- Group decision making
- Aggregation of preferences
- Social preference function
  Arrow's axioms
  Arrow's theorem

How to aggregate the preferences of a group of individuals

X set of alternatives that society has to choose from.

$S = \{1, 2, \ldots, n\}$ set of individuals

For every $i \in N$, $\quad \succsim_i \quad$ i's preference relation over X

- complete:
- transitive:

$x \succsim_i y$

$x \succ_i y$

$x \sim_i y$

Issue: how to aggregate the preferences of the individuals into a single ranking that can be viewed as "society's ranking".

$\succsim$ (without subscript) society's preference relation over X

$x \succsim y$

$x \succ y$

$x \sim y$

$$\text{function } f: \quad (\succsim_1, \succsim_2, \ldots, \succsim_n) \mapsto \succsim$$

Majority rule

Let $X = \{A, B, C\}$ and $S = \{1, 2, 3\}$ and

	1's ranking	2's ranking	3's ranking
best	A	C	B
	B	A	C
worst	C	B	A

$A \succ B$

$B \succ C$

$C \succ A$

Problem 1: $\succ$ not transitive

Problem 2: can be manipulated. Suppose Individual 2 sets the agenda ...

In his 1951 Ph.D. thesis Kenneth Arrow asked: what is a good *social preference function* (or aggregation rule)?

$$\text{function } f\colon \qquad (\succsim_1, \succsim_2, \ldots, \succsim_n) \mapsto \ \succsim$$

There are MANY possible social preference functions

E.g. let $X = \{A, B\}$ and $S = \{1, 2\}$

possible rankings of Individual 1:

possible rankings of Individual 2:

Thus 9 possible profiles of preferences:

		Individual 2's ranking		
		$A \succ_2 B$	$A \sim_2 B$	$B \succ_2 A$
Individual	$A \succ_1 B$			
1's	$A \sim_1 B$			
ranking	$B \succ_1 A$			

One of them is:
if 1 and 2 agree that x is better than y then $x \succ y$, otherwise $x \sim y$

		Individual 2's ranking		
		$A \succ_2 B$	$A \sim_2 B$	$B \succ_2 A$
Individual	$A \succ_1 B$			
1's	$A \sim_1 B$			
ranking	$B \succ_1 A$			

Second example: $X = \{A, B, C\}$ and $S = \{1, 2, 3\}$

and only strict rankings can be reported:

$$A \succ B \succ C$$

$$A \succ C \succ B$$

$$B \succ A \succ C$$

$$B \succ C \succ A$$

$$C \succ A \succ B$$

$$C \succ B \succ A$$

What is a good or reasonable SPF?

Establish some principles or *desiderata* or axioms

Example: $X = \{A, B\}$, $S = \{1, 2\}$
and only strict rankings: $A \succ B$ or $B \succ A$

Then 4 possible profiles and 16 possible functions:

profile $\rightarrow$ SPF $\downarrow$	$A \succ_1 B$ $A \succ_2 B$	$A \succ_1 B$ $B \succ_2 A$	$B \succ_1 A$ $A \succ_2 B$	$B \succ_1 A$ $B \succ_2 A$
SPF - 1	$A \succ B$	$A \succ B$	$A \succ B$	$A \succ B$
SPF - 2	$A \succ B$	$A \succ B$	$A \succ B$	$B \succ A$
SPF - 3	$A \succ B$	$A \succ B$	$B \succ A$	$A \succ B$
SPF - 4	$A \succ B$	$A \succ B$	$B \succ A$	$B \succ A$
SPF - 5	$A \succ B$	$B \succ A$	$A \succ B$	$A \succ B$
SPF - 6	$A \succ B$	$B \succ A$	$A \succ B$	$B \succ A$
SPF - 7	$A \succ B$	$B \succ A$	$B \succ A$	$A \succ B$
SPF - 8	$A \succ B$	$B \succ A$	$B \succ A$	$B \succ A$
SPF - 9	$B \succ A$	$A \succ B$	$A \succ B$	$A \succ B$
SPF - 10	$B \succ A$	$A \succ B$	$A \succ B$	$B \succ A$
SPF - 11	$B \succ A$	$A \succ B$	$B \succ A$	$A \succ B$
SPF - 12	$B \succ A$	$A \succ B$	$B \succ A$	$B \succ A$
SPF - 13	$B \succ A$	$B \succ A$	$A \succ B$	$A \succ B$
SPF - 14	$B \succ A$	$B \succ A$	$A \succ B$	$B \succ A$
SPF - 15	$B \succ A$	$B \succ A$	$B \succ A$	$A \succ B$
SPF - 16	$B \succ A$	$B \succ A$	$B \succ A$	$B \succ A$

UNANIMITY

By appealing to Unanimity we can discard all except:

profile $\rightarrow$ SPF $\downarrow$	$A{\succ_1}B$ $A{\succ_2}B$	$A{\succ_1}B$ $B{\succ_2}A$	$B{\succ_1}A$ $A{\succ_2}B$	$B{\succ_1}A$ $B{\succ_2}A$
SPF - 2	$A \succ B$	$A \succ B$	$A \succ B$	$B \succ A$
SPF - 4	$A \succ B$	$A \succ B$	$B \succ A$	$B \succ A$
SPF - 6	$A \succ B$	$B \succ A$	$A \succ B$	$B \succ A$
SPF - 8	$A \succ B$	$B \succ A$	$B \succ A$	$B \succ A$

NON-DICTATORHIP

$$\downarrow$$

profile $\rightarrow$ SPF $\downarrow$	$A{\succ_1}B$ $A{\succ_2}B$	$A{\succ_1}B$ $B{\succ_2}A$	$B{\succ_1}A$ $A{\succ_2}B$	$B{\succ_1}A$ $B{\succ_2}A$
SPF - 2	$A \succ B$	$A \succ B$	$A \succ B$	$B \succ A$
SPF - 8	$A \succ B$	$B \succ A$	$B \succ A$	$B \succ A$

Arrow's axioms

- **Axiom 1: Unrestricted Domain or Freedom of Expression**

- **Axiom 2: Rationality**

- **Axiom 3: Unanimity or Pareto Principle**

	1's ranking	2's ranking	3's ranking
best	A	C	B
	B	A	C
worst	C	B	A

	1's ranking	2's ranking	3's ranking
best	A	C	A, B
	B	A	
worst	C	B	C

	1's ranking	2's ranking	3's ranking
best	A	C	A
	B	A	C
worst	C	B	B

- **Axiom 4: Non-dictatorship**

• Axiom 5: Independence of Irrelevant Alternatives

(1)

	individual 1	individual 2
best	A	A, B
	B	
worst	C	C

suppose that $\mapsto$ $A \succ B$

	1	2
best	A	A, B, C
	B	
worst	C	

	1	2
best	A	C
	B	
worst	C	A, B

	1	2
best	C	A, B
	A	
worst	B	C

	1	2
best	A, C	A, B
worst	B	C

	1	2
best	A	A, B
	C	
worst	B	C

	1	2
best	A	A, B
worst	B, C	C

	1	2
best	C	A,B,C
	A	
worst	B	

	1	2
best	A, C	A,B,C
worst	B	

	1	2
best	A	A,B,C
	C	
worst	B	

	1	2
best	A	A,B,C
worst	B, C	

	1	2
best	C	C
	A	
worst	B	A,B

	1	2
best	A, C	C
worst	B	A,B

	1	2
best	A	C
	C	
worst	B	A,B

	1	2
best	A	C
worst	B, C	A,B

If there are only two alternatives the Independence of Irrelevant Alternatives axiom is trivially satisfied.

Remark 1. *If there are only two alternatives (and any number of individuals) then the method of majority voting satisfies all of Arrow's axioms.*

Arrow's Impossibility Theorem

If the number of alternatives is at least three,

there is no social preference function that satisfies the five axioms.

Arrow's axioms

Unrestricted Domain
or Freedom of Expression

Rationality **R**

Unanimity or Pareto **U**

Non-Dictatorship **ND**

Independence of Irrelevant Alternatives **IIA**

Majority Rule with 2 alternatives

Plurality Rule with 2 alternatives

Majority Rule with more than 2 alternatives

Plurality Rule with more than 2 alternatives

n voters

majority =

if n is even: number or individuals $\geq \frac{n}{2}+1$

if n is odd: number or individuals $\geq \frac{n+1}{2}$

Majority rule: **if a majority prefers x to y then society prefers x to y**
if a majority prefers y to x then society prefers y to x
otherwise society is indifferent between x and y

Plurality rule: **if the number of individuals who prefer x to y is grater than the number of individuals who prefer y to x then society prefers x to y**

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	1	2
best	A	C
	B	
worst	C	A, B

	1	2
best	C	A, B
	A	
worst	B	C

	1	2
best	A, C	A, B
worst	B	C

	1	2
best	A	A, B
	C	
worst	B	C

	1	2
best	A	A, B
worst	B, C	C

	1	2
best	C	A,B,C
	A	
worst	B	

	1	2
best	A, C	A,B,C
worst	B	

	1	2
best	A	A,B,C
	C	
worst	B	

	1	2
best	A	A,B,C
worst	B, C	

	1	2
best	C	C
	A	
worst	B	A,B

	1	2
best	A, C	C
worst	B	A,B

	1	2
best	A	C
	C	
worst	B	A,B

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