UNCERTAINTY, INFORMATION and BELIEFS

uncertainty = set of possibilities

Murder suspects:

initial uncertainty

Ann Boone	Amy Bloom	Arthur Bragg	Jane Singer	Jim Shore	John Smith

information = reduction of uncertainty = shrinking of the set of possibilities

A handkerchief with initials was found at the murder scene:

Ann Boone	Amy Bloom	Arthur Bragg

initials AB

Jane Singer	Jim Shore	John Smith

initials JS

Add a witness who can tell if the murderer was a man or a woman:

Ann Boone	Amy Bloom

initials AB & woman

Arthur Bragg

initials AB & man

Jane Singer

JS & woman

Jim Shore	John Smith

JS & man

Think of information not as a particular item of information but as the list of possible items of information that one might receive

Why? Because when you contemplate seeking or purchasing information you do not know yet what specific item of information you might receive.

INFORMATION = PARTITION OF THE SET OF STATES

- Initial state of uncertainty: $S = \{s_1, s_2, ..., s_n\}$
- Information: partition of S into two or more subsets (information sets): $\mathcal{I} = \{S_1, S_2, ..., S_m\}$

Initial state of uncertainty: $S = \{s_1, s_2, ..., s_n\}$

- PERFECT INFORMATION: $\mathcal{I} = \{\{s_1\}, \{s_2\}, ..., \{s_n\}\}$ you learn what the state is

- IMPERFECT information: $\mathcal{I} = \{S_1, S_2, ..., S_m\}$ where at least one of the information sets S_i is not a singleton.

Theorem: FREE INFORMATION IS ALWAYS VALUABLE

Free information will always make you better off or leave you just as well off.

EXAMPLE

Your symptoms are compatible with three diseases: A, B and C. For each disease there is a drug that is effective only for that disease. Only two outcomes: you heal (H) or you remain sick (S). The doctor gives you the following probabilistic assessment of the three diseases:

states:	A	B	C
	70%	20%	10%
TA (treat A)	H	S	S
TB (treat B)	S	H	S
TC (treat C)	S	S	H

		U
best	H	1
worst	S	0

Suppose you have von Neumann-Morgenstern preferences.

	A	B	C
	70%	20%	10%
TA (treat A)	1	0	0
TB (treat B)	0	1	0
TC (treat C)	0	0	1

$\mathbb{E}[U(TA)] = \frac{70}{100} \cdot 1 + \frac{20}{100} 0 + \frac{10}{100} 0 = \frac{7}{10}$

$\mathbb{E}[U(TB)] = \frac{2}{10}$

$\mathbb{E}[U(TC)] = \frac{1}{10}$

thus your best course of action is and your expected utility is

$\{\{A, B\}, \{C\}\}$

test −   test +

Suppose that there is a **free** blood test to check if you have disease C, that is, the blood test gives you information that corresponds to the following partition:

A	B	C
70%	20%	10%

1. Suppose I take the test and it comes back **negative**. What would I do then?

- Update the probabilities:

$\{A, \quad B\} \quad \cancel{C}$

$P(\bullet \mid negative)$:

initially:

A	B	C
$\frac{7}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

$P(A \mid \underbrace{\{A, B\}}_{\text{not } C})$

	A	B	C
probability	$\frac{7}{9}$	$\frac{2}{9}$	0
TA (treat A)	1	0	0
TB (treat B)	0	1	0
TC (treat C)	0	0	1

$\mathbb{E}[U(TA) \mid negative] = \frac{7}{9} \cdot 1 + \frac{2}{9} 0 + 0 \cdot 0 = \frac{7}{9}$

$\mathbb{E}[U(TB) \mid negative] = \frac{2}{9} \cdot 1 = \frac{2}{9}$

$\mathbb{E}[U(TC) \mid negative] = 0 \cdot 1 = 0$

thus your best course of action is and your expected utility is initial prob.

A	B	C
$\frac{7}{10}$	$\frac{2}{10}$	$\frac{1}{10}$

$$\begin{array}{lccc} & A & B & C \\ P(\bullet \mid positive): & 0 & 0 & 1 \end{array}$$

	A	B	C
probability	0	0	1
TA (treat A)	1	0	0
TB (treat B)	0	1	0
TC (treat C)	0	0	1

$\mathbb{E}[U(TA) \mid positive] =$ $0 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 0$

$\mathbb{E}[U(TB) \mid positive] =$ 0

$\mathbb{E}[U(TC) \mid positive] =$ $0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1 = 1$

thus your best course of action is and your expected utility is

Positive test: take action TA with expected utility

negative test: take action TC with expected utility

- Based on the initial assessment: $\begin{array}{ccc} A & B & C \\ 70\% & 20\% & 10\% \end{array}$, how likely is it that you will get a negative result and how likely is it that you will get a positive result?

$$P(negative) = \frac{7}{10} + \frac{2}{10} = \frac{9}{10}$$

$$P(positive) = P(C) = \frac{1}{10}$$

- Thus your expected utility if you take the blood test is:

+	−
Treat C	Treat A
get 1	get $\frac{7}{9}$
$\frac{1}{10}$	$\frac{9}{10}$

Exp. utility of taking blood test and reacting optimally to the result is $\frac{1}{10} 1 + \frac{\cancel{9}}{10} \frac{7}{\cancel{9}} = \frac{8}{10}$

WHAT IF INFORMATION IS COSTLY?

How much should one be prepared to pay for information?

CASE 1: monetary outcomes and perfect information

CASE 1.1: risk neutrality

CASE 1.2: risk aversion

CASE 2: monetary outcomes and imperfect information

CASE 2.1: risk neutrality

CASE 2.2: risk aversion

CASE 3: general outcomes and perfect information

CASE 4: general outcomes and imperfect information

How much should one be prepared to pay for information?

CASE 1: monetary outcomes and perfect information

$U(\$m) = m$

CASE 1.1: $\boxed{\text{risk neutrality}}$

initial	probability	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
	state $\rightarrow$	s_1	s_2	s_3
	act $\downarrow$			
Amounts are **changes** in her wealth.	a	\$4	\$36	\$244
	b	\$8	\$201	\$18
	c	\$124	\$12	\$24

$$\mathbb{E}[a] = \frac{1}{2}4 + \frac{1}{3}36 + \frac{1}{6}244 = 54.67$$

$$\mathbb{E}[b] = \frac{1}{2}8 + \frac{1}{3}201 + \frac{1}{6}18 = \boxed{74} \leftarrow$$

$$\mathbb{E}[c] = \frac{1}{2}124 + \frac{1}{3}12 + \frac{1}{6}24 = 70$$

Analysis in terms of **total wealth**. Let W be her initial wealth, then

probability	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
state $\rightarrow$	s_1	s_2	s_3
act $\downarrow$			
a	$\$(4+W)$	$\$(36+W)$	$\$(244+W)$
b	$\$(8+W)$	$\$(201+W)$	$\$(18+W)$
c	$\$(124+W)$	$\$(12+W)$	$\$(24+W)$

$\mathbb{E}[a] = \frac{1}{2}(4+W) + \frac{1}{3}(36+W) + \frac{1}{6}(244+W) = W + 54.67$

$\mathbb{E}[b] = \frac{1}{2}(8+W) + \frac{1}{3}(201+W) + \frac{1}{6}(18+W) = W + 74$ $\leftarrow$

$\mathbb{E}[c] = W + 70$

When a person is risk neutral there it does not matter whether we carry out the analysis in term of changes in wealth or in terms of total wealth.

Suppose now that Ann is offered to be given **perfect information** at price p, that is, she pays \$$p$ and then she will be told what the state is. Note that she must pay **before** she gets the information.

• If I am told that the state is s_1 then I will choose c and get 124	probability	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
• If I am told that the state is s_2 then I will choose b and get 201	state →	s_1	s_2	s_3
• If I am told that the state is s_3 then I will choose a and get 244	act ↓			
	a	\$4	\$36	\$244
	b	\$8	\$201	\$18
	c	\$124	\$12	\$24

Thus expected change in wealth is:

Yes to info: $\begin{pmatrix} \$124-p & \$201-p & \$244-p \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$

$$E[\text{info}] = \frac{1}{2}(124-p) + \frac{1}{3}(201-p) + \frac{1}{6}(244-p)$$

$$= \frac{1}{2}124 + \frac{1}{3}201 + \frac{1}{6}244 - p\underbrace{\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)}_{1} = \boxed{169.67-p}$$

instead of 74

With no information the expected change in wealth is $\mathbb{E}[b] = 74$. Thus, if $169.67 - p \geq 74$, that is, if $p \leq \$95.67$ it is worth buying the information. The **value of perfect information** for a risk-neutral person is \$(169.67–74) = \$95.67.