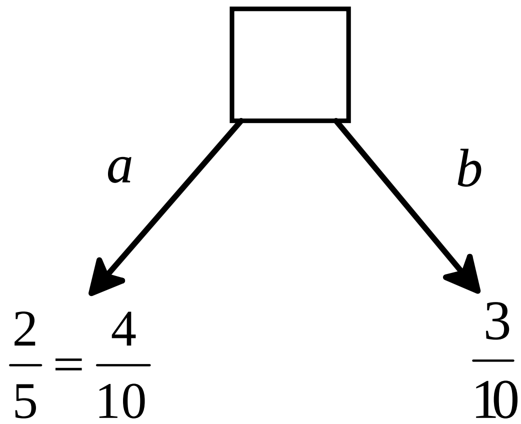


Two more questions and then you are done!

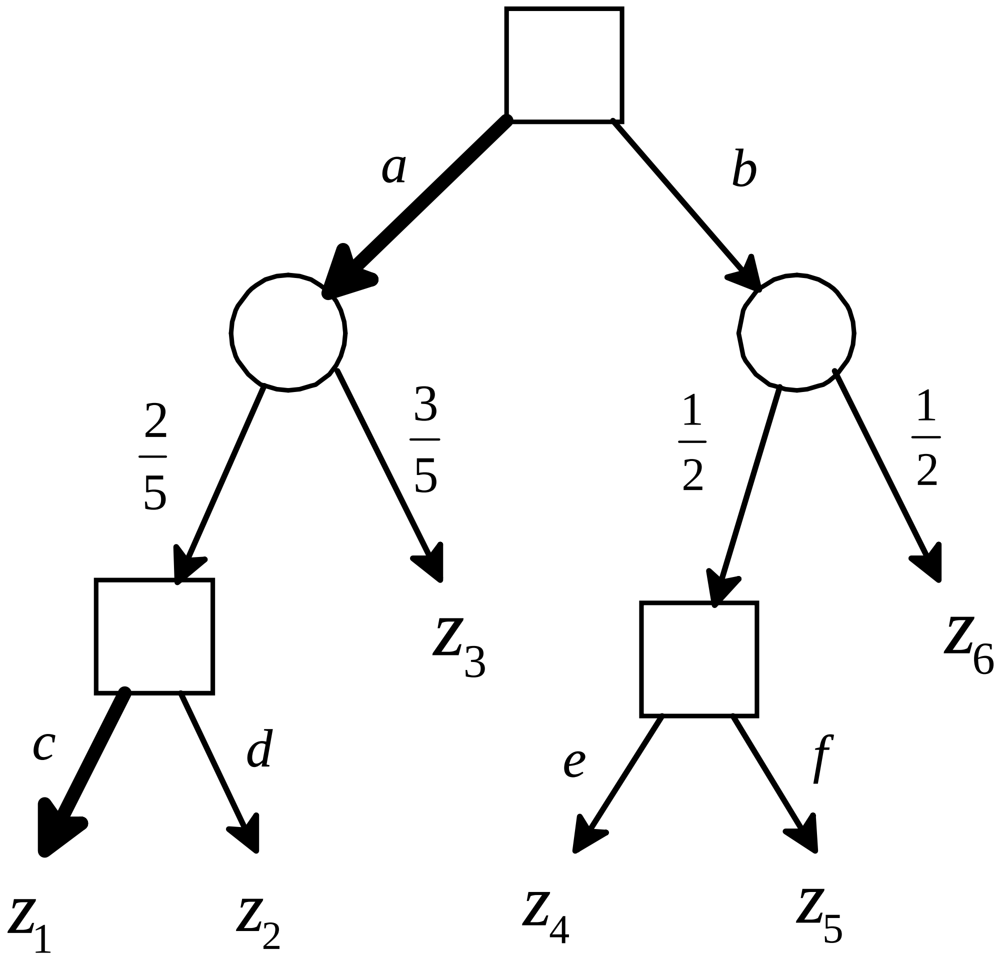
(4) What p is such that $\begin{pmatrix} z_6 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_2 & z_3 \\ p & 1-p \end{pmatrix}$? Suppose the answer is $p = \frac{1}{2}$.

(5) What p is such that $\begin{pmatrix} z_4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_2 & z_3 \\ p & 1-p \end{pmatrix}$? Suppose the answer is $p = \frac{1}{10}$.

Then the lottery corresponding to the random event on the right has an expected utility of



Hence the optimal decision is to first take action *a* and then, if a second choice is required between *c* and *d*, choose *d*:



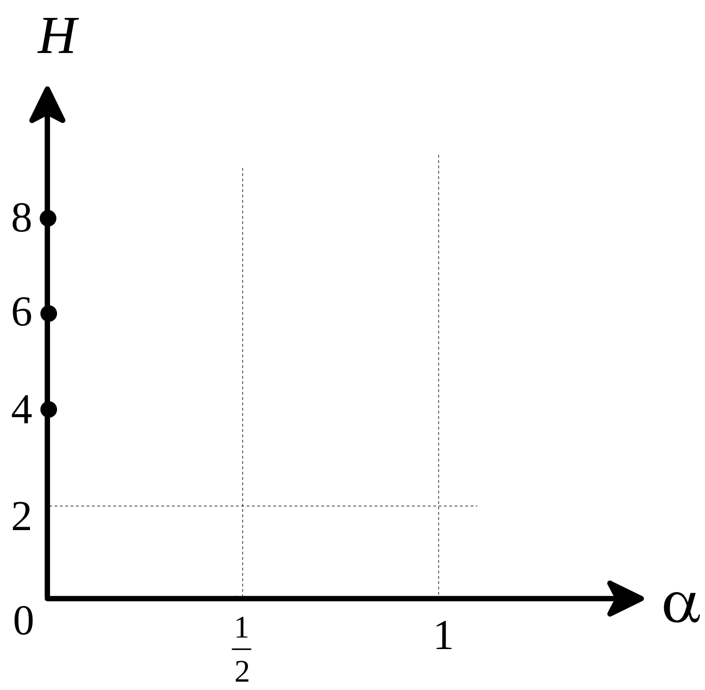
THE HURWICZ INDEX

	s_1	s_2	s_3
a_1	8	1	0
a_2	6	2	3
a_3	0	3	4

$$H_\alpha(a_1) = 0\alpha + 8(1-\alpha) = 8 - 8\alpha$$

$$H_\alpha(a_2) = 2\alpha + 6(1-\alpha) = 6 - 4\alpha$$

$$H_\alpha(a_3) = 0\alpha + 4(1-\alpha) = 4 - 4\alpha$$



Note: the Hurwicz index is invariant to allowed transformations of the utility function.

MinMax REGRET

	s_1	s_2	s_3
a_1	8	1	0
a_2	6	2	3
a_3	0	3	4

Define the **regret of taking action *a* under state *s*** as the difference between the maximum utility you could have got under state *s* (by taking the best action for that state) and the utility that you get with action *a*. We can then construct a **regret table:**

	s_1	s_2	s_3
a_1			
a_2			
a_3			

If I had chosen an alternative utility function, would I have reached the same conclusion in terms of MinMaxRegret? Consider a new decision problem:

	s_1	s_2	s_3
a_1	0	2	4
a_2	2	1	1
a_3	8	0	0

	s_1	s_2	s_3
a_1	z_1	z_2	z_3
a_2	z_4	z_5	z_6
a_3	z_7	z_8	z_9

we infer that the ranking is

	U	V
best	8	
	4	
	2	
	1	
worst	0	

Using U:

	s_1	s_2	s_3
a_1	0	2	4
a_2	2	1	1
a_3	8	0	0

regret:

	s_1	s_2	s_3
a_1			
a_2			
a_3			

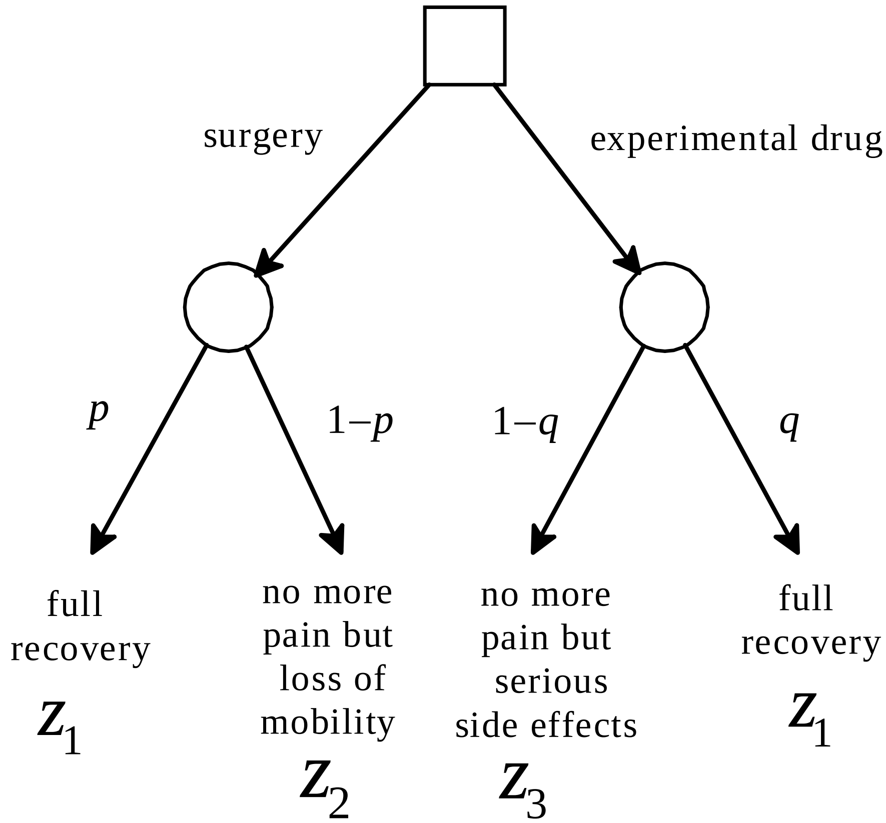
using V :

	s_1	s_2	s_3
a_1			
a_2			
a_3			

regret:

	s_1	s_2	s_3
a_1			
a_2			
a_3			

Example: knee injury



Suppose that:

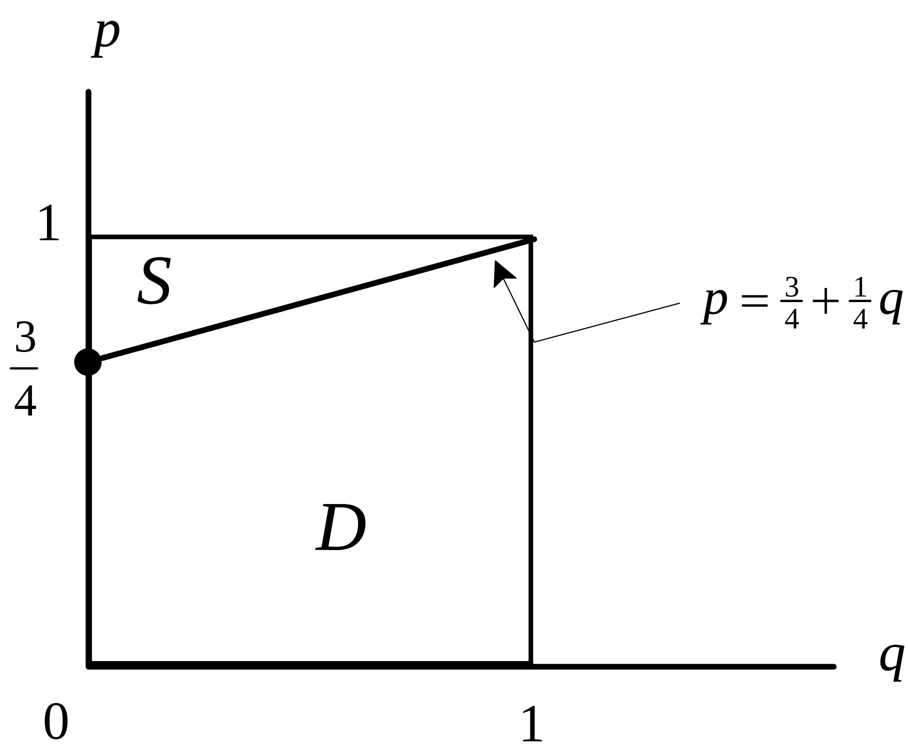
		U
best	z_1	100
	z_3	75
worst	z_2	0

The expected utility of surgery is

the expected utility of taking the drug is

So if you know the values of p and q then your optimal decision is:

- surgery if
- drug if
- either surgery or drug is



Suppose that the values of *p* and *q* are not available

	(S, D)	$(S, \neg D)$	$(\neg S, D)$	$(\neg S, \neg D)$
Surgery	z_1	z_1	z_2	z_2
Drug	z_1	z_3	z_1	z_3

		U
best	z_1	100
	z_3	75
worst	z_2	0

Replacing outcomes with utilities:

	(S, D)	$(S, \neg D)$	$(\neg S, D)$	$(\neg S, \neg D)$
Surgery				
Drug				

The corresponding regret table is:

	(S, D)	$(S, \neg D)$	$(\neg S, D)$	$(\neg S, \neg D)$
Surgery				
Drug				

What about the Hurwicz index?

$H_\alpha(Drug) =$ $\quad$ $H_\alpha(Surgery) =$