~~~~~~

HOMEWORK 1 : ANSWERS

1. **(a)** Let us focus on Player 1. First of all, for each hand there are three choices: *P, R* and *S.* Thus a total of $3 \times 3 = 9$ possibilities for the first-stage choice. There are 9 possible configurations of the two hands of Player 2 and, for each such configuration, Player 1 has two choices: *L* or *R*, thus a total of $2^9 = 512$ possibilities. Hence the number of possible strategies for Player 1 is $9 \times 2^9 = 4{,}608$.

(b) There are no pure-strategy Nash equilibria.

(c) **(c.1)** Consider the case where Player 1 chooses (1) the same shape for both hands, say, (*P*,*P*), and then (2) to remove the left hand, whatever hand configuration Player 2 displays. Then, clearly, the final outcome would be the same with any other second-stage hand-removing strategy of Player 1. Since there are 9 possible configurations of Player 2's hands at the end of stage 1, there are $2^9 = 512$ second-stage strategies of Player 1 that can be combined with choosing (*P*,*P*) in the first stage; all the resulting strategies are equivalent.

(c.2) The cardinality of the largest set of equivalent strategies is 512. One might think that there are more; for example, to the equivalent strategies described in part (c.1), one might think that one can add the following: choose (*P*,*R*) in the first stage and then always remove the right hand. However, the two strategies

$s_1 = \big((P,P),\text{ always remove right hand}\big)$ and

$\hat{s}_1 = \big((P,R),\text{ always remove right hand}\big)$ are **not** equivalent.

To show this, it is enough to find a strategy s_2 of Player 2 against which s_1 and $\hat{s}_1$ yield different outcomes. Let

$s_2 = \big((S,R),\ f_2(P,P) = \text{ remove right hand and otherwise remove left hand}\big)$ (in particular, $f_2(P,R) = \text{ remove left hand}$). Then the outcome of (s_1, s_2) is (*P*,*S*) so that Player 2 wins, while the outcome of $(\hat{s}_1, s_2)$ is (*P*,*R*) so that Player 1 wins.

(The reason why this does not happen in the set of 512 strategies described above is that in all of them Player 1's first-stage choice is constant, namely (*P*,*P*) and thus Player 2's second-stage choice is unique.)

(d) No, because any strategy involving (*S*,*S*) in the first stage is worse than $\hat{s}_1$ against the strategy of Player 2 of choosing (*R*,*R*) in the first stage and then always removing the left hand.

(e) No, Player 1 does not have any weakly dominated strategies. For example, let us show that the strategy $s_1 = \big((P,P),\text{ always remove right hand}\big)$ is not weakly dominated by another strategy. The proof is long and tedious, so let us prove the more modest claim that s_1 is not dominated by a strategy of the form $\hat{s}_1 = \big((P,R),\ f_1(\bullet)\big)$, that is, there is a strategy s_2 of Player 2 against which s_1 yields a better outcome for Player 1 than $\hat{s}_1$. Fix an arbitrary such strategy $\hat{s}_1$. Two cases are possible:

Case 1: $f_1(P,R) = \text{ remove the left hand}$. Let

$s_2 = \big((P,R),\ f_2(P,P) = \text{ remove left hand and otherwise remove right hand}\big)$ (in particular,
~~~~~~

$f_2(P,R) =$ remove right hand) . Then the outcome of (s_1, s_2) is (*P*,*R*) so that Player 1 wins, while the outcome of $(\hat{s}_1, s_2)$ is (*R*,*P*) so that Player 2 wins.

Case 2: $f_1(P,R) =$ remove the right hand . Let

$\tilde{s}_2 = \big((P,R),\ f_2(P,P) =$ remove leftt hand and otherwise remove right hand$\big)$ (in particular, $f_2(P,R) =$ remove right hand) . Then the outcome of $(s_1, \tilde{s}_2)$ is (*P*,*R*) so Player 1 wins, while the outcome of $(\hat{s}_1, \tilde{s}_2)$ is (*P*,*P*) so tit is a draw.

2. **(a)** It is the second-price auction (due to Vickrey).

(b) The weakly dominant strategy is to bid *V*. Let *M* be the m^{th} bid on the seller's list modified by removing the bid of player *i*. Three cases are possible. **Case 1:** $M < V$. In this case, by bidding *V* player *i* obtains one unit and pays a price of *M* (which is now the $(m+1)^{th}$ bid) and thus obtains a payoff of $V - M > 0$. The same happens if he chooses any other bid $b_i > M$. If he chooses $b_i = M$ then he either gets the same payoff as above or a payoff of zero (in case his index is higher than the index of the other player who submitted a bid of *M*). If he chooses $b_i < M$ then he does not get the object and his payoff is zero. **Case 2:** $M = V$. In this case, if he gets one unit he pays *V* and thus his payoff is zero; if he doesn't get the object, his payoff is zero. Thus, *any* bid gives him a payoff of zero. **Case 3:** $M > V$. In this case, if he bids *V* (or any other $b_i < M$) he does not get the good and his payoff is zero. If he chooses $b_i > M$ then he gets the object by paying *M* and his payoff is $V - M < 0$. If he chooses $b_i = M$, then he either does not get the object (in case his index is higher than the index of the other player who submitted a bid of *M*) and his payoff is zero, or he does get the object and pays *M*, obtaining a payoff of $V - M < 0$.
Thus in all cases bidding *V* is at least as good as submitting a different bid.

(c) If everybody else bids less than *V*, say $V - \varepsilon$ (with $0 < \varepsilon \le V$), then by bidding *V* player *i* gets one unit at the price of $V - \varepsilon$ (and thus obtains a payoff of $\varepsilon > 0$), but he can also get the unit, at the same price, by submitting a bid of $V - \frac{\varepsilon}{2}$ or a bid of $2V$ (or any other bid higher than $V - \varepsilon$).

(d) The seller sells *m* units at a price of *V*, thus her revenue is *mV*.