Set of alternatives among which society has to choose:

$$X = \{x_1, x_2, ..., x_m\}$$

Set of individuals (members of society or voters:

$$S = \{1, 2, ..., n\}$$

Each voter i has a complete and transitive ranking $\succsim_i$ of X

Social preference function: $\underbrace{(\succsim_1, \succsim_2, ..., \succsim_n)}_{input} \mapsto \underbrace{\succsim}_{output}$

Social choice function: $\underbrace{(\succsim_1, \succsim_2, ..., \succsim_n)}_{input} \mapsto \underbrace{x \in X}_{output}$

Social Choice Function

Two voters, two alternatives:

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	$\square$	$\square$
	$b \succ a$	$\square$	$\square$

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

		2	
		$a \succ b$	$b \succ a$
1	$a \succ b$	□	□
	$b \succ a$	□	□

First requirement: UNANIMITY. A good SCF should be such that if both voters put the same alternative at the top of their reported ranking then that alternative should be chosen.

(1)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	a	a

(2)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	a	b

(3)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	b	a

(4)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	a	a

(5)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	a	a

(6)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	a
1	$b \succ a$	b	b

(7)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	a	b

(8)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	b	a

(9)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	a	b

(10)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	b	a

(11)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	a	a

(12)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	a	b
1	$b \succ a$	b	b

(13)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	a
1	$b \succ a$	b	b

(14)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	a	b

(15)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	b	a

(16)

		2	
		$a \succ b$	$b \succ a$
	$a \succ b$	b	b
1	$b \succ a$	b	b

By imposing unanimity we are left with:

(2)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	a
$b \succ a$	a	b

(6)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	a
$b \succ a$	b	b

(7)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	b
$b \succ a$	a	b

(12)

1 \ 2	$a \succ b$	$b \succ a$
$a \succ b$	a	b
$b \succ a$	b	b

Second requirement: NON-DICTATORSHIP. A good SCF should be such that there is no individual whose top alternative is always chosen, that is, if he reports $a \succ b$ then a is chosen and if he reports $b \succ a$ then b is chosen.

By imposing **Unanimity** and **Non-Dictatorship** we are left with

(2)

		2: $a \succ b$	2: $b \succ a$
1	$a \succ b$	a	a
1	$b \succ a$	a	b

(a is chosen, except when both rank b at the top)

(12)

		2: $a \succ b$	2: $b \succ a$
1	$a \succ b$	a	b
1	$b \succ a$	b	b

(b is chosen, except when both rank a at the top)

Third requirement: NON-MANIPULABILITY. A good SCF should be such that there is no situation where an individual can gain by reporting a false ranking (that is, a ranking which is not her true ranking). Both of the remaining two rankings satisfy this requirement.

Now two voters but three alternatives: *a*, *b*, *c*.

2's ranking ➔ / 1's ranking ➔	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc						
acb						
bac						
bca						
cab						
cba						

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Does it satisfy **Unanimity**?

2's ranking ➔ *abc* *acb* *bac* *bca* *cab* *cba*

1's ranking ⬇

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Does it satisfy **Non-Dictatorship?**

Satisfies **Unanimity** and **Non-Dictatorship**, but **fails Non-Manipulability:**

1's ranking ↓ / 2's ranking →	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*c*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*c*
bac	*b*	*a*	*b*	*b*	*b*	*c*
bca	*a*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*c*	*c*	*b*	*c*	*c*
cba	*c*	*a*	*b*	*c*	*c*	*c*

Gibbard-Satterthwaite theorem:

MANIPULABILITY of the BORDA count

Four alternatives: a, b, c and d

Three voters

	1	2	3	score
best				
worst				

a:
b:
c:
d:

1 changes to:

	1	2	3	score
best				
worst				

a:
b:
c:
d:

MANIPULABILITY of the KEMENY-YOUNG method

The Kemeny-Young procedure is a social **preference** function. However, just like the Borda rule, it can be converted to a social **choice** function by picking the top-ranked alternative in the selected ranking.

Consider the following tie-breaking rule: if two or more rankings are selected by the Kemeny-Young procedure, then pick the one whose top alternative comes first in alphabetical order.

	voter 1	voter 2	voter 3
best	A	C	B
	B	A	C
worst	C	B	A

Ranking	Kemeny-Young score
$A \succ B \succ C$	$\#(A \succ B) + \#(A \succ C) + \#(B \succ C) =$
$A \succ C \succ B$	$\#(A \succ C) + \#(A \succ B) + \#(C \succ B) =$
$B \succ A \succ C$	$\#(B \succ A) + \#(B \succ C) + \#(A \succ C) =$
$B \succ C \succ A$	$\#(B \succ C) + \#(B \succ A) + \#(C \succ A) =$
$C \succ A \succ B$	$\#(C \succ A) + \#(C \succ B) + \#(A \succ B) =$
$C \succ B \succ A$	$\#(C \succ B) + \#(C \succ A) + \#(B \succ A) =$

If Voter 3 (for whom A is the worst alternative) lies and reports $C \succ B \succ A$ instead of the true $B \succ C \succ A$

	voter 1	voter 2	voter 3
best	A	C	C
	B	A	B
worst	C	B	A

Ranking	Kemeny-Young score
$A \succ B \succ C$	$\#(A \succ B) + \#(A \succ C) + \#(B \succ C) =$
$A \succ C \succ B$	$\#(A \succ C) + \#(A \succ B) + \#(C \succ B) =$
$B \succ A \succ C$	$\#(B \succ A) + \#(B \succ C) + \#(A \succ C) =$
$B \succ C \succ A$	$\#(B \succ C) + \#(B \succ A) + \#(C \succ A) =$
$C \succ A \succ B$	$\#(C \succ A) + \#(C \succ B) + \#(A \succ B) =$
$C \succ B \succ A$	$\#(C \succ B) + \#(C \succ A) + \#(B \succ A) =$