- Expected Utility Theory ✓
- The Allais Paradox
- Normalization of the utility function ✓
- Decision trees revisited
- Hurwicz index
- MinMax Regret

The Allais paradox

(Maurice Allais, 1952)

$$A = \begin{pmatrix} \$1M \\ 100\% \end{pmatrix} \quad \text{versus} \quad B = \begin{pmatrix} \$2.5M & \$1M & 0 \\ 10\% & 89\% & 1\% \end{pmatrix}$$

most answered $A \succ B$

$$\underbrace{C = \begin{pmatrix} \$1M & 0 \\ 11\% & 89\% \end{pmatrix}}_{E[U(C)] = \frac{11}{100}a} \;>\; \text{versus} \quad \underbrace{D = \begin{pmatrix} \$2.5M & 0 \\ 10\% & 90\% \end{pmatrix}}_{\frac{10}{100}}.$$

hence $C > D$

most answered $D \succ C$

	outcomes	normalized utility
best	\$2.5 M	1
	\$1 M	a
worst	\$0	0

$0 < a < 1$

$$\boxed{A \succ B} \quad \text{if and only if} \quad \mathbb{E}[U(A)] > \mathbb{E}[U(B)]$$

$\mathbb{E}[U(A)] = 1 \cdot U(\$1M) = a$

$\mathbb{E}[U(B)] = \frac{10}{100} \cdot U(\$2.5M) + \frac{89}{100} \cdot U(\$1M) + \frac{1}{100} U(\$0) =$

$= \frac{10}{100} 1 + \frac{89}{100} a + \frac{1}{100} 0 =$

$= \frac{10}{100} + \frac{89}{100} a$

$$\boxed{\frac{11a}{100} > \frac{10}{100}}$$

First question to ask yourself:
what is my ranking of the basic outcomes?

state → act ↓	s_1	s_2	s_3
a_1	z_1	z_2	z_3
a_2	z_4	z_5	z_6
a_3	z_7	z_8	z_9

state → act ↓	s_1	s_2	s_3
a_1	z_1	z_2	z_3
a_2	z_4	z_5	z_6
a_3	z_7	z_8	z_9

best	z_8
	z_3
	z_1, z_9
	z_2, z_6
	z_4, z_5
worst	z_7

Note:

- a_1 strictly dominates a_2 : $z_1 \succ z_4$, $z_2 \succ z_5$, $z_3 \succ z_6$

Thus …

is then a_1 a dominant act? No because a_1 does not dominate a_3 : with a_3 get z_8

with a_1 get z_2

$z_8 \succ z_2$

state →	s_1	s_2	s_3
act ↓			
a_1	z_1	z_2	z_3
a_3	z_7	z_8	z_9

		normalized Utility
best	z_8	1
	z_3	$\frac{3}{4}$
	z_1, z_9	$\frac{2}{3}$
	z_2	$\frac{2}{5}$
worst	z_7	0

Three questions to ask yourself:

Note that p is the probability of the worst outcome, not the best

(1) What p is such that $\begin{pmatrix} z_3 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_7 & z_8 \\ p & 1-p \end{pmatrix}$? Suppose the answer is $p = \frac{1}{4}$, $1-p = \frac{3}{4}$

(2) What p is such that $\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_7 & z_8 \\ p & 1-p \end{pmatrix}$? Suppose the answer is $p = \frac{1}{3}$, $1-p = \frac{2}{3}$

(3) What p is such that $\begin{pmatrix} z_2 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_7 & z_8 \\ p & 1-p \end{pmatrix}$? Suppose the answer is $p = \frac{3}{5}$, $1-p = \frac{2}{5}$

$\begin{pmatrix} a = 60 \\ b = 0 \end{pmatrix}$

		Utility	multiply by 60
best	z_8	1	60
	z_3	$\frac{3}{4}$	45
	z_1, z_9	$\frac{2}{3}$	40
	z_2	$\frac{2}{5}$	24
worst	z_7	0	0

In order not to deal with fractions, rescale the utility function by multiplying each number by 60:

$$E[U(a_1)] = \frac{31.4}{60}$$

$$E[U(a_3)] = \frac{44}{60}$$

		Utility
best	z_8	60
	z_3	45
	z_1, z_9	40
	z_2	24
worst	z_7	0

$a_1 = \begin{pmatrix} z_1 & z_2 & z_3 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$

$U(a_1) = \begin{pmatrix} 40 & 24 & 45 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$

$\mathbb{E}[U(a_1)] =$

$\frac{1}{5} 40 + \frac{3}{5} 24 + \frac{1}{5} 45$

$= 31.4$

probability:	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
state $\rightarrow$	s_1	s_2	s_3
act $\downarrow$			
a_1	40	24	45
a_3	0	60	40

so a_3 is better than a_1

$\mathbb{E}[U(a_3)] =$

$\frac{1}{5} \cdot 0 + \frac{3}{5} 60 + \frac{1}{5} 40$

$= 44$

Next step: try to assign probabilities to the states (from objective data or some subjective assessment). Suppose you assess the following:

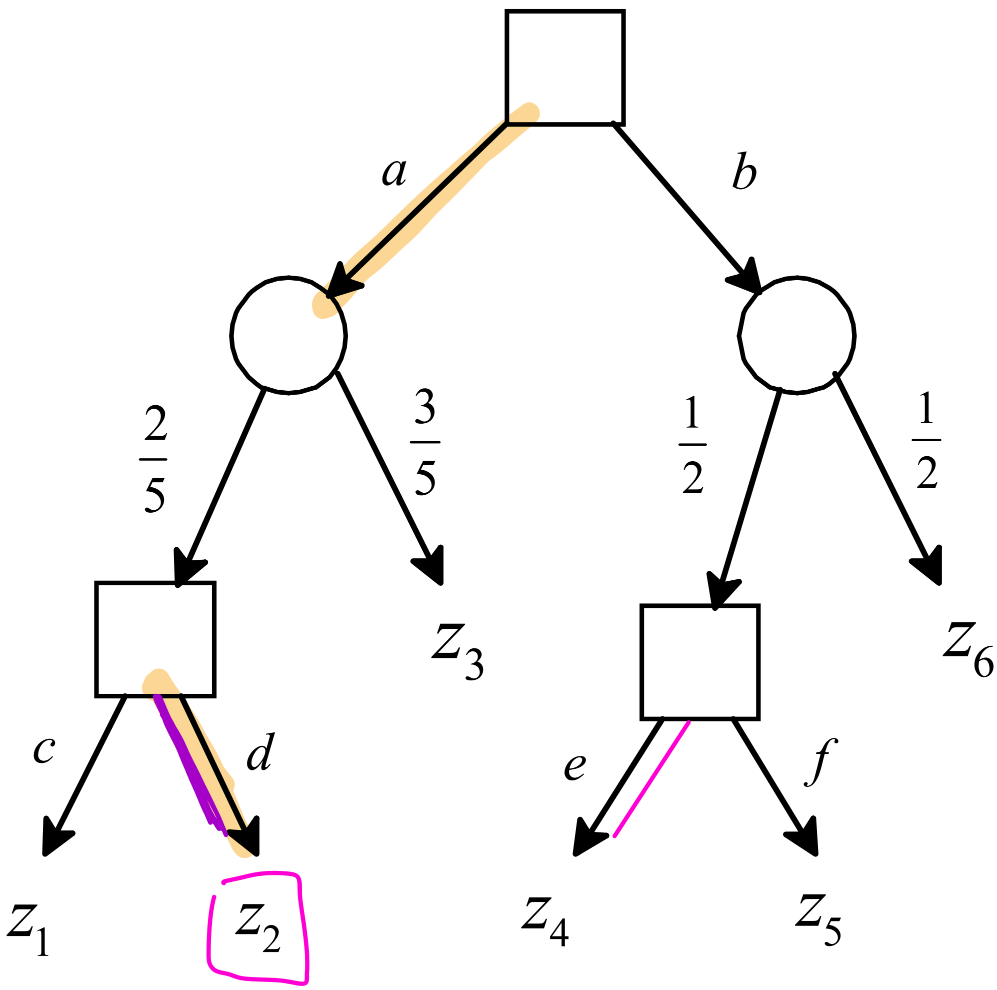
state:	s_1	s_2	s_3
probability:	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

Then: $\mathbb{E}[U(a_1)] =$

$\mathbb{E}[U(a_3)] =$

Hence you should take action a_3

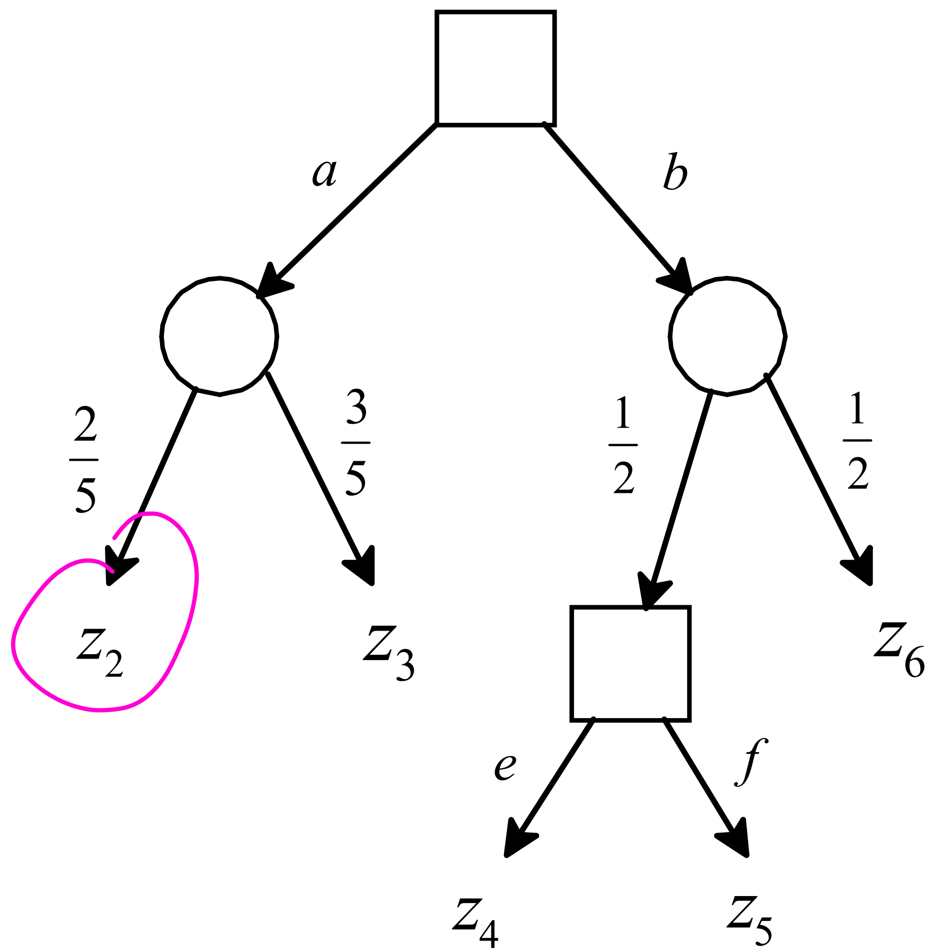
Decision tree



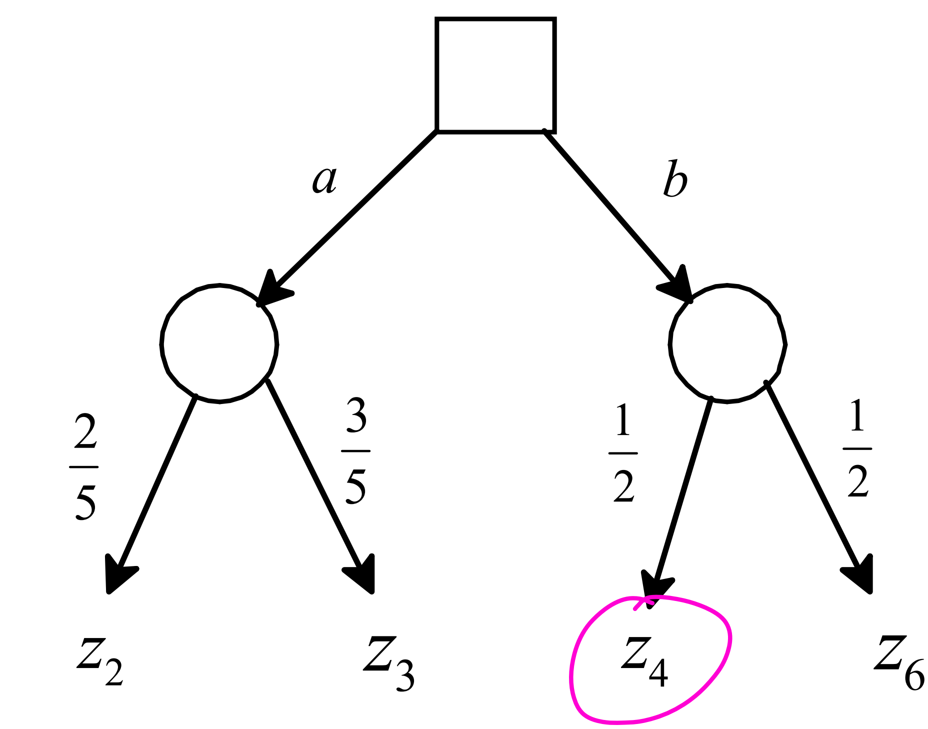
Suppose : $z_2 > z_1$

Suppose : $z_4 \succ z_5$

First question to ask yourself: how do I rank z_1 and z_2? Suppose that the answer is $z_2 \succ z_1$.



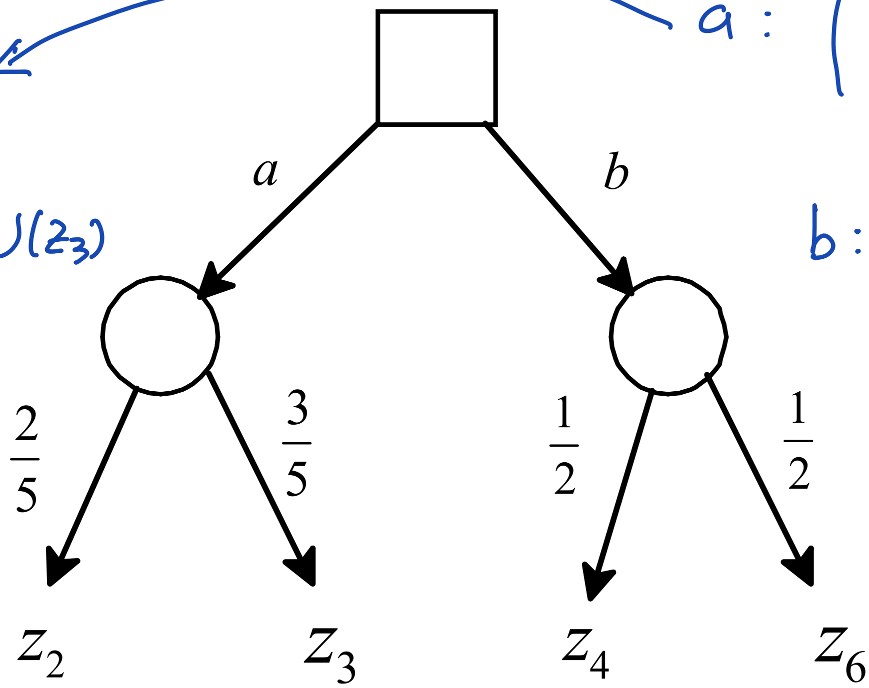
Second question to ask yourself: how do I rank z_4 and z_5? Suppose that the answer is $z_4 \succ z_5$.



$$a: \begin{pmatrix} z_2 & z_3 \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

$$b: \begin{pmatrix} z_4 & z_6 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

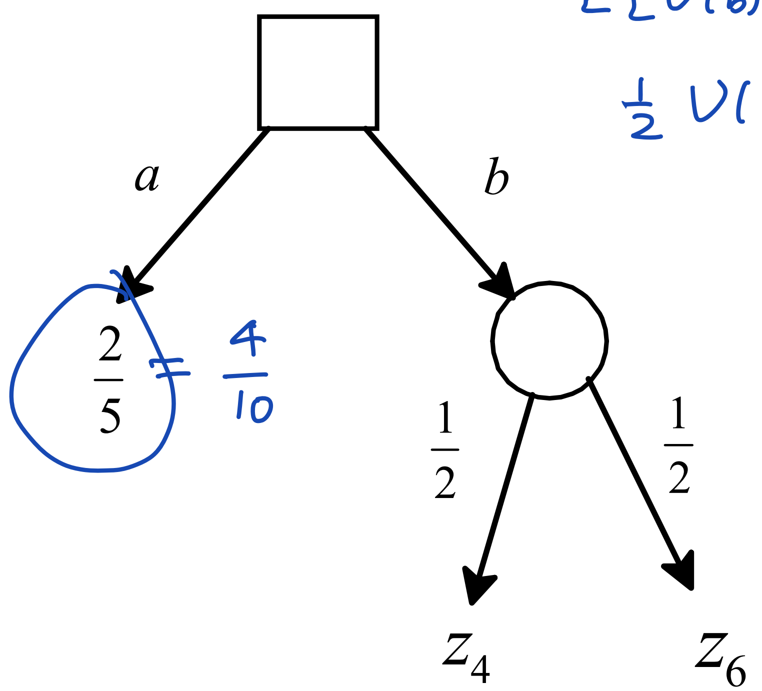
$$E[U(a)] = \frac{2}{5} U(z_2) + \frac{3}{5} U(z_3) = \frac{2}{5} \cdot 1 + \frac{3}{5} \cdot 0 = \frac{2}{5}$$



Next question: how do I rank the remaining four outcomes? Suppose:

		Utility
best	z_2	1
	z_6	$\frac{1}{2}$
	z_4	$\frac{1}{10}$
worst	z_3	0

This is sufficient to eliminate the random event on the left:



$E[U(b)] = \frac{1}{2} U(z_4) + \frac{1}{2} U(z_6) = \frac{1}{2} \frac{1}{10} + \frac{1}{2} \frac{1}{2} = \frac{3}{10}$

$E[U(a)] > E[U(b)]$

so $a \succ b$

so I should choose a

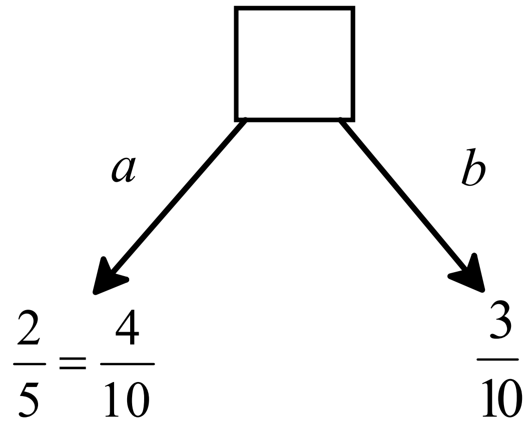
Two more questions and then you are done!

(4) What p is such that $\begin{pmatrix} z_6 \\ 1 \end{pmatrix} \sim \begin{array}{c} \begin{array}{cc} \text{best} & \text{worst} \end{array} \\ \begin{pmatrix} z_2 & z_3 \\ p & 1-p \end{pmatrix} \end{array}$? Suppose the answer is $p = \frac{1}{2}$.

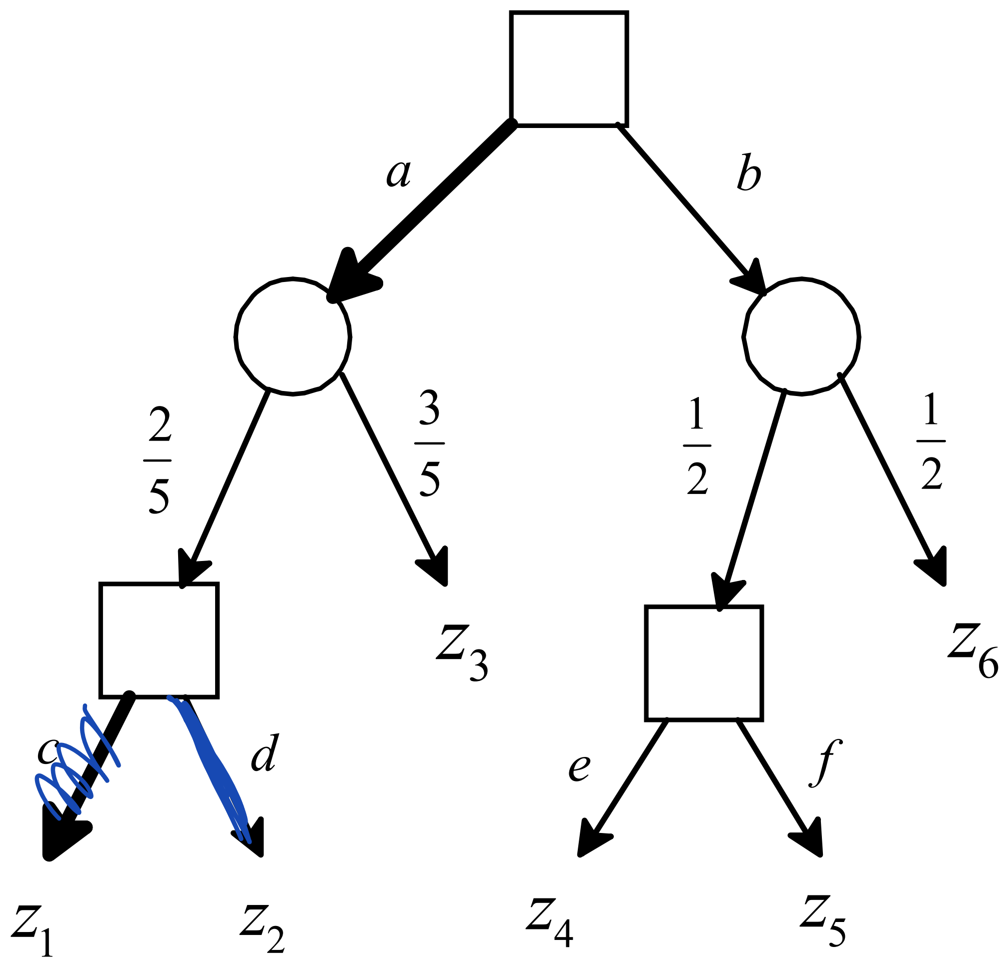
(5) What p is such that $\begin{pmatrix} z_4 \\ 1 \end{pmatrix} \sim \begin{pmatrix} z_2 & z_3 \\ p & 1-p \end{pmatrix}$? Suppose the answer is $p = \frac{1}{10}$.

Then the lottery corresponding to the random event on the right has an expected utility of

Decision: First a, then if I have to choose again between c and d choose d



Hence the optimal decision is to first take action a and then, if a second choice is required between c and d, choose d:



THE HURWICZ INDEX

utilities

	s_1	s_2	s_3
a_1	8	1	0
a_2	6	2	3
a_3	0	3	4

MaxiMin $= a_2$

$0 \le \alpha \le 1$

$$H_\alpha(a_1) = 0\alpha + 8(1-\alpha) = 8 - 8\alpha$$

$$H_\alpha(a_2) = 2\alpha + 6(1-\alpha) = 6 - 4\alpha$$

$$H_\alpha(a_3) = 0\alpha + 4(1-\alpha) = 4 - 4\alpha$$

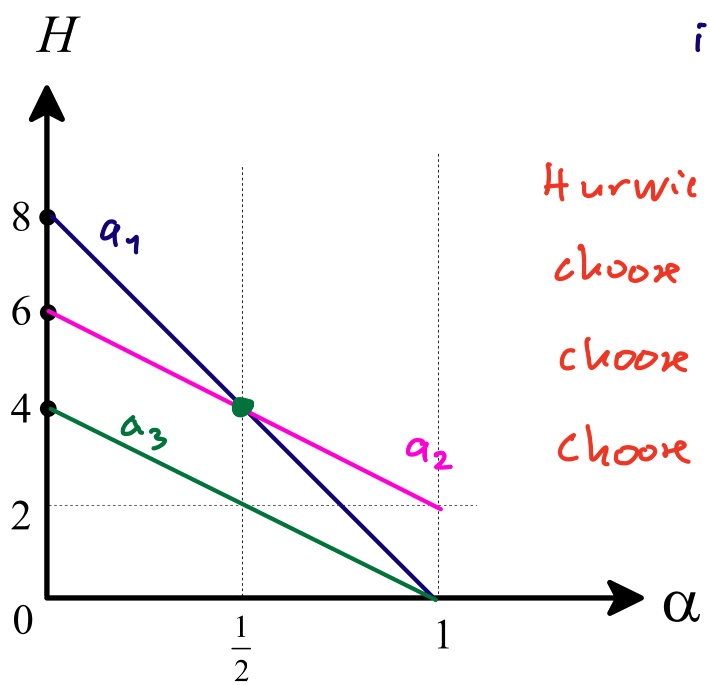
if I take $\alpha = \frac{1}{2}$

$H_{\frac{1}{2}}(a_1) = 8 - 4 = 4$

$H_{\frac{1}{2}}(a_2) = 6 - 2 = 4$

$H_{\frac{1}{2}}(a_3) = 4 - 2 = 2$

Hurwicz criterion coincides with MaxiMin if $\alpha = 1$



Hurwicz criterion:

choose a_1 if $\alpha < \frac{1}{2}$

choose a_2 if $\alpha > \frac{1}{2}$

choose either a_1 or a_2 if $\alpha = \frac{1}{2}$

Note: the Hurwicz index is invariant to allowed transformations of the utility function.