- Conditional reasoning: the frequency tree
- Conditional probability
- Updating probabilities
- Independence
- Bayes' formulas

Example from the first class

How to process information

- In the US, **1% of women of age 40 have breast cancer**.
- If a woman **has** breast cancer, the probability that she tests **positive** on a screening mammogram is **90%**.
- If she **does not have** breast cancer, the probability that she tests **negative** on a screening mammogram is **90%**.

That is, mammograms have a **90% accuracy**.

Susan is a 40-year old woman who tested **positive** *on a mammogram.*

What are the chances that she actually has breast cancer?

CONDITIONAL REASONING: the FREQUENCY approach

- Suppose there is a new variant of COVID
- The fraction p of the population is infected
- Typical symptoms: nasal congestion
- 80% of those infected have the symptoms
- 10% of those **not** infected have the symptoms

Suppose that p = 5%. You wake up with nasal congestion.

How likely is it that you are infected?

- 5% of the population are infected
- 80% of those infected have the symptoms
- 10% of those **not** infected have the symptoms

A test is now available. The probability of testing positive is independent of whether or not you have symptoms:

- If you are infected, the probability of testing positive is 80% (whether or not you have the symptoms)
- If you are **not** infected, the probability of testing positive is 10% (whether or not you have the symptoms)

Since you woke up with symptoms, you decided to get tested and the result was positive. How likely is it that you are infected?

- If you are infected, the probability of testing positive is 80% (whether or not you have the symptoms)
- If you are **not** infected, the probability of testing positive is 10% (whether or not you have the symptoms)

One more example

Base rate of a disease: percentage of the population that has the disease

Sensitivity of a test: percentage of those who have the disease that tests positive

Specificity of a test: percentage of those who **do not** have the disease that tests negative

Suppose:

Base rate = 6%

Sensitivity = 88%

Specificity = 93%

Suppose you test positive. What is the probability that you have the disease?

MORE THAN TWO CATEGORIES

Enrollment in a class

ECN	*ARE*	*PSY*	*Other*
38%	20%	12%	30%

Percentages of those who passed:

major	*ECN*	*ARE*	*PSY*	*Other*
percentage who passed	70%	60%	40%	35%

You learn that Ann passed the class. How likely is it that Ann is a PSY major?

major	*ECN*	*ARE*	*PSY*	*Other*	**Ann passed the class. How likely is it that she is a PSY major?**
enrollment	38%	20%	12%	30%	
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Probability and conditional probability

Finite set of ***states*** $S = \{s_1, s_2, ..., s_n\}$. Subsets of S are called ***events***.

Probability distribution over *S:*

$$\begin{array}{cccc} s_1 & s_2 & \ldots & s_n \\ p_1 & p_2 & \ldots & p_n \end{array}$$

Denote the probability of state *s* by $p(s)$.

Given an event $E \subseteq S$, the probability of E is:

$$P(E) = \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

Denote by $\neg E$ the complement of $E \subseteq S$.

Example

$S = \{a, b, c, d, e, f, g\}$ $A = \{a, c, d, e\}$ $B = \{a, e, g\}$

$\neg A =$ $\neg B =$

Given

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A) =$ $P(B) =$

$A \cap B =$ $P(A \cap B) =$

$A \cup B =$ $P(A \cup B) =$.

Note: for every two events E and F:

$$P(E \cup F) =$$

We denote by $P(E\,|\,F)$ the probability of E **conditional on** F and define it as:

$$P(E\,|\,F) = \frac{P(E \cap F)}{P(F)}$$

Continuing the example above where $\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{array}$ $\quad A = \{a, c, d, e\} \qquad B = \{a, e, g\}$

$P(A) = \frac{1}{14} + 0 + \frac{1}{14} + \frac{6}{14} = \frac{8}{14}$, $\quad P(B) = \frac{1}{14} + \frac{6}{14} + \frac{3}{14} = \frac{10}{14}$, $\quad P(A \cap B) = \frac{1}{14} + \frac{6}{14} = \frac{7}{14}$

$$P(A\,|\,B) =$$

$$P(B\,|\,A) = \qquad .$$

The conditional probability formula can also be applied to individual states:

$$p(s\,|\,E) = \begin{cases} & \text{if} \\ & \text{if} \end{cases}$$

We can think of $p(\bullet|E)$ as a probability distribution on the entire set S. Continuing the example above

where $S=\{a,b,c,d,e,f,g\}$, $A=\{a,c,d,e\}$ and $\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{array}$ (so that $P(A)=\frac{8}{14}$)

$$\begin{array}{lccccccc} & a & b & c & d & e & f & g \\ p(\bullet|A): & & & & & & & \end{array}$$

Shortcut to obtain the revised or updated probabilities:

Initial or prior probabilities. Note that here they all have the **same denominator**.	$\begin{pmatrix} a & b & c & d \\ \frac{15}{100} & \frac{70}{100} & \frac{5}{100} & \frac{10}{100} \end{pmatrix}$
Information or conditioning event: $F=\{a,b,d\}$	
STEP 1. Set the probability of every state which is not in F to zero:	$\begin{pmatrix} a & b & c & d \\ & & 0 & \end{pmatrix}$
STEP 2. For the other states write new fractions with the same numerators as before:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{\dots} & \frac{70}{\dots} & 0 & \frac{10}{\dots} \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 15+70+10=95. Thus the updated probabilities are:	$\begin{pmatrix} a & b & c & d \\ \frac{15}{95} & \frac{70}{95} & 0 & \frac{10}{95} \end{pmatrix}$

In the above example, where $\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{14} & \frac{2}{14} & 0 & \frac{1}{14} & \frac{6}{14} & \frac{1}{14} & \frac{3}{14} \end{array}$ and $A = \{a, c, d, e\}$, to compute $p(\bullet | A)$

Step 1: assign zero probability to states in $\neg A$:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$

Step 2: keep the same numerators for the states in A:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ & 0 & & & & 0 & 0 \end{array}$$

Step 3: since the sum of the numerators is 8, put 8 as the denominator:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{6}{8} & 0 & 0 \end{array}$$

EXAMPLE 2. Sample space or set of states: $\{a, b, c, d, e, f\}$.

Initial or prior probabilities:	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{20} & \frac{3}{10} & \frac{1}{20} & 0 & \frac{2}{5} & \frac{1}{10} \end{pmatrix}$
Information:	$F = \{a, b, d, e\}$
STEP 0. Rewrite all the probabilities **with the same denominator:**	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \\ & & & & & \end{pmatrix}$
STEP 1. Change the probability of every state which is not in F to zero:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \\ & & & & & \end{pmatrix}$
STEP 2. Write new fractions which have the same numerators as before:	$\begin{pmatrix} a & b & c & d & e & f \\ & & 0 & & & 0 \\ & & & & & \end{pmatrix}$
STEP 3. In every denominator put the sum of the numerators: 3+6+8=17.	$\begin{pmatrix} a & b & c & d & e & f \\ \frac{3}{17} & \frac{6}{17} & 0 & 0 & \frac{8}{17} & 0 \end{pmatrix}$

INDEPENDENT EVENTS.

We say that two events *A* and *B* are independent if

$$P(A \cap B) = P(A)\,P(B) \qquad (*)$$

It follows from this and the definition of conditional probability that if *A* and *B* are independent then

$$P(A \mid B) = P(A) \;\text{ and }\; P(B \mid A) = P(B) \qquad (**)$$

Alternatively, one can take one of the two equalities in (**) as definition of independence and derive both the other and (*). Thus (*) and (**) are equivalent.

Going back to our example where $S=\{a,b,c,d,e,f,g\}$, $A=\{a,c,d,e\}$, $B=\{a,e,g\}$, $A\cap B=\{a,e\}$ and

a	b	c	d	e	f	g
$\frac{1}{14}$	$\frac{2}{14}$	0	$\frac{1}{14}$	$\frac{6}{14}$	$\frac{1}{14}$	$\frac{3}{14}$

$P(A)=$, $P(B)=$, $P(A\cap B)=$ $P(A)P(B)=$

On the other hand, if $S=\{a,b,c,d,e,f,g,h,i\}$ and

a	b	c	d	e	f	g	h	i
$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{2}{9}$	0	$\frac{1}{9}$	$\frac{1}{9}$

Then $E=\{a,b,c,e\}$ and $F=\{c,d,e,g\}$ are independent. In fact, $P(E)=$, $P(F)=$,
$E\cap F=\{c,e\}$, $P(E\cap F)=$ and thus $P(E\cap F)=P(E)P(F)$.

Bayes' formula

Let E and F be two events such that $P(E) > 0$ and $P(F) > 0$. Then

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \qquad (1)$$

and

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)} \qquad (2)$$

From (2) we get that

(3)

Substituting (3) into (1) we get

Bayes' formula (4)

$\boxed{\text{Bayes' theorem}}$

Bayes’ formula says that $\boxed{P(E \mid F) = \dfrac{P(F \mid E)\, P(E)}{P(F)}}$. From set theory we have that, given any two sets *A* and *B*,

$$A = \left(A \cap B\right) \cup \left(A \cap \neg B\right) \tag{5}$$

and the two sets $\left(A \cap B\right)$ and $\left(A \cap \neg B\right)$ are disjoint. Thus $P(A) = P\left(A \cap B\right) + P\left(A \cap \neg B\right)$.

Hence in the denominator of Bayes’ formula we can replace $P(F)$ with

Then, using conditional probability we get that $P\left(F \cap E\right) =$ and

$P\left(F \cap \neg E\right) =$

Thus $P(F) =$.

Replacing this in Bayes’ formula we get

Bayes’ theorem (6)

$$\boxed{P(E\mid F)=\frac{P(F\mid E)\,P(E)}{P(F\mid E)P(E)+P(F\mid\neg E)P(\neg E)}}$$

EXAMPLE.

Enrollment in a class is as follows: 60% econ majors (E), 40% other majors ($\neg E$). In the past, 80% of the econ majors passed and 65% of the other majors passed. A student tells you that she passed the class. What is the probability that she is an econ major? Let P stand for "Pass the class".

$$P(E\mid P)=$$