- Arrow's approach to preference aggregation
- Borda count as a SPF
- Kemeny-Young SPF
- Social choice functions.
  Gibbard-Satterthwaite theorem

Arrow's axioms

- **Axiom 1: Unrestricted Domain or Freedom of Expression**

- **Axiom 2: Rationality**

- **Axiom 3: Unanimity or Pareto Principle**

	1's ranking	2's ranking	3's ranking
best	A	C	B
	B	A	C
worst	C	B	A

	1's ranking	2's ranking	3's ranking
best	A	C	A, B
	B	A	
worst	C	B	C

	1's ranking	2's ranking	3's ranking
best	A	C	A
	B	A	C
worst	C	B	B

- **Axiom 4: Non-dictatorship**

• Axiom 5: Independence of Irrelevant Alternatives

(1)

	individual 1	individual 2
best	A	A, B
	B	
worst	C	C

suppose that $\mapsto$ $A \succ B$

	1	2
best	A	A, B, C
	B	
worst	C	

	1	2
best	A	C
	B	
worst	C	A, B

	1	2
best	C	A, B
	A	
worst	B	C

	1	2
best	A, C	A, B
worst	B	C

	1	2
best	A	A, B
	C	
worst	B	C

	1	2
best	A	A, B
worst	B, C	C

	1	2
best	C	A,B,C
	A	
worst	B	

	1	2
best	A, C	A,B,C
worst	B	

	1	2
best	A	A,B,C
	C	
worst	B	

	1	2
best	A	A,B,C
worst	B, C	

	1	2
best	C	C
	A	
worst	B	A,B

	1	2
best	A, C	C
worst	B	A,B

	1	2
best	A	C
	C	
worst	B	A,B

	1	2
best	A	C
worst	B, C	A,B

If there are only two alternatives the Independence of Irrelevant Alternatives axiom is trivially satisfied.

Remark 1. *If there are only two alternatives (and any number of individuals) then the method of majority voting satisfies all of Arrow's axioms.*

Arrow's Impossibility Theorem

If the number of alternatives is at least three,

there is no social preference function that satisfies the five axioms.

Arrow's axioms

Unrestricted Domain
or Freedom of Expression

Rationality **R**

Unanimity or Pareto **U**

Non-Dictatorship **ND**

Independence of Irrelevant Alternatives **IIA**

Majority Rule with 2 alternatives

Plurality Rule with 2 alternatives

Majority Rule with more than 2 alternatives

Plurality Rule with more than 2 alternatives

n voters

majority =

- **if n is even:** number or individuals $\geq \frac{n}{2}+1$
- **if n is odd:** number or individuals $\geq \frac{n+1}{2}$

Majority rule: **if a majority prefers x to y then society prefers x to y**
if a majority prefers y to x then society prefers y to x
otherwise society is indifferent between x and y

Plurality rule: **if the number of individuals who prefer x to y is grater than the number of individuals who prefer y to x then society prefers x to y**

if the number of individuals who prefer y to x is grater than the number of individuals who prefer x to y then society prefers y to x

otherwise society is indifferent between x and y

Arrow's Impossibility Theorem

If the number of alternatives is at least three,
there is no social preference function that satisfies the five axioms.

===

Borda count

- n alternatives, m voters
- each voter submits a *strict* ranking of the alternatives
- for each voter the top alternative receives n points,
  the second $(n-1)$ points, etc.
- for each alternative we take the sum of each individual score
- alternatives are ranked according to the computed score

	Voter 1	Voter 2	Voter 3	score
best	a	b	c	
	b	a	b	
worst	c	c	a	

Social ranking:

Which of Arrow's axioms does the Borda count satisfy?

1. Unrestricted domain?

2. Rationality?

3. Unanimity?

4. Non-dictatorship?

5. Independence of irrelevant alternatives?

Voter:	1	2	3	4	5	6	7
best	x	a	b	x	a	b	x
	c	x	a	c	x	a	c
	b	c	x	b	c	x	b
worst	a	b	c	a	b	c	a

Social ranking:

Voter:	1	2	3	4	5	6	7
best	c	a	b	c	a	b	c
	b	c	a	b	c	a	b
	a	b	c	a	b	c	a
worst	x	x	x	x	x	x	x

Social ranking:

Kemeny-Young method

For each pair of alternatives, x and y, count:
(1) the number of individuals for whom $x \succ y$; denote it by $\#(x \succ y)$,
(2) the number of individuals for whom $x \sim y$; denote it by $\#(x \sim y)$,
(3) the number of individuals from whom $y \succ x$ denote it by $\#(y \succ x)$.

Next go through all the complete and transitive rankings of X and for each compute a total score by adding up the scores of each pairwise ranking.

Example: $X = \{A, B, C\}$, $S = \{1, 2, 3, 4, 5\}$

	voter 1	voter 2	voter 3	voter 4	voter 5
best	A	B	B	C	B
	B	C	C	A	A
worst	C	A	A	B	C

Ranking	Score
$A \succ B \succ C$	
$A \succ C \succ B$	
$B \succ A \succ C$	
$B \succ C \succ A$	
$C \succ A \succ B$	
$C \succ B \succ A$	

Social ranking:

Which of Arrow's axioms does Kemeny-Young satisfy?

1. Unrestricted domain?

2. Rationality?

3. Unanimity?

requires some proof: see textbook

4. Non-dictatorship?

5. Independence of irrelevant alternatives?

	1	2	3	4	5	6	7
best	A	A	A	B	B	C	C
	B	B	B	C	C	A	A
worst	C	C	C	A	A	B	B

Ranking	Score
$A \succ B \succ C$	
$A \succ C \succ B$	
$B \succ A \succ C$	
$B \succ C \succ A$	
$C \succ A \succ B$	
$C \succ B \succ A$	

Social ranking:

	1	2	3	**4**	**5**	6	7
best	A	A	A	**C**	**C**	C	C
	B	B	B	**B**	**B**	A	A
worst	C	C	C	**A**	**A**	B	B

Ranking	Score
$A \succ B \succ C$	
$A \succ C \succ B$	
$B \succ A \succ C$	
$B \succ C \succ A$	
$C \succ A \succ B$	
$C \succ B \succ A$	

Social ranking:

Set of alternatives among which society has to choose:

$$X = \{x_1, x_2, ..., x_m\}$$

Set of individuals (members of society or voters:

$$S = \{1, 2, ..., n\}$$

Each voter i has a complete and transitive ranking $\succsim_i$ of X

Social preference function: $\underbrace{(\succsim_1, \succsim_2, ..., \succsim_n)}_{input} \mapsto \underbrace{\succsim}_{output}$

Social choice function: $\underbrace{(\succsim_1, \succsim_2, ..., \succsim_n)}_{input} \mapsto \underbrace{x \in X}_{output}$

Social Choice Function

Two voters, two alternatives:

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

First requirement: UNANIMITY. A good SCF should be such that if both voters put the same alternative at the top of their reported ranking then that alternative should be chosen.

(1)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	a	a

(2)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	a	b

(3)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	b	a

(4)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	a	a

(5)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	a	a

(6)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	b	b

(7)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	a	b

(8)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	b	a

(9)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	a	b

(10)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	b	a

(11)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	a	a

(12)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	b	b

(13)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	b	b

(14)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	a	b

(15)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	b	a

(16)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	b	b

By imposing unanimity we are left with:

(2)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	a
$b \succ a$	a	b

(6)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	a
$b \succ a$	b	b

(7)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	b
$b \succ a$	a	b

(12)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	b
$b \succ a$	b	b

Second requirement: NON-DICTATORSHIP. A good SCF should be such that there is no individual whose top alternative is always chosen, that is, if he reports $a \succ b$ then a is chosen and if he reports $b \succ a$ then b is chosen.

By imposing **Unanimity** and **Non-Dictatorship** we are left with

(2)

		2: $a \succ b$	2: $b \succ a$
1	$a \succ b$	a	a
1	$b \succ a$	a	b

(a is chosen, except when both rank b at the top)

(12)

		2: $a \succ b$	2: $b \succ a$
1	$a \succ b$	a	b
1	$b \succ a$	b	b

(b is chosen, except when both rank a at the top)

Third requirement: NON-MANIPULABILITY. A good SCF should be such that there is no situation where an individual can gain by reporting a false ranking (that is, a ranking which is not her true ranking). Both of the remaining two rankings satisfy this requirement.

Now two voters but three alternatives: *a*, *b*, *c*.

2's ranking ➔ / 1's ranking ➔	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc						
acb						
bac						
bca						
cab						
cba						

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Does it satisfy **Unanimity**?

2's ranking ➔ *abc* *acb* *bac* *bca* *cab* *cba*

1's ranking ⬇

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Does it satisfy **Non-Dictatorship?**

Satisfies **Unanimity** and **Non-Dictatorship**, but **fails Non-Manipulability:**

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Gibbard-Satterthwaite theorem:

MANIPULABILITY of the BORDA count

Four alternatives: a, b, c and d

Three voters

	1	2	3	score
best				
worst				

a:

b:

c:

d:

1 changes to:

	1	2	3	score
best				
worst				

a:

b:

c:

d:

MANIPULABILITY of the KEMENY-YOUNG method

The Kemeny-Young procedure is a social **preference** function. However, just like the Borda rule, it can be converted to a social **choice** function by picking the top-ranked alternative in the selected ranking.

Consider the following tie-breaking rule: if two or more rankings are selected by the Kemeny-Young procedure, then pick the one whose top alternative comes first in alphabetical order.

	voter 1	voter 2	voter 3
best	A	C	B
	B	A	C
worst	C	B	A

Ranking	Kemeny-Young score
$A \succ B \succ C$	$\#(A \succ B) + \#(A \succ C) + \#(B \succ C) =$
$A \succ C \succ B$	$\#(A \succ C) + \#(A \succ B) + \#(C \succ B) =$
$B \succ A \succ C$	$\#(B \succ A) + \#(B \succ C) + \#(A \succ C) =$
$B \succ C \succ A$	$\#(B \succ C) + \#(B \succ A) + \#(C \succ A) =$
$C \succ A \succ B$	$\#(C \succ A) + \#(C \succ B) + \#(A \succ B) =$
$C \succ B \succ A$	$\#(C \succ B) + \#(C \succ A) + \#(B \succ A) =$

If Voter 3 (for whom A is the worst alternative) lies and reports $C \succ B \succ A$ instead of the true $B \succ C \succ A$

	voter 1	voter 2	voter 3
best	A	C	C
	B	A	B
worst	C	B	A

Ranking	Kemeny-Young score
$A \succ B \succ C$	$\#(A \succ B) + \#(A \succ C) + \#(B \succ C) =$
$A \succ C \succ B$	$\#(A \succ C) + \#(A \succ B) + \#(C \succ B) =$
$B \succ A \succ C$	$\#(B \succ A) + \#(B \succ C) + \#(A \succ C) =$
$B \succ C \succ A$	$\#(B \succ C) + \#(B \succ A) + \#(C \succ A) =$
$C \succ A \succ B$	$\#(C \succ A) + \#(C \succ B) + \#(A \succ B) =$
$C \succ B \succ A$	$\#(C \succ B) + \#(C \succ A) + \#(B \succ A) =$